OCTAHEDRAL SHEAR STRESS CRITERION (VON MISES)

Since hydrostatic stress alone does not cause yielding, we can find a material plane called the octahedral plane, where the stress state can be decoupled into dilation strain energy and distortion strain energy[1]. On the octahedral plane, the octahedral normal stress solely contributes to the dilation strain energy and is

$$s_h = \frac{s_1 + s_2 + s_3}{3} \tag{1}$$

This is the average of the three principal stresses. For example, if $\sigma_1 = \sigma_2 = \sigma_3 = p$ where p is the pressure, then $\sigma_h = p$. The remaining stain energy in the state of stress is determined by the octahedral shear stress and is given by

$$t_h = \tfrac{1}{3}\sqrt{(s_1 - s_2)^2 + (s_2 - s_3)^2 + (s_3 - s_1)^2} \tag{2}$$

We expect yielding when the octahedral shear stress is equal to or exceeds a stress criterion value for failure for a given material, which is the octahedral stress criterion t_{h0}:

$$t_h \geq t_{h0} \qquad \text{(failure)} \tag{3}$$

$$t_h = t_{h0} \qquad \text{(at yielding)} \tag{4}$$

The octahedral stress criterion for say 1080 Al is not likely to be reported in the literature so we need to relate it to the axial yield strength σ_0. For a given material under axial load where $\sigma_1 = \sigma_0$ and $\sigma_2 = \sigma_3 = 0$, we assume that yielding occurs when the octahedral shear stress is equivalent to the octahedral stress criterion. This means we can combine Eq. 2 and 4 to get the octahedral stress criterion in terms of the yield strength:

$$t_{h0} = t_h = \tfrac{1}{3}\sqrt{(s_0 - 0)^2 + (0 - 0)^2 + (0 - s_1)^2} = \tfrac{\sqrt{2}}{3} s_0 \tag{5}$$

With $s_0 = \tfrac{3}{\sqrt{2}} t_{h0}$, we expect to observe yielding in a material under 3-D loading when, as before, we combine Eq. 2 and 4 to get

$$s_0 = \tfrac{1}{\sqrt{2}}\sqrt{(s_1 - s_2)^2 + (s_2 - s_3)^2 + (s_3 - s_1)^2} \tag{6}$$

As a result, we can define the effective stress for von Mises theory to be equivalent to Eq. 6.

$$\bar{s}_H = \tfrac{1}{\sqrt{2}}\sqrt{(s_1 - s_2)^2 + (s_2 - s_3)^2 + (s_3 - s_1)^2} \tag{7}$$

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[1] For dilation, stresses are the same in all directions and there is no shear. For distortion, stresses are different in magnitude and/or direction and so there exists shear stress. See full derivation in Popov, E.P., 1968 *Introduction to Mechanics of Solids*, 1st edition, Prentice Hall, Englewood Cliffs, NJ.

We can express Eq. 7 in terms of the stress invariants (I_1 and I_2):

$$\bar{s}_H = \tfrac{1}{\sqrt{2}}\sqrt{2(I_1 - 3I_2)^2} \tag{8}$$

Failure is likely when

$$\bar{s}_H \geq s_0 \tag{9}$$

For plane stress ($s_3 = 0$), we expect yielding when $\bar{s}_H = s_0$ and so

$$
\begin{aligned}
s_0 &= \tfrac{1}{\sqrt{2}}\sqrt{(s_1 - s_2)^2 + (s_2 - 0)^2 + (0 - s_1)^2} \\
s_0^2 &= \tfrac{1}{2}\left[(s_1 - s_2)^2 + s_2^2 + s_1^2\right] \\
s_0^2 &= \tfrac{1}{2}\left[s_1^2 - 2s_1 s_2 + s_2^2 + s_2^2 + s_1^2\right] \\
s_0^2 &= s_1^2 - s_1 s_2 + s_2^2
\end{aligned}
\tag{10}
$$

The last form in Eq. 10 is an ellipse with its major axis along the $\sigma_1 = \sigma_2$ line. We can solve this and graph it in MATHEMATICA:

```
Eqn = Solve[σ1^2-σ1*σ2+σ2^2==1,σ2]
  {{σ2 → 1/2 (σ1 - √(4 - 3 σ1^2))}, {σ2 → 1/2 (σ1 + √(4 - 3 σ1^2))}}
Eqn[[1]][[1]][[2]]
  1/2 (σ1 - √(4 - 3 σ1^2))
Plot[{Eqn[[1]][[1]][[2]], Eqn[[2]][[1]][[2]]}, {σ1, -2√3/3, 2√3/3},
 AxesLabel → {σ1, σ2}, AspectRatio → 1]
```

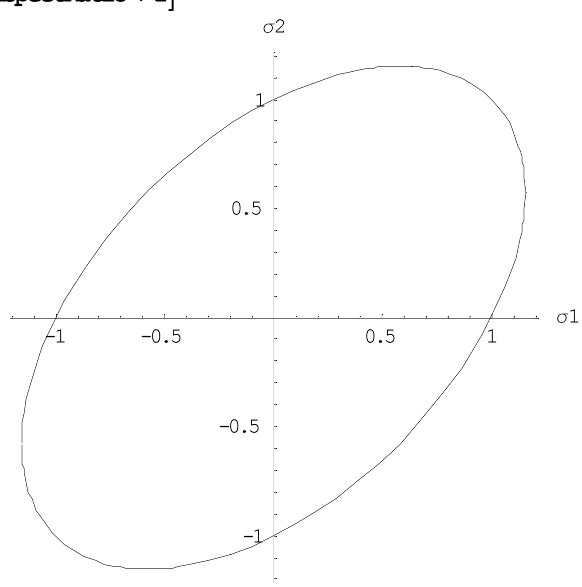

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Figure 1. Mathematica code to plot octahedral stress failure for plane stress.