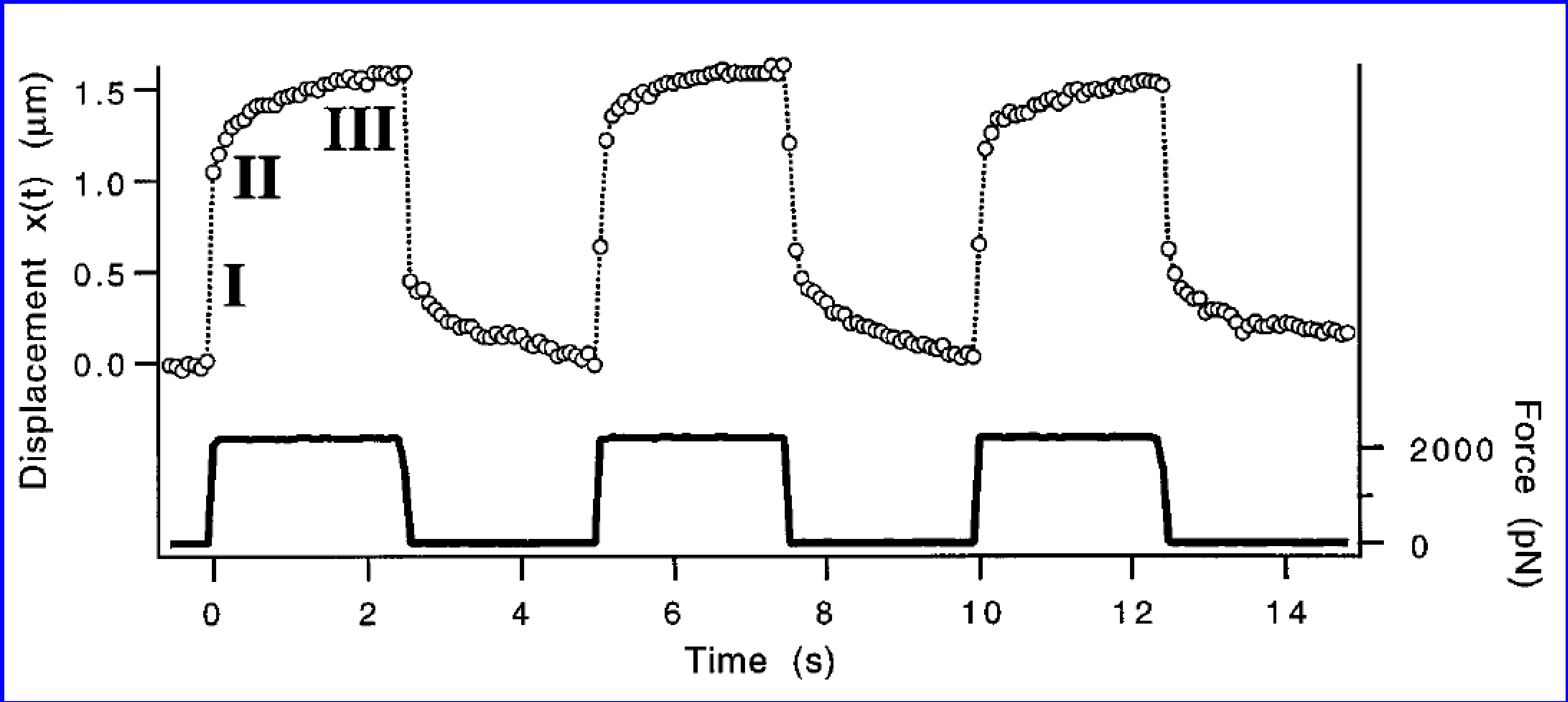
Session 12

LUMPED PARAMETER MODELS

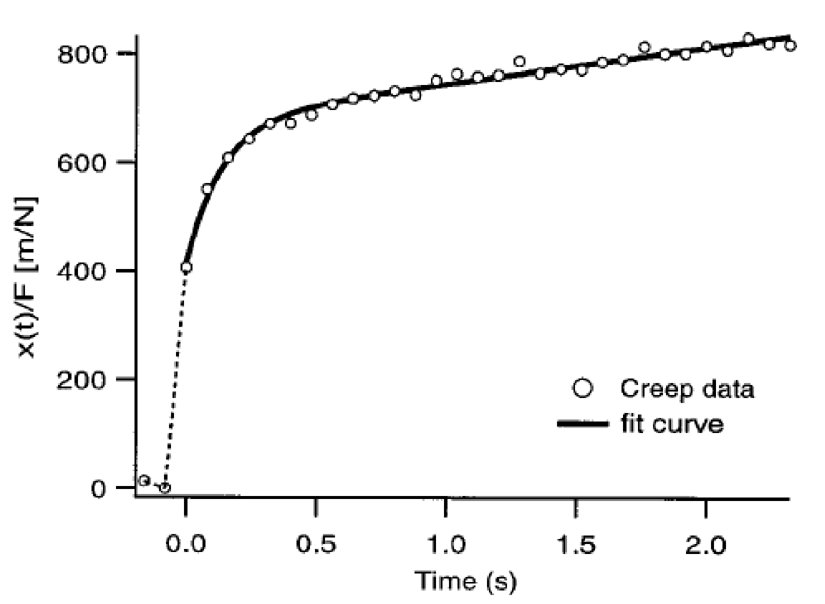
Recall Bausch et al.

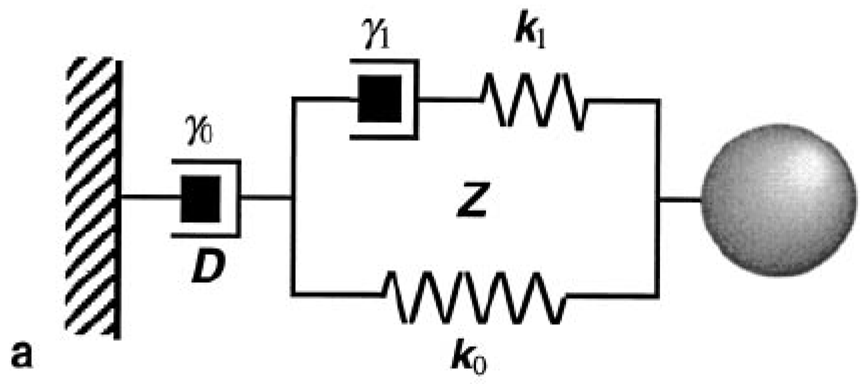
- Classic Creep Response
  - Elastic displacement (Regime I)
  - Relaxation (Regime II)
  - Steady-state Flow (Regime III)

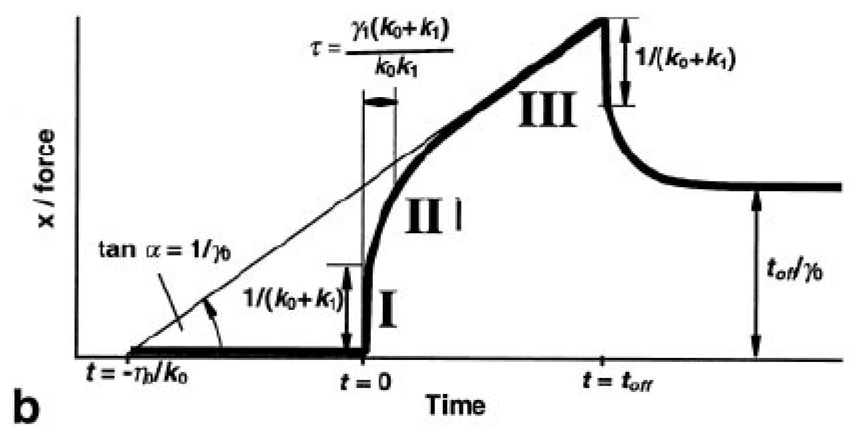


The Model

- Effective elastic modulus $k = k_1 + k_2$
- Viscosity γ_0
- Relaxation time τ

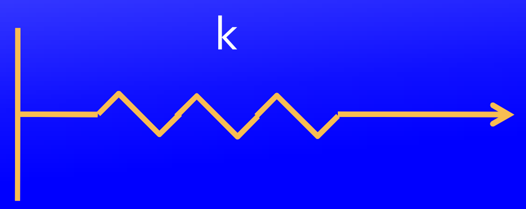






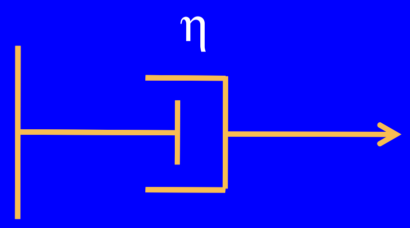
Springs and Dashpots

- Force is proportional (k) with distance



$F = k\,x$

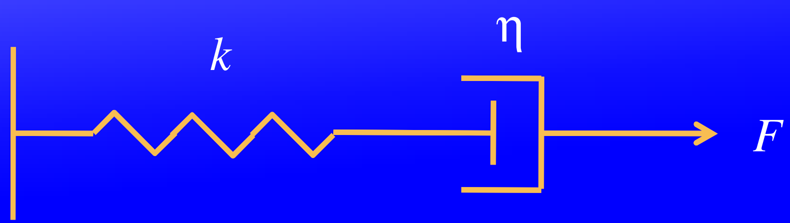
- Force is proportional (η) with velocity



$F = \eta\,\dot{x}$

Maxwell Body

- Spring and dashpot in series



- Displacement is combined

$$x = x_{sp} + x_{dp}$$

$$\dot{x} = \dot{x}_{sp} + \dot{x}_{dp}$$

$$\dot{x} = \frac{\dot{F}}{k} + \frac{F}{\eta}$$

Maxwell Body

- Consider applying a step force

$$F(t>0)=F$$

$$\dot{F}=0$$

- Differential equation becomes

$$\dot{x}=\frac{\dot{F}}{k}+\frac{F}{\eta}$$

$$\dot{x}=0+\frac{F}{\eta}$$

$$x=\frac{F}{\eta}t+C$$

Maxwell Body

- Initially, dashpot has zero velocity so displacement is only from spring

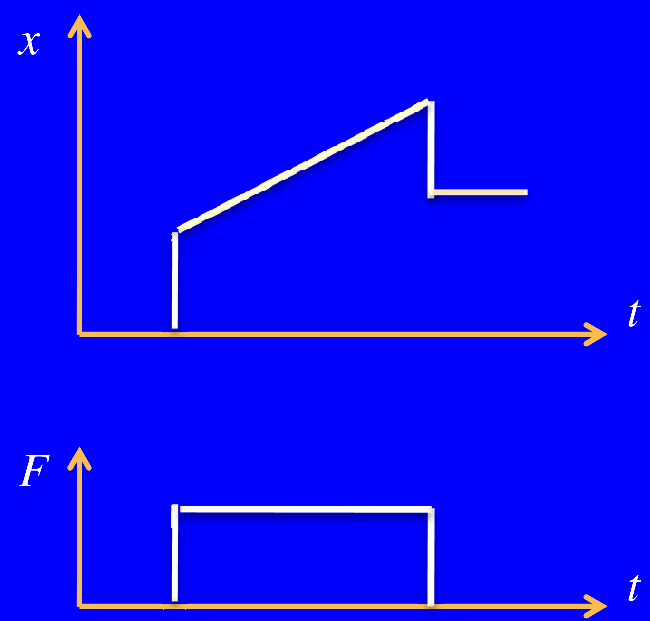
$$x(0^+) = \frac{F}{k} = 0^+ + C$$

- Thus,

$$x = \frac{F}{\eta} t + C$$

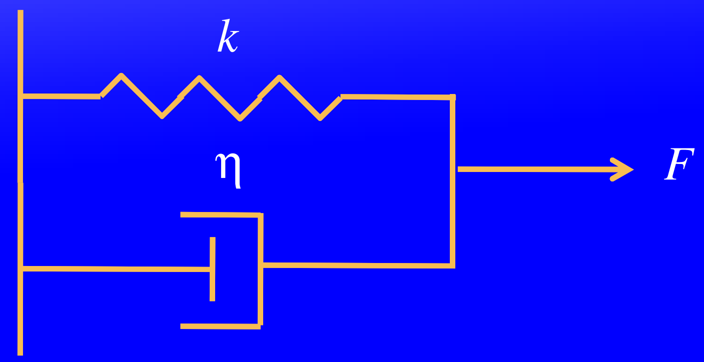
$$x = \frac{F}{\eta} t + \frac{F}{k}$$

$$x = \left(\frac{1}{\eta} t + \frac{1}{k} \right) F$$



Voigt Body

- Spring and dashpot in parallel



- Sum of forces at cross-bar

$$F = F_{sp} + F_{dp}$$

- Since displacements are the same

$$F = kx + \eta \dot{x}$$

Voigt Body

- Have 1st ODE

$$\dot{x} + \frac{k}{\eta}x = \frac{F}{\eta}, \qquad x(0) = 0$$

- Use integration factor

$$\mu = \exp\left(\int \frac{k}{\eta} dt\right) = \exp\left(kt/\eta\right)$$

$$\exp\left(kt/\eta\right)\left(\dot{x} + \frac{k}{\eta}x\right) = \frac{F}{\eta}\exp\left(kt/\eta\right)$$

$$\frac{d}{dt}\left(\exp(kt/\eta)x\right) = \frac{F}{\eta}\exp\left(kt/\eta\right)$$

$$\exp(kt/\eta)x = \frac{F}{k}\exp\left(kt/\eta\right) + C$$

Voigt Body

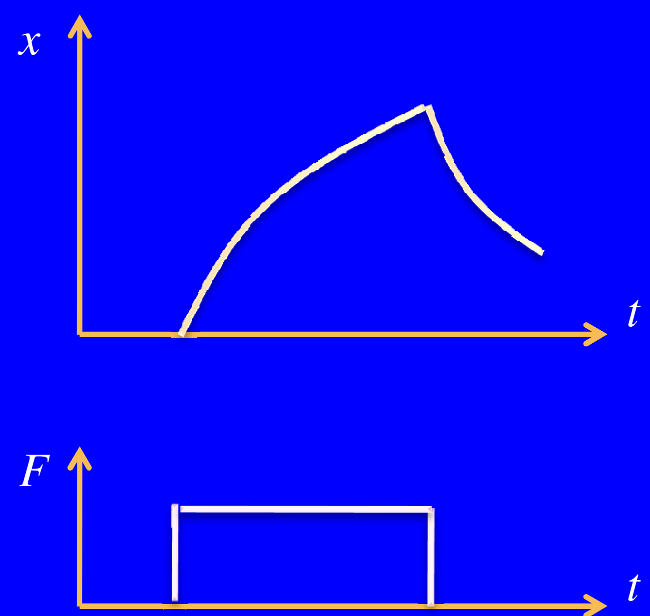
- Use initial condition

$$\exp(kt/\eta)x = \frac{F}{k}\exp\left(kt/\eta\right) + C$$

$$\Rightarrow 0 = \frac{F}{k} + C$$

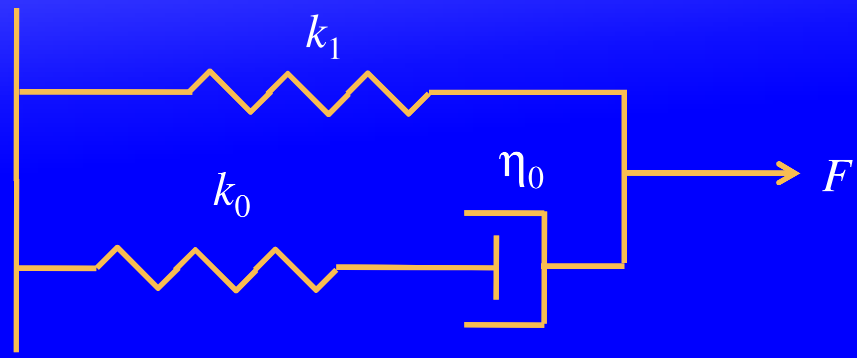
- Thus

$$x = \frac{F}{k}\left(1 - \exp\left(-kt/\eta\right)\right)$$



Kelvin Body

- (Standard Linear Solid)



- Force is combined

$$F = F_{k1} + F_{maxwell}$$
$$\Rightarrow F_{maxwell} = F - k_1 x$$

Kelvin Body

- Recall Maxwell body equation

$$\dot{x} = \frac{\dot{F}_{maxwell}}{k_0} + \frac{F_{maxwell}}{\eta_0}$$

- Using previous equation

$$\dot{x} = \frac{\left(\dot{F} - k_1 \dot{x}\right)}{k_0} + \frac{\left(F - k_1 x\right)}{\eta_0}$$

- Rearranging

$$k_1 \left(x + \frac{\eta_0}{k_1} \left(1 + \frac{k_1}{k_0} \right) \dot{x} \right) = F + \frac{\eta_0}{k_0} \dot{F}$$

Kelvin Body

- Substitute τ for 1st ODE

$$\tau = \frac{\eta_0}{k_1}\left(1 + \frac{k_1}{k_0}\right)$$

$$k_1\left(x + \tau\dot{x}\right) = F + \frac{\eta_0}{k_0}\dot{F}$$

- For "step" force, simplifies to

$$F(t > 0) = F$$

$$\dot{F} = 0$$

$$k_1\left(x + \tau\dot{x}\right) = F + 0$$

$$\dot{x} + \frac{1}{\tau}x = \frac{F}{k_1\tau}$$

Kelvin Body

- Use integration factor

$$\dot{x} + \frac{1}{\tau} x = \frac{F}{k_1 \tau}$$

$$\mu = \exp(t/\tau)$$

$$\exp(t/\tau)\left(\dot{x} + \frac{1}{\tau} x\right) = \exp(t/\tau)\frac{F}{k_1 \tau}$$

$$\exp(t/\tau)\,x = \int \exp(t/\tau)\frac{F}{k_1 \tau}\,dt$$

$$\exp(t/\tau)\,x = \exp(t/\tau)\frac{F}{k_1} + C$$

Kelvin Body

- Initially (again), dashpot has zero velocity so displacement is only from springs

$$F = k_0 x(0^+) + k_1 x(0^+)$$

$$x(0^+) = \frac{F}{k_0 + k_1}$$

- Thus

$$\exp(t/\tau)\, x = \exp(t/\tau)\frac{F}{k_1} + C$$

$$\frac{F}{k_0 + k_1} = \frac{F}{k_1} + C \Rightarrow C = F\left(\frac{1}{k_0 + k_1} - \frac{1}{k_1}\right)$$

Kelvin Body

- Simplifying

$$x = \frac{F}{k_1} + F\left(\frac{1}{k_0 + k_1} - \frac{1}{k_1}\right)\exp\left(-t/\tau\right)$$

$$x = \frac{F}{k_1} + \frac{F}{k_1}\left(\frac{k_1}{k_0 + k_1} - 1\right)\exp\left(-t/\tau\right)$$

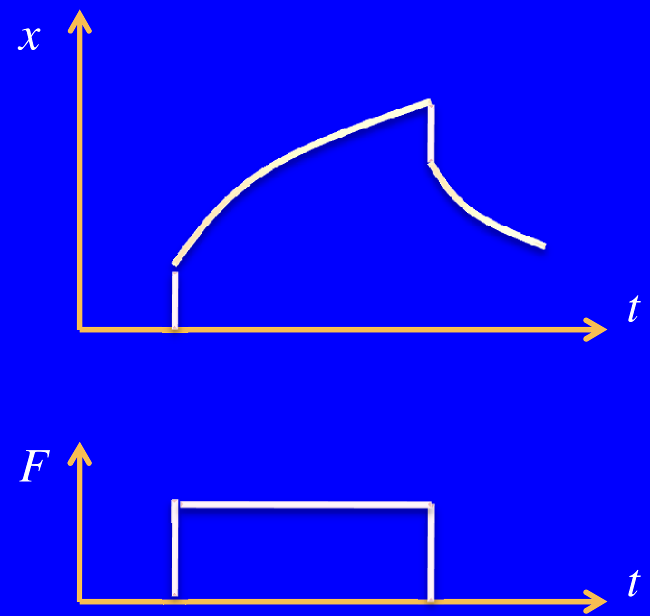
$$x = \frac{F}{k_1}\left(1 + \left(\frac{k_1}{k_0 + k_1} - \frac{k_0 + k_1}{k_0 + k_1}\right)\exp\left(-t/\tau\right)\right)$$

$$x = \frac{F}{k_1}\left(1 - \left(\frac{k_0}{k_0 + k_1}\right)\exp\left(-t/\tau\right)\right)$$

Kelvin Body

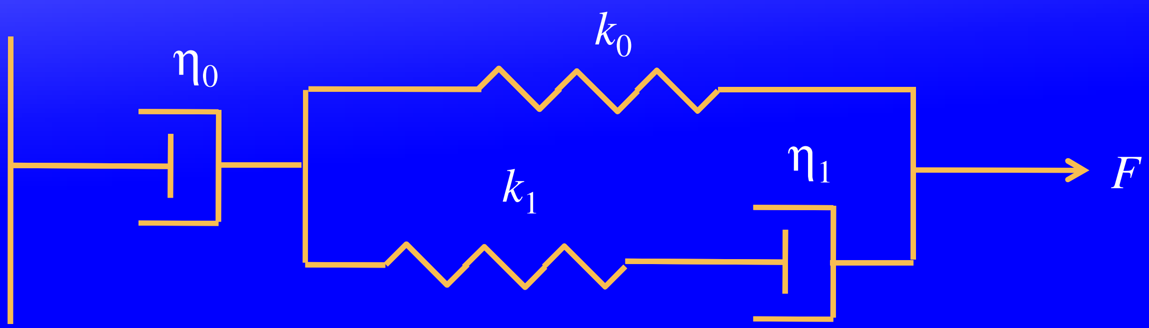
- Graphically

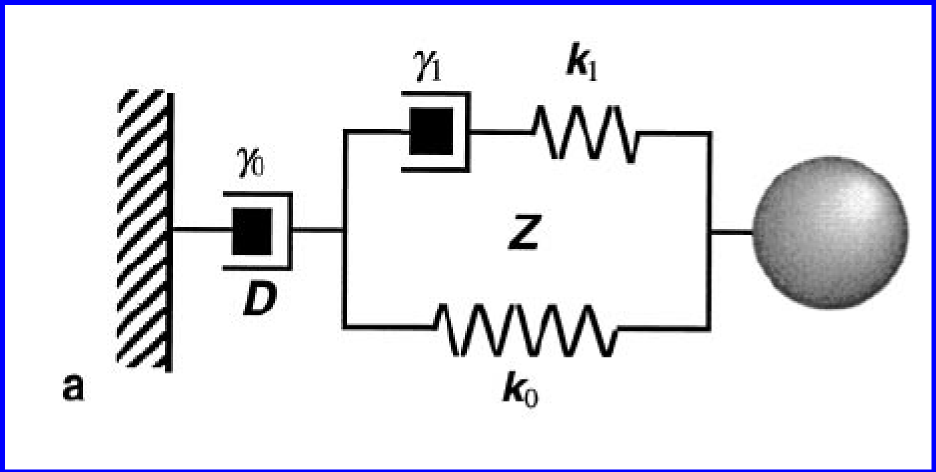
$$x = \frac{F}{k_1}\left(1 - \left(\frac{k_0}{k_0 + k_1}\right)\exp\left(-t/\tau\right)\right)$$



Bausch et al.

- Two spring, two dashpots





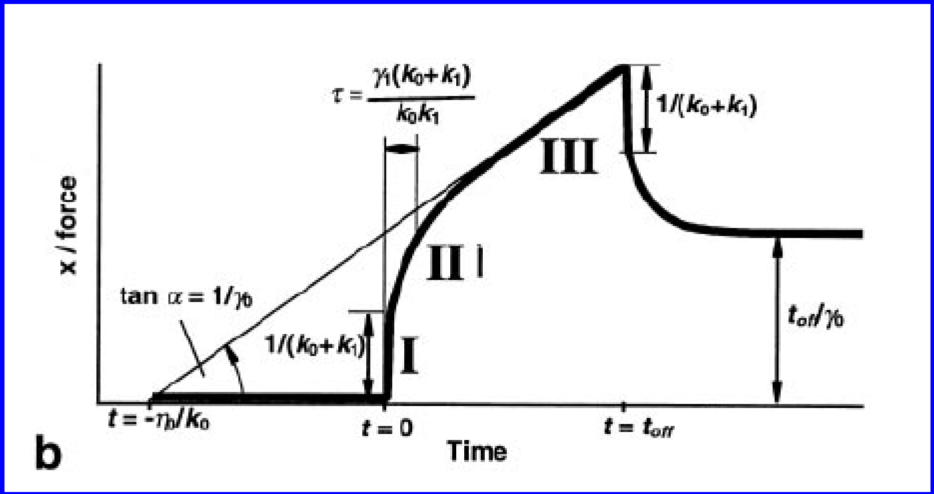
Bausch et al.

- By superposition

$$\frac{x}{F}=\frac{1}{k_0}\left(1-\left(\frac{k_1}{k_0+k_1}\right)\exp\left(-t/\tau\right)\right)+\frac{t}{\eta_0}$$

Note: difference in subscripts from Kelvin body

- Graphically



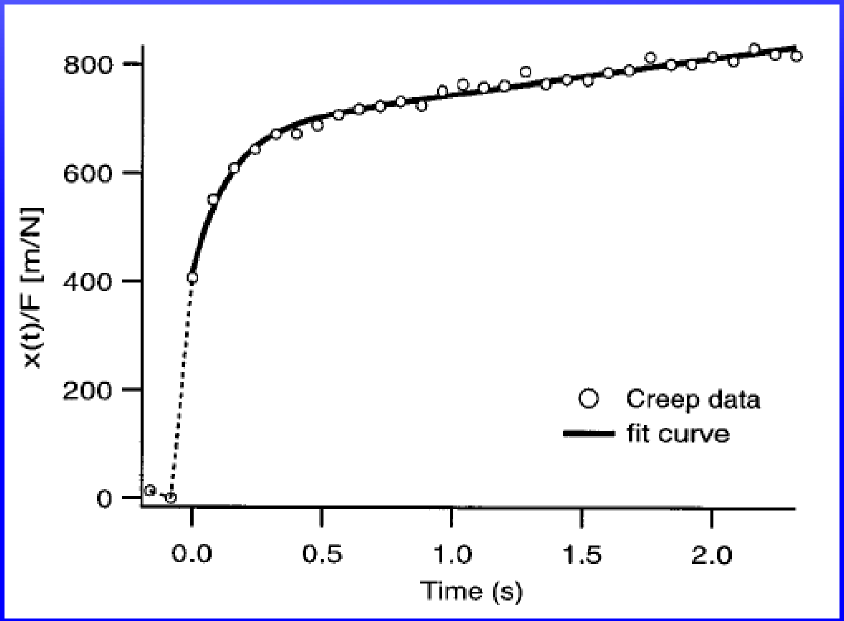
Regression Analysis

- Least Squares Fit

$$y = Ax + B$$

$$\Rightarrow A = \frac{\sum x^2 \sum y - \sum x \sum xy}{N\sum x^2 - \left(\sum x\right)^2}$$

$$\Rightarrow B = \frac{N\sum xy - \sum x \sum y}{N\sum x^2 - \left(\sum x\right)^2}$$



- More complex fitting requires statistical package

$$\frac{x}{F} = \frac{1}{k_0}\left(1 - \left(\frac{k_1}{k_0 + k_1}\right)\exp\left(-\frac{\eta_1\left(k_0 + k_1\right)}{k_0 k_1}t\right)\right) + \frac{t}{\eta_0}$$