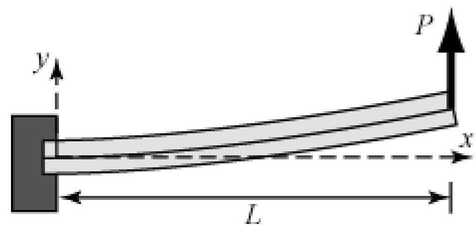
Homework #3 (due 2/9/13)

The learning objective of this homework is to understand the bending mechanics of cantilever beams since they are used frequently in cell mechanics studies.

Euler-Bernoulli Beam Theory



For a cantilever beam, the relationship between an applied force P at its tip and the beam's local radius of curvature ρ is given by

$$\rho = \frac{EI}{P(L-x)}, \quad (1)$$

where E is the modulus of elasticity of the beam's material and I is the second moment of area. As a consequence, a cantilever has its greatest curvature at the wall, $p(0) = EI/PL$, and has no local curvature at its tip, $p(L) = \infty$.

From calculus, the radius of curvature ρ can be expressed in terms of the infinitesimal deflection dy and infinitesimal distance dx as follows

$$\frac{1}{\rho} = \frac{\dfrac{d^2y}{dx^2}}{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{3/2}} . \quad (2)$$

For forces that induce a small degree of bending in the beam, we can assume that dy/dx is small. As a consequence, we can also assume that $(dy/dx)^2 \approx 0$. Thus, equation 2 simplifies to

$$\frac{1}{\rho} \approx \frac{d^2y}{dx^2} \quad (3)$$

Combining equations 1 and 3 yields

$$\frac{d^2y}{dx^2} = \frac{P}{EI}\left(L - x\right) . \quad (4)$$

This second order linear differential equation is the governing equation for the deflection of a cantilever beam.

Problem 1

Determine the equation for the defection $y(x)$ in the cantilever beam shown above. Find the equation for bending stiffness by the determining the deflection at its end $y(L)$ for a given load P.

Problem 2

Calculate the bending stiffness for an AFM tip with a V-shape, where two cantilevers with rectangular cross-sections join together at the tip. Each arm of the V-shape acts as a spring and together, they act as springs in series. Assume that an AFM tip made from silicon nitride has a modulus of elasticity of 100 GPa and each of its arms have a length of 200 μm, width of 30 μm, and thickness of 1 μm. The second moment of area for a rectangular beam is given by

$$I = \frac{wt^3}{12} . \quad (5)$$

Problem 3

Make a contour plot of bending stiffness for a PDMS cantilever as a function of its diameter and length. Plot the contour lines for 0.1, 1, 5, 10, 25, 50, 100, 200, and 400 nN/μm for values of diameter between 0.3 to 5 μm and length between 1 and 20 μm. Assume that PDMS has a modulus of elasticity of 2.5 MPa. The second moment of area for a cylindrical beam is given by

$$I = \frac{\pi d^4}{64} . \quad (6)$$