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Risk management in uncapacitated facility location models with random demands

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Abstract

In this paper we consider a location-optimization problem where the classical uncapacitated facility location model is recast in a stochastic environment with several risk factors that make demand at each customer site probabilistic and correlated with demands at the other customer sites. Our primary contribution is to introduce a new solution methodology that adopts the mean–variance approach, borrowed from the finance literature, to optimize the "Value-at-Risk" (VaR) measure in a location problem. Specifically, the objective of locating the facilities is to maximize the lower limit of future earnings based on a stated confidence level. We derive a nonlinear integer program whose solution gives the optimal locations for the p facilities under the new objective. We design a branch-and-bound algorithm that utilizes a second-order cone program (SOCP) solver as a subroutine. We also provide computational results that show excellent solution times on small to medium sized problems.

Keywords: Facility location; Risk management; Second-order cone programming; Value-at-Risk

1. Introduction

Classical uncapacitated facility location and p-median location models (e.g., Hakimi [1], Erlenkotter [2], Körkel [3]; see Cornuejols et al. [4] for a review) study the problem of locating a set of facilities to maximize the earnings for serving a set of markets with known demands. Because the demands in markets tend to change in relation to updated economic factors, in reality, at the time that an allocation decision is made, the future demands might not be known, except for their probability distributions. In this study, we consider a situation where a firm wishes to select facility locations that will serve a set of market points with uncertain and correlated demands. When facing uncertain demands, the objective function of classical p-median models becomes a first-order moment approximation (i.e., expected value). Following the mean–variance approach in a finance application (Markowitz et al. [5]), we use both first- and second-order moments to analyze the distribution of future earnings. Therefore, rather than using a risk-neutral objective that simply maximizes the expected earnings, our model is able to manage the monetary risk in a p-median problem by optimizing the "Value-at-Risk" (VaR) measure of future earnings. More precisely, given that the earnings remain probabilistic at the time the allocation decision is made, our model aims to maximize the lower limit of future earnings based on a confidence level required by the decision maker. The confidence level is defined as the probability that a lower limit can be met or surpassed by future earnings.

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Making strategic decisions, such as facility location, may require a firm to commit resources in a manner that is costly to reverse. As a result, associated risk becomes an important concern when uncertainty is significant (Doherty [6]). The seminal work on using a mean–variance approach to optimize portfolios (Markowitz [7]), for which Harry Markowitz received the 1990 Nobel Prize in Economics, has inspired much research toward the systematic treatment of two often-conflicting objectives: profit versus risk (see Steinbach [8] for a review). Analogous to using a financial portfolio to control risk, our intention is to use the allocation decision to influence the mean and standard deviation of future earnings. Nevertheless, conventional mean–variance models usually rely on an artificially defined utility function to describe the decision maker's preference under uncertainty; thus, the typical objective function is to maximize the expected utility (see Schoemaker [9] for a review), which, in a facility location model, would imply expected future earnings. Although it is theoretically sound to explain the behavior of individuals facing choices under risk, empirical support for the expected utility rationale has not been strong (see Tversky et al. [10] for a review). Even more severe limitations apply to using the utility function to explain the behavior at the organizational level. The mechanism through which utility is aggregated into a collective function for the organization has been open to wide speculation. The impossibility theorem (Arrow [11], Mas-Collell et al. [12], Chapter 21) even suggested that such a function might never be developed.

To avoid the pitfalls related to expected utility objective, in this paper we describe the decision maker's objective with an explicit measure named Value-at-Risk (VaR). VaR is defined as the lower limit of future earnings at a given confidence level chosen by the decision maker. In a p-median setting, we show that optimizing the VaR is equivalent to solving a bi-objective location problem (Daskin [13], Chapter 8) in which expected profit needs to be traded off with the standard deviation of future earnings. VaR has become one of the most important measures for managing risk in the financial industry (e.g., RiskMetrics [14]) as well as non-financial firms (Bodnar et al. [15]). The popularity of VaR is primarily related to a simple and easily understood representation of risk and value. For a comprehensive introduction to risk management and applications using VaR, we refer the reader to Jorion [16,17] and Duffie and Pan [18].

Quite naturally, different stochastic location models require different risk measures for their objective functions. Our model considers the uncapacitated facility location setting. Such problems with stochastic weights were first introduced by Frank [19,20], which consider only the single facility allocation decision. As a result, the risk measure adopted in this paper is associated with the uncertainty of overall future earnings. In stochastic facility location models such as covering and center problems, the risk measures are usually associated with the worst service quality among demand points. For example, in a center problem with probabilistic weights, Berman et al. [21] propose a model that minimizes the maximum weighted distance between a facility and demand points exceeding a given target. Snyder and Daskin [22] propose a location model that considers the situation where the reliability of each facility is stochastic and the objective function needs to consider the potential failure of certain facilities. When facility capacity is limited, location problems with stochastic demand and congestion (LPSDC) usually incorporate Queuing Theory to model the timing and actual demand generated by each location, as well as a possible loss of demand due to that facility's inability to provide adequate service (Berman et al. [23]). An extensive review of other capacitated stochastic location models can be found in Berman and Krass [24]. The assumption of uncapacitated facility is appropriate when demand can be served without specialized servers and when the replenishment of supply is frequent (e.g., a shopping center or grocery store).

Generally, there are two major complexities when solving median problems with correlated demands, such as the one considered in this paper. The first is associated with calculating correlation coefficients with a large number of demand points. In the recent finance literature, most approaches to calculating VaR assume a joint normal distribution of the underlying risk factors instead of calculating the tedious joint variance of demands from exhaustive historical data (Simons [25], Stambaugh [26], Pritsker [27], Alexander and Baptista [28]). In this paper, we replace the traditional variance–covariance matrix approach with one that involves systematic and nonsystematic risk factors.

The second complexity is attributed to the nonlinear nature of optimizing a quadratic function related to variance. Usually, simplification of demand correlation is necessary for the multi-facility version of location problems. For example, in their model of optimizing the location of p distribution centers (DCs), Shen et al. [29] assume that the random demands among nodes are independent and the variance of demand at each node is in a constant proportion of the mean of demand. However, in our paper we utilize concepts from nonlinear programming to exactly solve our combinatorial model. In particular, we formulate the p-median problem under the VaR objective as a nonlinear integer program. To solve this optimization problem, we design a branch-and-bound algorithm that utilizes a second-order cone program (SOCP) (e.g., see Alizadeh and Goldfarb [30]) solver as a subroutine. SOCPs are a generalization of linear programming that are solvable in polynomial time by interior point methods (e.g., see Renegar [31]). Thus, our model allows a decision maker, who is interested in solving a p-median model, a feasible way to manage the profit uncertainty when the demands at the nodes are stochastic and correlated.

Outline: The paper is organized as follows. Section 2 introduces notation, models random demand with a set of risk factors, and describes the objective function. In Section 3, we summarize SOCPs and show how they can be used to solve the p-median problem under the VaR objective. Section 4 presents the computational results to evaluate the performance of our algorithm. Conclusions and recommendations are addressed in Section 5.

2. Model formulation

In keeping with the p-median problem, we assume that the market is given by a graph $G = (N, A)$, where $N = \{1, 2, \ldots, n\}$ represents the set of nodes (the demand points) and $A = (i, j)$ represents the set of arcs with $i, j \in N$. The decision maker seeks to locate p (where $p \leqslant n$) facilities and these locations are to be chosen from the entire set of nodes N.

2.1. Modeling stochastic demands with risk factors

We model the demand correlations by using a set of risk factors. Let D_i be the random demand originating from node $i = 1, 2, \ldots, n$. We assume that D_i is represented by a mean given by $\overline{D}_i$ plus a random component Δ_i, where $\mathbb{E}[\Delta_i] = 0$:

$$D_i = \overline{D}_i + \Delta_i. \tag{1}$$

In order to understand what determines the composition of Δ_i, we now introduce the concept of systematic and nonsystematic risk factors. A systematic risk factor is one that affects the demands of a large number of nodes. To understand this better, let us assume that the firm in question is a grocery firm that wishes to locate p DCs to serve the customers in G (see Table 1). At the time the DC allocation decision needs to be made, the grocery chain faces several uncertainties that may affect the demands on a large scale. For example, one uncertain factor could be the chain's ongoing plan to cooperate with a coffee chain to provide express service for customers. If this plan goes well, it may increase node demands, depending on local demographic characteristics. A second factor could be the chain's decision to expand product lines to include certain domestic appliances in some stores. A third possible factor could be the potential movement of competing grocery chains. For example, competitors might enter or exit certain regions in the future and affect the demands in those regions accordingly. A fourth factor could be about general economic conditions, such as gross domestic productivity (GDP) and the unemployment rate. Because systematic risk factors often have market-wide effects, they are also called market risk in the financial literature. In contrast, an unsystematic risk factor could be one that affects only one node. For instance, local demand may be affected by local zoning by-laws and population changes. Because nonsystematic risk factors represent the independent part of uncertainty of each node, it often has a much smaller influence on the demand uncertainty than the systematic ones.

Table 1
Systematic and nonsystematic risk factors of demands (a grocery chain example)

Type	Risk factor (V_i or S^k)	Influence	Data needed for estimation of v_i or s_i^k
Systematic	Benefit from strategic alliance with a coffee chain	Demography segmented by age, gender, etc.	Linear combination of demographic data at each node
Systematic	Plan to expand product lines in certain stores	Market position and shopping traffic at participating stores	Conduct market survey and consumer behavior analysis
Systematic	Impact of competitor's movement	Geographic regions where competitors plan to enter/exit	Competitors' potential sites and market strategy at each area
Systematic	U.S./Canadian economy in the future	Demography segmented by job type and household income	GDP, unemployment rate, inflation, demographic data at each node
Nonsystematic	Local zoning or population growth	Number of local residents	Historical data; local manager's assessment

Assume that there are K ($K=4$ in Table 1) systematic risk factors that apply to each demand node. Each systematic risk factor is denoted by S^k, $k=1,\ldots,K$. In addition, we let V_i represent the nonsystematic risk factor at node i. To make the model tractable, for each node i, we assume that each risk factor V_i or S^k is a normal random variable. For each risk factor, the coefficient v_i or s_i^k represents the weight of random variable V_i or S^k, respectively, in the demand D_i. Once again, without loss of generality, we assume that V_i and S^k have zero mean and unit variance (i.e. $var(V_i)=var(S^k)=1$). For the systematic risk factor S^k, we allow the coefficient s_i^k to be a real number because the demands at the various nodes may have positive or negative correlations among them. Similarly, for the random variable V_i, we let the coefficient v_i be any real number. As a result, the distribution functions of D_i can be completely described by the weights v_i and $s_i^1,\ldots,s_i^K$. In practice, the estimation of the risk factor weights v_i and $s_i^1,\ldots,s_i^K$ is achieved by using combinations of demographic data and market intelligence. Table 1 summarizes the characteristics of each risk factor, the scope of each factor's influence, and the relevant information for estimating v_i and $s_i^1,\ldots,s_i^K$.

Since we have assumed that the risk factors V_i and S^k are normally distributed (with zero mean and unit variance) random variables, this permits demand value at node i to be negative. In order to assure positive demand, we require truncation on the lower tail. With the definitions and examples above, we have implicitly defined Δ_i. We next explicitly give a functional form for Δ_i, which we accomplish by restating Eq. (1), which represents demand D_i at node i, as the following nonnegative random variable:

$$D_i=\overline{D}_i+\Delta_i=\overline{D}_i+\max\left(-\overline{D}_i,\, v_iV_i+\sum_{k=1}^{K}s_i^kS^k\right). \tag{2}$$

However, in an application under a similar mean–standard deviation framework, Carr and Lovejoy [32] assume that all demands are normally distributed and demonstrate that explicitly dealing with the lower truncation complicates the analysis but with little gain in accuracy or insight. This is further corroborated in Petruzzi and Dada [33], who remark that if $\overline{D}_i$ is large relative to the variance of Δ_i, unbounded probability distributions such as the normal distribution provide adequate approximations. Therefore, to keep our model tractable and to focus on exploring managerial insight, in the rest of the paper we will adopt the same assumption made by Carr and Lovejoy [32]. In other words, we will assume that the effect of truncating the demand distribution below zero is sufficiently small and hence Eq. (2) will be approximated as $\overline{D}_i$ and a weighted sum of risk factors V_i and S^k:

$$D_i\approx\overline{D}_i+v_iV_i+\sum_{k=1}^{K}s_i^kS^k. \tag{3}$$

While the transition from Eq. (2) to (3) might seem drastic (since we still implicitly assume $D_i\geqslant 0$), we argue that a reliable estimation of the risk factor weights v_i and s^k from real data will give a model that results in realistic (nonnegative) demands. In any case, any negative demands can easily be rounded up to zero in an actual implementation.

The covariance between the demands of two nodes can therefore be derived as follows:

$$cov(D_i,D_j)=\sum_{k=1}^{K}s_i^ks_j^k. \tag{4}$$

2.2. Formulation of the objective function and the constraints

Having developed the risk-factor based approach for modeling stochastic demands, we now turn our attention to the formulation of the objective function and the constraints. All matrices and vectors will appear in boldface type. Assume that the decision maker needs to locate p facilities and the fixed cost of establishing a facility at node i is a constant f_i. We further assume there is no capacity constraint on the demand served by a facility. Once the p facilities have been located, each of the n nodes will get its shipments from the least costly facility. We let l_{ij} denote the smallest cost of transporting a unit good between any two nodes $i,j\in N$. For example, l_{ij} can be defined as the shortest path between nodes i and j, which can be easily calculated using Dijkstra's algorithm. We let r_i denote the unit sale price at node i. Given such an allocation decision, in accordance with the standard notation of the p-median problem, we let y_j, $j=1,\ldots,m$ be a binary variable that is defined as follows: $y_j=1$ if a facility is located at site j and $y_j=0$ otherwise.

Let $\mathbf{y}$ be a vector of the y_j variables. Similarly, x_{ij} is defined as follows: $x_{ij}=1$ if node i is served by a facility at site j and $x_{ij}=0$ otherwise.

We use an $n\times n$ matrix $\mathbf{X}=\{x_{ij}\}$ as an allocation matrix to represent an allocation decision of p facilities. One can see that an allocation matrix $\mathbf{X}$, with entries of zeros and ones, shall have exactly p nonzero columns and have one nonzero entry in each row. Hence, if we let $\Pi_i(\mathbf{X})$ represent the profit contributed by node i under this given allocation decision, assuming a facility is located at node i, then

$$\Pi_i(\mathbf{X}) = -f_i + \sum_{j=1}^{n} D_j(r_i - l_{ij})x_{ji}. \tag{5}$$

Finally, incorporating the $\mathbf{y}$ decisions, we let $\Pi(\mathbf{X},\mathbf{y})$ denote the sum of profits from all nodes and it can be seen that

$$\Pi(\mathbf{X},\mathbf{y}) = -\sum_{i=1}^{n} f_i y_i + \sum_{i=1}^{n}\sum_{j=1}^{n}\left[\overline{D}_j(r_i - l_{ij}) + \left(v_j V_j + \sum_{k=1}^{K} s_j^k S^k\right)(r_i - l_{ij})\right]x_{ji}. \tag{6}$$

Given that the demand D_i is approximated by a normal random variable as in Eq. (4), $\Pi(\mathbf{X},\mathbf{y})$ can then be represented as a multivariate normal random variable. The mean and standard deviation of the overall profit can be computed as

$$\mathbb{E}[\Pi(\mathbf{X},\mathbf{y})] = -\sum_{i=1}^{n} f_i y_i + \sum_{i=1}^{n}\sum_{j=1}^{n}\overline{D}_j(r_i - l_{ij})x_{ji} \tag{7}$$

and

$$\mathbb{STD}[\Pi(\mathbf{X},\mathbf{y})] = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}\left(v_j(r_i - l_{ij})x_{ji}\right)^2 + \sum_{k=1}^{K}\left(\sum_{i=1}^{n}\sum_{j=1}^{n} s_j^k (r_i - l_{ij})x_{ji}\right)^2}. \tag{8}$$

We now discuss the VaR approach, where the objective is to maximize the lower limit of future earnings based on a stated confidence level. Fixing the variables x_{ij} and y_j for all i, j, the VaR objective is to find the largest value of π such that $\mathbb{P}[\Pi \leqslant \pi] \leqslant \varepsilon$, for some given confidence level ε:

$$\begin{aligned} \pi^* = \max \quad & \pi \\ \text{s.t.} \quad & \mathbb{P}[\Pi \leqslant \pi] \leqslant \varepsilon. \end{aligned} \tag{9}$$

Since the profit Π is a normal random variable with mean $\mathbb{E}[\Pi]$ and standard deviation $\mathbb{STD}[\Pi]$, we can perform the following analysis, where Z is a standard normal random variable (zero mean, unit variance) and $\Phi(\cdot)$ is the distribution function of a standard normal random variable:

$$\begin{aligned} \mathbb{P}[\Pi \leqslant \pi] \leqslant \varepsilon &\Leftrightarrow \mathbb{P}\left[Z \leqslant \frac{\pi - \mathbb{E}[\Pi]}{\mathbb{STD}[\Pi]}\right] \leqslant \varepsilon \\ &\Leftrightarrow \Phi\left(\frac{\pi - \mathbb{E}[\Pi]}{\mathbb{STD}[\Pi]}\right) \leqslant \varepsilon \\ &\Leftrightarrow \frac{\pi - \mathbb{E}[\Pi]}{\mathbb{STD}[\Pi]} \leqslant \Phi^{-1}(\varepsilon) \quad \text{(since } \Phi(\cdot) \text{ is monotonously increasing)} \\ &\Leftrightarrow \pi \leqslant \mathbb{E}[\Pi] + \Phi^{-1}(\varepsilon)\mathbb{STD}[\Pi]. \end{aligned}$$

Therefore, $\pi^* = \mathbb{E}[\Pi] + \Phi^{-1}(\varepsilon)\mathbb{STD}[\Pi]$. Our VaR p-median problem is concerned with the location of facilities and allocations of demands in order to maximize the value of π^*. Consequently, our VaR p-median problem is formulated as follows:

$$\begin{aligned}
\max_{\mathbf{X},\mathbf{y}} \quad & \mathbb{E}[\Pi(\mathbf{X},\mathbf{y})] + \Phi^{-1}(\varepsilon)\mathbb{STD}[\Pi(\mathbf{X},\mathbf{y})] \\
\text{s.t.} \quad & \sum_{j=1}^{n} x_{ij} = 1, \quad \forall i \in N, \\
& \sum_{j=1}^{n} y_j = p, \\
& x_{ij} \leqslant y_j, \quad \forall i, j, \\
& x_{ij}, y_j \in \{0,1\}, \quad \forall i, j.
\end{aligned} \tag{10}$$

For the remainder of the paper, we assume that $\varepsilon \leqslant \frac{1}{2}$ so that $\Phi^{-1}(\varepsilon) \leqslant 0$, a necessary requirement for our solution approach. If $\varepsilon \leqslant \frac{1}{2}$ and the integer constraints are relaxed, Problem 10 is a convex optimization problem. If $\varepsilon > \frac{1}{2}$, the convexity is lost.

3. A solution procedure

3.1. Second-order cone programs

If $\mathbf{z} = (z_1, \ldots, z_d)$ is a d-dimensional vector of variables, the following mathematical program is an SOCP:

$$\begin{aligned}
\min_{\mathbf{z}} \quad & \mathbf{f}'\mathbf{z} \\
\text{s.t.} \quad & \|\mathbf{A}_i\mathbf{z} + \mathbf{b}_i\| \leqslant \mathbf{c}_i'\mathbf{z} + d_i, \quad i \in I,
\end{aligned} \tag{11}$$

where $\mathbf{f}, \mathbf{b}_i, \mathbf{c}_i$ are real-valued vectors, d_i is a real number, and $\mathbf{A}_i$ are real-valued matrices of appropriate dimensions. The norm in the constraints is the standard Euclidean norm: $\|\mathbf{x}\| = \sqrt{\mathbf{x}'\mathbf{x}}$. A property of SOCPs that is useful for our analysis is that they can be solved efficiently (both theoretically and practically) via interior point algorithms; see Renegar [31] for an introduction to appropriate algorithms.

3.2. Application of SOCPs to the VaR p-median problem

Next, we show how a SOCP can be used in computing a solution to our VaR p-median problem (10). We let the variable vector $\mathbf{z} = (z_1, \ldots, z_d)$ contain all variables y_j and x_{ij} (i.e., $d = n^2 + n$). In particular,

$$\mathbf{z} = (y_1, \ldots, y_n, x_{11}, \ldots, x_{1n}, x_{21}, \ldots, x_{2n}, x_{31}, \ldots, x_{nn});$$

we denote this correspondence concisely as $\mathbf{z} = (\mathbf{y}, \mathbf{X})$. The expected profit can be written as

$$\mathbb{E}[\Pi(\mathbf{X},\mathbf{y})] = \boldsymbol{v}'\mathbf{z}, \tag{12}$$

where $\boldsymbol{v}$ is a vector whose elements are derived from Eq. (7). Likewise, the standard deviation of the profit can be written as

$$\mathbb{STD}[\Pi(\mathbf{X},\mathbf{y})] = \sqrt{\mathbf{z}'\boldsymbol{\Sigma}\mathbf{z}}, \tag{13}$$

where $\boldsymbol{\Sigma}$ is the covariance matrix derived from Eq. (8). More specifically, using the shorthand $\mathbf{z} = (\mathbf{y}, \mathbf{X})$, note that the profit function Π can be written as $\Pi(\mathbf{z}) = \sum_{i=1}^{d} W_i z_i$, where the W_i are random variables that can be identified from Eq. (6). Consequently, $\boldsymbol{\Sigma}_{ij} = cov(W_i, W_j)$ and $\boldsymbol{\Sigma}_{ii} = var(W_i)$. Detailed derivations of $\boldsymbol{v}$ and $\boldsymbol{\Sigma}$ are given in Section 4. Let

$$P = \left\{ \mathbf{z} \;\middle|\; \mathbf{z} = (\mathbf{y}, \mathbf{X}), \sum_{j=1}^{n} x_{ij} = 1, \forall i \in N, \sum_{j=1}^{n} y_j = p, x_{ij} \leqslant y_j, \forall i, j \in N \right\} \tag{14}$$

denote the feasible region of a linear programming relaxation of the classic p-median problem. Therefore, the VaR p-median problem (10) can be written as

$$\begin{aligned} \max_{\mathbf{z}} \quad & \boldsymbol{v}'\mathbf{z} + \Phi^{-1}(\varepsilon)\sqrt{\mathbf{z}'\boldsymbol{\Sigma}\mathbf{z}} \\ \text{s.t.} \quad & \mathbf{z} \in P, \\ & \mathbf{z} \in \{0,1\}^{n^2+n}. \end{aligned} \tag{15}$$

A relaxation of formulation (15) can be obtained by removing the binary variable condition:

$$\begin{aligned} \max_{\mathbf{z}} \quad & \boldsymbol{v}'\mathbf{z} + \Phi^{-1}(\varepsilon)\sqrt{\mathbf{z}'\boldsymbol{\Sigma}\mathbf{z}} \\ \text{s.t.} \quad & \mathbf{z} \in P. \end{aligned} \tag{16}$$

By introducing an extra variable w, we see that the relaxation (16) can be rewritten as

$$\begin{aligned} \max \quad & w \\ \text{s.t.} \quad & \sqrt{\mathbf{z}'\boldsymbol{\Sigma}\mathbf{z}} \leqslant \frac{\boldsymbol{v}'\mathbf{z} - w}{-\Phi^{-1}(\varepsilon)}, \\ & \mathbf{z} \in P, \end{aligned} \tag{17}$$

an SOCP, since $\sqrt{\mathbf{z}'\boldsymbol{\Sigma}\mathbf{z}} = \|\boldsymbol{\Sigma}^{1/2}\mathbf{z}\|$, assuming the square root of the covariance matrix exists. Note that, by definition, the covariance matrix $\boldsymbol{\Sigma}$ is positive semidefinite (i.e., for any $\mathbf{z}$, $\mathbf{z}'\boldsymbol{\Sigma}\mathbf{z} = var(\Pi(\mathbf{z})) \geqslant 0$). Since $\boldsymbol{\Sigma}$ is a symmetric matrix, it has real eigenvalues λ_i, $i = 1, \ldots, d$, and d mutually orthogonal real (column) eigenvectors $\mathbf{e}_i$, $i = 1, \ldots, d$. Furthermore, we can write $\boldsymbol{\Sigma} = \sum_{i=1}^{d} \lambda_i \mathbf{e}_i \mathbf{e}_i'$. We show the existence of the square root matrix by writing $\boldsymbol{\Sigma}^{1/2} = \sum_{i=1}^{d} \sqrt{\lambda_i}\mathbf{e}_i\mathbf{e}_i'$. Finally, note that if $\varepsilon > \frac{1}{2}$, the inequality of the first constraint of formulation (17) would have been inverted and the problem would no longer be an SOCP.

3.3. A branch-and-bound algorithm to solve the VaR p-median problem

In this section we assume that the VaR p-median problem (10) is feasible. Let OPT denote the value of problem (10) and $K = \{0,1\}^{n^2+n}$ denote the set of possible integer solutions. If $Z = \{\mathbf{z} | z_i \in \{0,1\}, i \in I, \mathbf{z} \in Y\}$, for some continuous set Y, let $R(Z) = \{\mathbf{z} | 0 \leqslant z_i \leqslant 1, i \in I, \mathbf{z} \in Y\}$ denote the set that relaxes the binary constraints of Z. Define

$$\begin{aligned} \max_{\mathbf{z}, w} \quad & w \\ \text{s.t.} \quad & \sqrt{\mathbf{z}'\boldsymbol{\Sigma}\mathbf{z}} \leqslant \frac{\boldsymbol{v}'\mathbf{z} - w}{-\Phi^{-1}(\varepsilon)}, \\ & \mathbf{z} \in P, \\ & \mathbf{z} \in Z \end{aligned} \tag{18}$$

and let $OPT(Z)$ denote its optimal value. Note that $OPT(\{0,1\}^{n^2+n}) = OPT$ and $OPT(R(Z))$ is a SOCP for any set $Z \subseteq [0,1]^{n^2+n}$. The branch-and-bound algorithm `BB` for solving the VaR p-median problem is as follows.

`BB`:

- `Initialization`:
  (1) `Set the initial queue` $Q = \{K\}$.
  (2) `Find an initial lower bound`[1] L `for` OPT.
- `Loop`:
  (1) `If` Q `is empty, stop. The current value in` $\mathbf{z}^*$ `is optimal.`
  (2) `Choose the first element` Z `of the queue` Q `and remove it. If` $OPT(R(Z)) \leqslant L$ `or if` $OPT(R(Z))$ `is infeasible, go to step 1.`

[1] For example, L can be equal to the VaR cost of any feasible allocation, which can be calculated using standard algorithms for the classic p-median problem.

(3) `If the solution to` $OPT(R(Z))$ `is integer, set` $\mathbf{z}^*$ `to be this solution and set` $L = OPT(R(Z))$. `Go to step 1.`
(4) `Find the smallest index` i `such that` z_i `is not integer. Insert` $Z_1 = Z \cap \{z_i = 0\}$ `and` $Z_2 = Z \cap \{z_i = 1\}$ `into the queue` Q. `Go to step 1.`

Remarks. Since we assumed the VaR p-median problem is feasible, the branch-and-bound algorithm is guaranteed to find the optimal solution. Note that there is also an implicit degree of freedom in managing the queue Q. Different policies, such as FIFO and LIFO, will lead to different exploration strategies of the underlying tree, such as depth-first or breath-first.

4. Computational study

We implement `BB` in MATLAB [34]. More specifically, we modify the MATLAB `bintprog`[2] procedure to utilize the Disciplined Convex Programming [35] package to solve the SOCP relaxations. Our implementation of `BB` utilizes the default options for `bintprog`: The branching strategy is to choose the variable with the maximum infeasibility (i.e., the variable with value closest to 0.5) and the node search strategy is to choose the node with the lowest bound on the objective function.

4.1. Experimental design

Recall that the aggregate vector of variables $\mathbf{z}$ is defined as

$$\mathbf{z} = (y_1, \ldots, y_n, x_{11}, \ldots, x_{1n}, x_{21}, \ldots, x_{2n}, x_{31}, \ldots, x_{nn}).$$

We also give equations for the mean and variance of the profit, with slightly different indexing:

$$\mathbf{v}'\mathbf{z} = -\sum_{i=1}^{n} f_i y_i + \sum_{i=1}^{n}\sum_{j=1}^{n} \overline{D}_i (r_j - l_{ji}) x_{ij} \tag{19}$$

and

$$\mathbf{z}'\boldsymbol{\Sigma}\mathbf{z} = \sum_{i=1}^{n}\sum_{j=1}^{n} (v_i (r_j - l_{ji}) x_{ij})^2 + \sum_{k=1}^{K} \left(\sum_{i=1}^{n}\sum_{j=1}^{n} s_i^k (r_j - l_{ji}) x_{ij} \right)^2. \tag{20}$$

It is clear that

$$v_k = \begin{cases} -f_k, & k = 1, \ldots, n, \\ \overline{D}_i (r_j - l_{ji}), & k = n + ij, \quad i, j = 1, \ldots, n. \end{cases} \tag{21}$$

Similarly, we can derive the coefficients of the covariance matrix to be

$$\Sigma_{pq} = \begin{cases} 0, & p \leqslant n \quad \text{or} \quad q \leqslant n, \\ \sum_{k=1}^{K} s_i^k (r_j - l_{ji}) s_l^k (r_m - l_{ml}), & p = n + ij, \quad q = n + lm, \\ & i \neq l \ \text{ or } \ j \neq m, \ \ i, j, l, m = 1, \ldots, n, \\ \sum_{k=1}^{K} s_i^k (r_j - l_{ji}) s_l^k (r_m - l_{ml}) + v_i^2 (r_j - l_{ji})^2, & p = q = n + ij, \ \ i, j = 1, \ldots, n. \end{cases} \tag{22}$$

We generate the data for our simulation study in the following manner. We let the facility location fixed costs $f_i, \forall i$ be uniformly distributed in the interval [1000,1200]. Similarly, the average node demands $\overline{D}_i, \forall i$ are uniformly distributed in the interval [200,300]. The revenues $r_i, \forall i$ are uniformly distributed in [50,100] and the loads $l_{ij}, \forall i, j$ are independently uniformly distributed in the interval [20,60]; note that we do not assume symmetric loads. Finally, we similarly model the risk factors: we set $K = 4$ and let $s_i^k, \forall i, k$ and $v_i, \forall i$ be independently uniformly distributed in [5,10].

[2] `bintprog` is a branch-and-bound algorithm for solving 0–1 integer linear programs.

Table 2
For each combination of n and ε, we record the following data points: average objective function value, average time in s required to calculate solution

	$n=5$	$n=10$	$n=15$
$\varepsilon=0.01$	(21,356; 0.72 s)	(30,589; 8.45 s)	(21,075; 76.83 s)
$\varepsilon=0.05$	(26,088; 0.83 s)	(53,366; 4.41 s)	(61,018; 25.93 s)
$\varepsilon=0.10$	(25,799; 0.79 s)	(56,461; 9.12 s)	(78,999; 7.68 s)

The averages are taken over 10 simulations.

We vary n and the confidence level ε; in particular, we let $n \in \{5, 10, 15\}$ and $\varepsilon \in \{0.01, 0.05, 0.10\}$. We let $p = n/5$. For each combination of n and ε, we perform 10 simulations and record the average VaR objective function value and the average time required to solve the problem, in seconds.

Table 2 tabulates the results of our computational experiment. We notice that for small values of n and realistic confidence levels ε, our algorithm computes the exact solution very quickly. An interesting observation for the $n = 15$ case is that the computation time increases significantly as the confidence level ε is decreased; this behavior was not present for the other values of n studied.

We next briefly mention the time required by our branch-and-bound algorithm for larger values of n for $\varepsilon = 0.10$; we report the performance of a single simulation run. For $n = 20$, our algorithm output a cost of 55,680, which took 7.27 min. For $n=25$, our algorithm output a cost of 51,226, which took 47.67 min. For larger values of n, our simulation setup resulted in infeasible problem instances. In order to test our algorithm for larger values of n (for $\varepsilon = 0.10$), we instead generated the covariance matrix as

$$\boldsymbol{\Sigma} = \boldsymbol{\Lambda}'\boldsymbol{\Lambda},$$

where $\boldsymbol{\Lambda}$ is an $(n^2+n) \times (n^2+n)$ matrix whose elements are i.i.d. realizations of the standard normal distribution; this design results in a positive semidefinite matrix $\boldsymbol{\Sigma}$, which is necessary for a covariance matrix. We again report the performance of a single simulation run. For $n \in \{30, 35, 40\}$, our algorithm successfully converged in 9.25, 43.51 min and 1.76 h, respectively. Therefore, these computational results encourage the application of our algorithm in practice.

Further modifications of our algorithm are also possible to create a more efficient solution approach tailored to specific applications, especially for larger values of n. In particular, more sophisticated branching strategies can be implemented to take advantage of problem structure and the sparsity of the covariance matrix $\boldsymbol{\Sigma}$ might also be exploited. However, detailed algorithmic analyses of this type are outside the scope of this paper.

5. Conclusion

In this paper, we have developed and examined a new algorithm for solving the p-median problem when the demands are probabilistic and correlated. The best allocation decision was selected by balancing two often-conflicting objectives: profit and associated uncertainty. We utilized concepts from nonlinear programming to design a branch-and-bound algorithm to solve the problem exactly. The results presented in this paper open up a number of new applications in location analysis. One such possibility is the connection between location decision and financial risk control, which is an application currently dominated only by financial researchers. Due to the strategic nature of location selection, the commitment of facility sites could have significant impact on a firm's financial uncertainty. Therefore, it is apparent that our model has the potential to improve corporate financial planning if the decision maker has the objective of profit maximization, but also has concerns about risk control. Another natural extension is to investigate the capacitated models in which capacity investment needs to include financial risk control when making decisions.

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