ANALYSIS AND MODELING OF CELL MECHANICS

Homework #4 (due 4/27/09)

The learning objective of this homework is to understand the elastic response of contact loading on the surface of a cell. The load can be due to a normal force from an AFM tip or a tangential traction from a microbead under optical or magnetic force. The intent is to determine the spatial displacements of the cell and the stresses in the cell membrane.

We will consider the cell under loading as an elastic half-space bounded by a surface plane ($z=0$) under the action of a normal p and tangential q applied loads on a closed area S in the neighborhood about the origin O. Outside of S, the tractions are zero ($p=0$, $q=0$). The loading is two-dimensional and can vary in the x and y direction such that

$$p(x,y),\quad q_x(x,y),\quad q_y(x,y)$$

The stress system is three-dimensional with six components of stress: σ_x, σ_y, σ_z, τ_{xy}, τ_{yx}, τ_{xz}.

Take $C(\xi,\eta)$ to be a general point on the surface S, while $A(x,y,z)$ is a general point within the body of the solid. Let ρ be the distance between $\overline{CA}$

$$\rho=\sqrt{(\xi-x)^2+(\eta-y)^2+z^2} \tag{1}$$

We now define the following potential functions which are harmonic functions and satisfy Laplace's Equation.

$$F_1=\iint_S q_x(\xi,\eta)\Omega d\xi d\eta \tag{2}$$

$$G_1=\iint_S q_y(\xi,\eta)\Omega d\xi d\eta \tag{3}$$

$$H_1=\iint_S p(\xi,\eta)\Omega d\xi d\eta \tag{4}$$

where

$$\Omega=z\ln(\rho+z)-\rho \tag{5}$$

In addition, we define the potential functions

$$F = \frac{\partial F_1}{\partial z} = \iint_S q_x(\xi,\eta)\ln(\rho+z)d\xi d\eta \tag{6}$$

$$G = \frac{\partial G_1}{\partial z} = \iint_S q_y(\xi,\eta)\ln(\rho+z)d\xi d\eta \tag{7}$$

$$H = \frac{\partial H_1}{\partial z} = \iint_S p(\xi,\eta)\ln(\rho+z)d\xi d\eta \tag{8}$$

We can write

$$\psi_1 = \frac{\partial F_1}{\partial x} + \frac{\partial G_1}{\partial y} + \frac{\partial H_1}{\partial z} \tag{9}$$

$$\psi = \frac{\partial \psi_1}{\partial z} = \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} + \frac{\partial H}{\partial z} \tag{10}$$

It can be shown that the elastic deformations u_x, u_y, u_z at any point $A(x,y,z)$ in the elastic half-space can be expressed by

$$u_x = \frac{1}{4\pi G}\left\{2\frac{\partial F}{\partial z} - \frac{\partial H}{\partial x} + 2\nu\frac{\partial \psi_1}{\partial x} - z\frac{\partial \psi}{\partial x}\right\} \tag{11}$$

$$u_y = \frac{1}{4\pi G}\left\{2\frac{\partial G}{\partial z} - \frac{\partial H}{\partial y} + 2\nu\frac{\partial \psi_1}{\partial y} - z\frac{\partial \psi}{\partial y}\right\} \tag{12}$$

$$u_z = \frac{1}{4\pi G}\left\{\frac{\partial H}{\partial z} + (1-2\nu)\psi - z\frac{\partial \psi}{\partial z}\right\} \tag{13}$$

By Hooke's Law, the corresponding stresses are

$$\sigma_x = \frac{2\nu G}{1-2\nu}\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right) + 2G\frac{\partial u_x}{\partial x} \tag{14}$$

$$\sigma_y = \frac{2\nu G}{1-2\nu}\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right) + 2G\frac{\partial u_y}{\partial y} \tag{15}$$

$$\sigma_z = \frac{2\nu G}{1-2\nu}\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right) + 2G\frac{\partial u_z}{\partial z} \tag{16}$$

$$\tau_{xy} = G\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) \tag{17}$$

$$\tau_{yz} = G\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right) \tag{18}$$

$$\tau_{xz} = G\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) \tag{19}$$

1. Concentrated Pressure Load

Consider that case that area S over which a normal traction acts on the cell's surface is made to approach zero. This yields a concentrated point force P defined as

$$\iint_S p(\xi,\eta)d\xi d\eta = P \tag{20}$$

a) Show that the elastic displacements at any point in the elastic half-space (a/k/a cell) are

$$u_x = \frac{P}{4\pi G}\left(\frac{xz}{\rho^3} - \frac{(1-2\nu)x}{\rho(\rho+z)}\right) \tag{21}$$

$$u_y = \frac{P}{4\pi G}\left(\frac{yz}{\rho^3} - \frac{(1-2\nu)y}{\rho(\rho+z)}\right) \tag{22}$$

$$u_z = \frac{P}{4\pi G}\left(\frac{2(1-\nu)}{\rho} - \frac{z^2}{\rho^3}\right) \tag{23}$$

b) What are the stresses at the origin for a load P acting on the surface of a cell with Poisson's ratio ν = 0.5?

2. Concentrated Tangential Traction

Now, consider the case where a tangential traction acting in the x-direction is concentrated on a vanishingly small area S at the origin O. This yields a concentrated point force Q_x defined as

$$\iint_S q_x(\xi,\eta)\,d\xi\,d\eta = Q_x \tag{24}$$

a) Show that the elastic displacements at any point in the elastic half-space are

$$u_x = \frac{Q_x}{4\pi G}\left(\frac{1}{\rho}+\frac{x^2}{\rho^3}+(1-2\nu)\left(\frac{1}{\rho+z}-\frac{x^2}{\rho(\rho+z)^2}\right)\right) \tag{25}$$

$$u_y = \frac{Q_x}{4\pi G}\left(\frac{xy}{\rho^3}-(1-2\nu)\frac{xy}{\rho(\rho+z)^2}\right) \tag{26}$$

$$u_z = \frac{Q_x}{4\pi G}\left(\frac{xz}{\rho^3}+(1-2\nu)\frac{x}{\rho(\rho+z)}\right) \tag{27}$$

b) What are the stresses at a surface position 100 nm behind the line of action for an applied load Q_x = 2π/3 nN (≈ 2.1 nN) acting on the surface of a cell, i.e. at $A(-10^{-7},0,0)$? Assume that the cell has Poisson's ratio ν = 0.5.

c) What are the stresses at a position on the surface 10 μm behind the same load?

Reference:

K.L. Johnson, (1985) "Contact Mechanics", Cambridge University Press