ANALYSIS AND MODELING OF CELL MECHANICS

Homework #5 (due 5/6/09)

The learning objective of this homework is to understand the rheological properties of a cell as it relates to the different viscoelastic models, complex modulus, storage modulus, and loss modulus.

Recall Euler's Formula:

$$e^{ix} = \cos x + i \sin x$$

where i is the imaginary unit and x is given in radians. Consider the simple harmonic motion of a microbead on the surface of a cell under external force. The displacement will be of the form

$$x = A\cos(wt) \tag{1}$$

where A is the amplitude and w is the frequency in radians per second. Harmonic motion can be due a magnetic bead moving laterally moves to a magnetic twisting field or a latex bead moving under an oscillating beam of an optical trap.

We can write eq. 1 in polar form,

$$x = Ae^{iwt} \;. \tag{2}$$

The derivative with respect to time is

$$\dot{x} = \frac{d}{dt}\left(Ae^{iwt}\right) = Aiwe^{iwt} \tag{3}$$

Notice that by combining eq. 2 and 3,

$$\dot{x} = iwx \tag{4}$$

This is a useful expression for derivatives that we will apply to the lumped parameter viscoelastic models. Recall the differential equation for a Maxwell body

$$\dot{x} = \frac{\dot{F}}{k} + \frac{F}{\eta} \tag{5}$$

This can be written in polar form of a complex number

$$iwx = \frac{iwF}{k} + \frac{F}{\eta} \tag{6}$$

We will now introduce the complex modulus G^* by noting that it is can be regarded as the ratio of force to displacement.*

$$F = G^* x \tag{7}$$

With this, we can write eq. 6 as

$$iwx = \frac{iwG^* x}{k} + \frac{G^* x}{\eta} \tag{8}$$

$$iw = \left(\frac{iw}{k} + \frac{1}{\eta}\right) G^* \tag{9}$$

This expression can be rearranged into the following

$$G^* = \frac{w^2 k \eta^2}{w^2\eta^2 + k^2} + \frac{wk^2\eta}{w^2\eta^2 + k^2} i \tag{10}$$

The real part of the complex modulus is the storage modulus and the imaginary part is the loss modulus ($G^* = G' + iG''$). We can easily read off the two distinct terms in eq. 10 to obtain the rheological properties for the model

$$G' = \frac{w^2 k \eta^2}{w^2\eta^2 + k^2} \tag{11}$$

$$G'' = \frac{wk^2\eta}{w^2\eta^2 + k^2} \tag{12}$$

Using a similar approach, what is the storage modulus and loss modulus for a) Voigt and b) Kelvin bodies?†

Reference:

Y.C. Fung, (1993) "Biomechanics: Mechanical Properties of Living Tissues", Springer. (Chapter 2.12)

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* A more formal definition for the complex modulus is the ratio of shear stress to shear strain under oscillatory conditions.

† For a Kelvin viscoelastic model, reported model parameters from micropipette aspiration data have a range from k_1 = 5–360 Pa, k_0 = 22–2300 Pa, h_0 = 6–4,300 Pa·s. Lim, C.T., et al. (2006) "Mechanical models for living cells – a review", J. Biomechanics, 39:195–216.