* 11.4. ECCENTRIC LOADING; THE SECANT FORMULA

In this section we shall approach the problem of column buckling in a different way, by observing that the load **P** applied to a column is never perfectly centric. Denoting by e the ecentricity of the load, i.e., the distance between the line of action of **P** and the axis of the column (Fig. 11.18a), we shall replace the given eccentric load by a centric force **P** and a couple $\mathbf{M}_A$ of moment $M_A = Pe$ (Fig. 11.18b). It is clear that, no matter how small the load **P** and the eccentricity e, the couple $\mathbf{M}_A$ will cause some bending of the beam (Fig. 11.19). As the eccentric load is increased, both the couple $\mathbf{M}_A$ and the axial force **P** increase, and both cause the beam to bend further. Viewed in this way, the problem of buckling is not a question of determining how long the column can remain straight and stable under an increasing load, but rather how much the column can be permitted to bend under the increasing load, if the allowable stress is not to be exceeded and if the deflection y_{max} is not to become excessive.

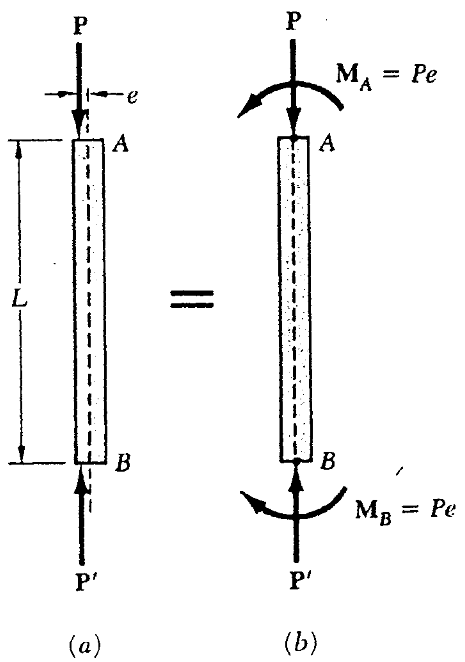

Fig. 11.18

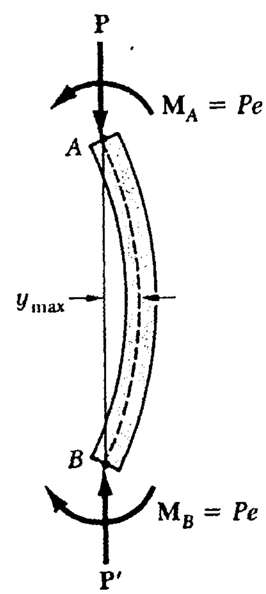

Fig. 11.19

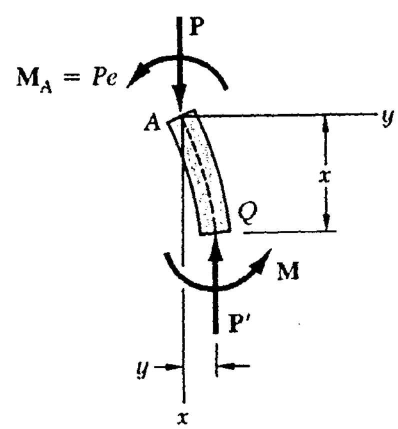

Fig. 11.20

We shall first write and solve the differential equation of the elastic curve, proceeding in the same manner as we did earlier in Secs. 11.2 and 11.3. Drawing the free-body diagram of a portion AQ of the column and choosing the coordinate axes as shown (Fig. 11.20), we find that the bending moment at Q is

$$M = -Py - M_A = -Py - Pe \qquad (11.22)$$

Substituting the value of M into Eq. (8.4) of Sec. 8.2, we write

$$\frac{d^2y}{dx^2} = \frac{M}{EI} = -\frac{P}{EI}y - \frac{Pe}{EI}$$

Transposing the term containing y and setting

$$p^2 = \frac{P}{EI} = K^2 \quad (NS) \tag{11.6}$$

as done earlier, we write

$$\frac{d^2y}{dx^2} + p^2y = -p^2e \tag{11.23}$$

Since the left-hand member of this equation is the same as that of Eq. (11.7), which was solved in Sec. 11.2, we write the general solution of Eq. (11.23) as

$$y = A \sin px + B \cos px - e \tag{11.24}$$

where the last term is a particular solution of Eq. (11.23).

The constants A and B are obtained from the boundary conditions shown in Fig. 11.21. Making first $x = 0$, $y = 0$ in Eq. (11.24), we have

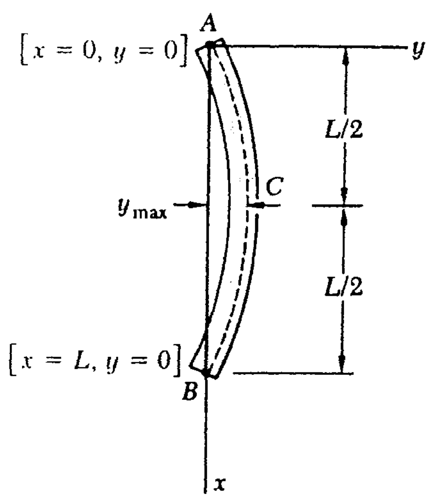

Fig. 11.21

$$B = e$$

Making next $x = L$, $y = 0$, we write

$$A \sin pL = e(1 - \cos pL) \tag{11.25}$$

Recalling that

$$\sin pL = 2 \sin \frac{pL}{2} \cos \frac{pL}{2}$$

and

$$1 - \cos pL = 2 \sin^2 \frac{pL}{2}$$

and substituting into Eq. (11.25), we obtain, after reductions,

$$A = e \tan \frac{pL}{2}$$

Substituting for A and B into Eq. (11.24), we write the equation of the elastic curve:

$$\boxed{y = e\left(\tan \frac{pL}{2} \sin px + \cos px - 1\right)} \tag{11.26}$$

The value of the maximum deflection is obtained by setting $x = L/2$ in Eq. (11.26). We have

$$y_{\max} = e\left(\tan\frac{pL}{2}\sin\frac{pL}{2} + \cos\frac{pL}{2} - 1\right)$$

$$= e\left(\frac{\sin^2\frac{pL}{2} + \cos^2\frac{pL}{2}}{\cos\frac{pL}{2}} - 1\right)$$

$$y_{\max} = e\left(\sec\frac{pL}{2} - 1\right) \tag{11.27}$$

Recalling Eq. (11.6), we write

Recall:

$\sec\alpha = \dfrac{1}{\cos\alpha}$

$$\boxed{y_{\max} = e\left[\sec\left(\sqrt{\frac{P}{EI}}\frac{L}{2}\right) - 1\right]} \tag{11.28}$$

So,

$\sec\alpha \to \infty$ as $\alpha \to \pi/2$

We note from the expression obtained that $y_{\max}$ becomes infinite when

$$\sqrt{\frac{P}{EI}}\frac{L}{2} = \frac{\pi}{2} \tag{11.29}$$

While the deflection does not actually become infinite, it nevertheless becomes unacceptably large, and P should not be allowed to reach the critical value which satisfies Eq. (11.29). Solving (11.29) for P, we have

$$\boxed{P_{\text{cr}} = \frac{\pi^2 EI}{L^2} = \frac{\pi^2 EI}{L_e^2}} \tag{11.30}$$

L_e is effective length

Pinned-Pinned $L_e = L$

Fixed-Free $L_e = 2L$

Fixed-Fixed $L_e = \tfrac{1}{2}L$

Pinned-Fixed $L_e = 0.7L$

which is the value that we obtained in Sec. 11.2 for a beam under a centric load. Solving (11.30) for EI and substituting into (11.28), we may express the maximum deflection in the alternate form

$$y_{\max} = e\left(\sec\frac{\pi}{2}\sqrt{\frac{P}{P_{\text{cr}}}} - 1\right) \tag{11.31}$$

The maximum stress $\sigma_{\max}$ occurs in the section of the column where the bending moment is maximum, i.e., in the transverse section through the midpoint C, and may be obtained by adding the normal stresses due, respectively, to the axial force and the bending couple exerted on that section (cf. Sec. 4.12). We have

$$\sigma_{\max} = \frac{P}{A} + \frac{M_{\max}c}{I} \tag{11.32}$$

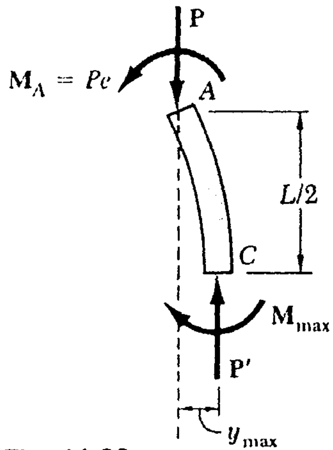

Fig. 11.22

From the free-body diagram of the portion AC of the column (Fig. 11.22), we find that

$$M_{max} = Py_{max} + M_A = P(y_{max} + e)$$

Substituting this value into (11.32) and recalling that $I = Ar^2$, we write
or $I = Ak^2$ (NS)

$$\sigma_{max} = \frac{P}{A}\left[1 + \frac{(y_{max} + e)c}{r^2}\right] \tag{11.33}$$

Substituting for y_{max} the value obtained in (11.28), we write

$$\boxed{\sigma_{max} = \frac{P}{A}\left[1 + \frac{ec}{r^2}\sec\left(\sqrt{\frac{P}{EI}}\frac{L}{2}\right)\right]} \tag{11.34}$$

An alternate form for σ_{max} may be obtained by substituting for y_{max} from (11.31) into (11.33). We have

$$\sigma_{max} = \frac{P}{A}\left(1 + \frac{ec}{r^2}\sec\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right) \tag{11.35}$$

The equation obtained may be used with any end conditions, as long as the appropriate value is used for the critical load (cf. Sec. 11.3).