BIOLOGICAL FRAMEWORKS FOR ENGINEERS

Homework #7 (due 12/5/11)

ASSIGNMENT

1. A normal human contains 5 liters of blood, approximately 2% of which is resident in the systemic (i.e. non-pulmonary) capillaries at any given time.

   (a) Assuming that the capillaries are 8 μm in diameter, estimate the total length of capillaries in the body (excluding the lungs).
   (b) If an average capillary length is 1 mm, how many capillaries are there in the body?
   (c) Cardiac output is 5 liters/min. Assuming that flow rate is evenly distributed throughout a parallel network of capillaries found in (b), estimate the pressure drop across the capillary bed. Assume Newtonian, laminar flow with viscosity μ = 3.5 cP.

2. The pulsatile behavior of the cardiac cycle causes blood vessels to rhythmically expand to store blood by distention and primarily occurs in the elastic arterial vessels. The *Windkessel* model, developed by Otto Frank in 1899, roughly approximates the vascular biomechanics as a chamber with compliance C and volume V (Figure 1A). Beating of the heart cause a temporal varying arterial pressure $p_{art}(t)$ and flow rate from the heart $Q_H(t)$ to enter into the arterial "chamber" and subsequenty flow into the peripheral circulation (arterioles, capillaries, and venules) with flow rate Q(t). The contribution of the peripheral system is modeled as pure flow resistance R due to the narrow lumen diameter and relatively minimal compliance. This "lumped parameter" model is equivalent to an classic RC circuit and is shown in Figure 1B.

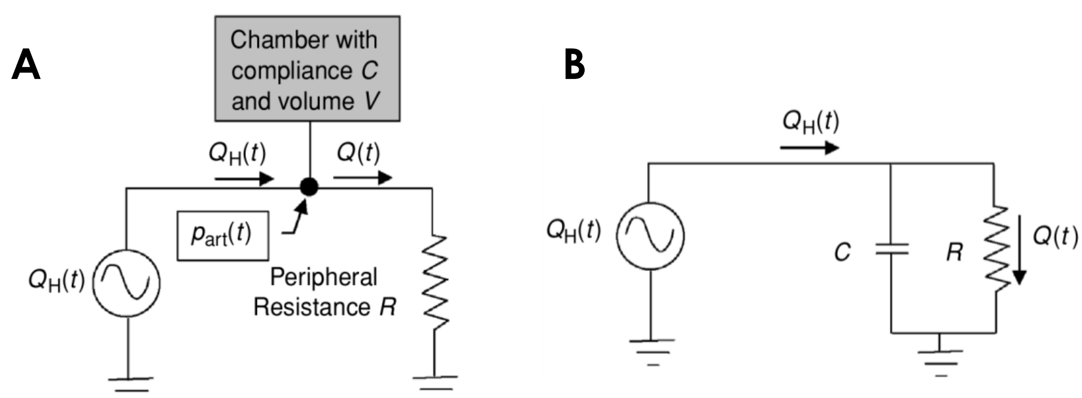

Figure 1. (A) Windkessel model of the vasculature. (B) Equivalent circuit.

Let's assume that an incremental change in the vessel volume (dV) is proportional to a change in the arterial pressure.

$$dV = C * dp_{art} \qquad (1)$$

where C is assumed to be constant. If we neglect back pressure in the veins (p_{veins}= 0), the arterial pressure at any given time can be expressed as

$$p_{art}(t)= Q(t) * R \qquad (2)$$

The difference in the output from the heart Q_H and the delivery to the peripheral system Q must be from the blood accumulating in the compliant artery. Since blood is effectively incompressible, the accumulation/loss of blood must correspond to a change in arterial volume. Conservation of mass then yields

$$dV/dt = Q_H – Q \qquad (3)$$

Let us <u>crudely</u> assume that the cardiac output is a simple sinusoid function

$$Q_H(t) = Q_{max} (1 - \cos(a\, t)) \qquad (4)$$

You can see that this produces a function that oscillates from 0 to 2*Q_{max} with angular frequency a. We can combining Eq. 1-4 into a differential equation whose solution at steady state is

$$Q(t) = Q_{max} - \frac{Q_{max}}{1+(R\,C\,a)^2}\left[\cos(a\,t) + R\,C\,a\sin(a\,t)\right] \qquad (5)$$

Please answer the following.

(a) What is the differential equation whose solution is Eq. 5?
(b) Solve this differential equation and solve for Eq. 5.
Hints: 1) Assume initial condition Q(0)=0, 2) use integration by parts, 3) steady state is where t → infinity, and 4) beware of MATHEMATICA!
(c) We know that the adult human heart has an average flow rate of Q_H = 5 L/min and heart rate of 72 beats/min. Graph the flow rate over a 10 s period. Suppose you have an artery with compliance C = 1.25 x 10^{-2} cm^3/Pa that feeds a peripheral system with resistance R = 2 Pa*min/cm^3.
(d) Atherosclerosis causes hardening of the arteries. Graph what happens to the system when the compliance is decreased by one half (C/2) and by one third (C/3). What are the implications to the shear stresses in the vasculature?