Homework #4 (due 2/22/13)

The learning objective of this homework is to understand the elastic response of contact loading on the surface of a cell. The load can be due to a normal force from an AFM tip or a tangential traction from a microbead under optical or magnetic force. The intent is to determine the spatial displacements of the cell and the stresses at the surface of a cell.

We will consider the cell under loading as an elastic half-space bounded by a surface plane ($z=0$) under the action of a normal p and tangential q applied loads on a closed area S in the neighborhood about the origin O. Outside of S, the tractions are zero ($p=0$, $q=0$). The loading is two-dimensional and can vary in the x and y direction such that

$$p(x,y), \quad q_x(x,y), \quad q_y(x,y)$$

The stress system is three-dimensional with six components of stress: σ_x, σ_y, σ_z, τ_{xy}, τ_{yx}, τ_{xz}.

Take $C(\xi,\eta)$ to be a general point on the surface S, while $A(x,y,z)$ is a general point within the body of the solid. Let ρ be the distance between $\overline{CA}$

$$\rho = \sqrt{(\xi - x)^2 + (\eta - y)^2 + z^2} \qquad (1)$$

We now define the following potential functions which are harmonic functions and satisfy Laplace's Equation.

$$F_1 = \iint_S q_x(\xi,\eta)\Omega d\xi d\eta \qquad (2)$$

$$G_1 = \iint_S q_y(\xi,\eta)\Omega d\xi d\eta \qquad (3)$$

$$H_1 = \iint_S p(\xi,\eta)\Omega d\xi d\eta \qquad (4)$$

where

$$\Omega = z\ln(\rho + z) - \rho \qquad (5)$$

In addition, we define the potential functions

$$F = \frac{\partial F_1}{\partial z} = \iint_S q_x(\xi,\eta)\ln(\rho+z)d\xi d\eta \tag{6}$$

$$G = \frac{\partial G_1}{\partial z} = \iint_S q_y(\xi,\eta)\ln(\rho+z)d\xi d\eta \tag{7}$$

$$H = \frac{\partial H_1}{\partial z} = \iint_S p(\xi,\eta)\ln(\rho+z)d\xi d\eta \tag{8}$$

We can write

$$\psi_1 = \frac{\partial F_1}{\partial x} + \frac{\partial G_1}{\partial y} + \frac{\partial H_1}{\partial z} \tag{9}$$

$$\psi = \frac{\partial \psi_1}{\partial z} = \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} + \frac{\partial H}{\partial z} \tag{10}$$

It can be shown that the elastic deformations u_x, u_y, u_z at any point $A(x,y,z)$ in the elastic half-space can be expressed by

$$u_x = \frac{1}{4\pi G}\left\{2\frac{\partial F}{\partial z} - \frac{\partial H}{\partial x} + 2\nu\frac{\partial \psi_1}{\partial x} - z\frac{\partial \psi}{\partial x}\right\} \tag{11}$$

$$u_y = \frac{1}{4\pi G}\left\{2\frac{\partial G}{\partial z} - \frac{\partial H}{\partial y} + 2\nu\frac{\partial \psi_1}{\partial y} - z\frac{\partial \psi}{\partial y}\right\} \tag{12}$$

$$u_z = \frac{1}{4\pi G}\left\{\frac{\partial H}{\partial z} + (1-2\nu)\psi - z\frac{\partial \psi}{\partial z}\right\} \tag{13}$$

By Hooke’s Law, the corresponding stresses are

$$\sigma_x = \frac{2\nu G}{1-2\nu}\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right) + 2G\frac{\partial u_x}{\partial x} \tag{14}$$

$$\sigma_y = \frac{2\nu G}{1-2\nu}\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right) + 2G\frac{\partial u_y}{\partial y} \tag{15}$$

$$\sigma_z = \frac{2\nu G}{1-2\nu}\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right) + 2G\frac{\partial u_z}{\partial z} \tag{16}$$

$$\tau_{xy} = G\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) \tag{17}$$

$$\tau_{yz} = G\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right) \tag{18}$$

$$\tau_{xz} = G\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) \tag{19}$$

1. Concentrated Pressure Load

Consider that case that area S over which a normal traction acts on the cell's surface is made to approach zero. This yields a concentrated point force P defined as

$$\iint_S p(\xi,\eta)d\xi d\eta = P \tag{20}$$

a) Show that the elastic displacements at any point in the elastic half-space (a/k/a cell) are

$$u_x = \frac{P}{4\pi G}\left(\frac{xz}{\rho^3} - \frac{(1-2\nu)x}{\rho(\rho+z)}\right) \tag{21}$$

$$u_y = \frac{P}{4\pi G}\left(\frac{yz}{\rho^3} - \frac{(1-2\nu)y}{\rho(\rho+z)}\right) \tag{22}$$

$$u_z = \frac{P}{4\pi G}\left(\frac{2(1-\nu)}{\rho} + \frac{z^2}{\rho^3}\right) \tag{23}$$

b) What are the stresses at the origin for a load P acting on the surface of a cell with Poisson's ratio ν = 0.5?

2. Concentrated Tangential Traction

Now, consider the case where a tangential traction acting in the x-direction is concentrated on a vanishingly small area S at the origin O. This yields a concentrated point force Q_x defined as

$$\iint_S q_x(\xi,\eta)\,d\xi\,d\eta = Q_x \qquad (24)$$

a) Show that the elastic displacements at any point in the elastic half-space are

$$u_x = \frac{Q_x}{4\pi G}\left(\frac{1}{\rho} + \frac{x^2}{\rho^3} + (1-2\nu)\left(\frac{1}{\rho+z} - \frac{x^2}{\rho(\rho+z)^2}\right)\right) \qquad (25)$$

$$u_y = \frac{Q_x}{4\pi G}\left(\frac{xy}{\rho^3} - (1-2\nu)\frac{xy}{\rho(\rho+z)^2}\right) \qquad (26)$$

$$u_z = \frac{Q_x}{4\pi G}\left(\frac{xz}{\rho^3} + (1-2\nu)\frac{x}{\rho(\rho+z)}\right) \qquad (27)$$

b) What are the stresses at a surface position 100 nm behind the line of action for an applied load Q_x = 2π/3 nN (≈ 2.1 nN) acting on the surface of a cell, i.e. at $A(-10^{-7},0,0)$? Assume that the cell has Poisson's ratio ν = 0.5.

c) What are the stresses at a position on the surface 10 μm behind the same load?

Reference:

K.L. Johnson, (1985) "Contact Mechanics", Cambridge University Press