Survey of Dark Energy and Quintessence

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Abstract

In this paper we discuss dark energy and its potential manifestation as a scalar field. Recent data from the Wilkinson Microwave Anisotropy Probe (WMAP), the type Ia supernova measurements, and other observational evidence for the increased expansion rate of the universe will be briefly discussed. Dark energy, energy not readily detectable by interactions with baryons, appears to have a dominant effect on the dynamics of the universe. We will explore models for this dark energy including the cosmological constant and scalar fields (quintessence). Quintessence will be investigated in greater detail than other models.

CONTENTS

I. OBSERVATION

The term "quintessence" is derived from the ancient word for fifth element; from mediaeval metaphysicians supposition that the universe was composed of earth, air, fire, water, and an additional all-pervasive ephemeral component that created the motion of the moon and planets. Currently it is being used to describe the fifth component responsible for the large-scale motion of the cosmos, in combination with the other four known components: baryons, leptons, photons, and dark matter.[30]

Recent observations appear to indicate that the universe is filled (~70%) with mass-energy that produces an overall negative pressure but interacts weakly with baryons and leptons (the stuff of which we are made).[16,29,31] This energy is apparently invisible to us and thus has been dubbed "dark energy." Also, the effect of gravitation on it is repulsive: seventy percent of the stuff in the universe is apparently repelled by gravity, rather than attracted! This dark energy has effects that have only been detected on the largest scales of our Universe and then only in the past ten years. Dark energy is certainly ephemeral stuff and its exact nature is far from certain. A large number of models have been presented to account for its gravitational effects and they include a background vacuum energy dubbed a "cosmological constant" from its initial use by Albert Einstein in his gravitational field equations. Another is in the form of a scalar field, dubbed quintessence.[10] These two will differ slightly in their properties and thus differences could be detected.

That space is filled with energy whose effect is the same as that of a cosmological constant, Λ, or of a scalar field, "quintessence", is becoming well established. In both cases the dark energy may be dynamical approaching zero with time, or it may be slowly increasing. It is now dominating the universe because the reduction of mass and radiation energy density with the scale factor ("size" of the universe) is greater than the decrease in dark energy density in the present epoch. The cosmological constant and quintessence are also often termed vacuum energy.[16,29,31]

An important measure is the rate of expansion of the universe, by which we mean the change in distance between two points with time. It should be stated that the universe itself is not expanding into anything, like a balloon, but the distance between all points within the universe is increasing. In order to accurately measure the rate of expansion we need to measure the recessional velocities of objects of varying distances. We will often state the distance to an object in terms of its redshift, z, or the amount its radiated electromagnetic radiation has shifted toward the red. One needs a "standard candle" to measure the distance to assure that what you have measured at one distance is the same as at another. An example of a standard candle is the type Ia supernova.

Another important measure is the anisotropies in the 3-K cosmic microwave background (CMB) radiation. From the correlation between regions in the CMB we can derive much knowledge about the universe. Current data from the CMB as measured by WMAP is given below. This gives one of the most compelling arguments for both dark energy and dark matter as well.[7,16,29,31]

From measurements it seems the density of matter in the universe is about 30% of the critical value, where baryonic matter accounts for roughly 4.4%, and the rest is some sort of "dark" matter. The current measured value of the Hubble constant is $H=72\pm3$ km/(s Mpc), which we will discuss below.[9,16,29,31]

The question remains what models for dark energy are the most viable? The low background level of the dark energy and, if it is a scalar field, its very weak interactions with baryons makes it difficult to measure here on the surface of the Earth. In the future we can hope for more accurate observations to rule out unsuitable models.

It will be reiterated that the interactions of dark energy with ordinary matter are very weak so we have only detected it by its gravitational interactions on large scales. Correspondingly, dark matter has been detected in the same manner but much earlier due to its suspected effects on the motion of galaxies. The lack of interactions with ordinary stuff makes dark energy and matter elusive indeed.

II. PHYSICAL INTRODUCTION

Precision data from satellites and other methods has recently narrowed the range of cosmological models describing our universe. The precise measurement of the anisotropies in the cosmic microwave background (CMB) has led to increased precision in the Hubble constant and more accurate measurements of curvature. Much of the following is found in many cosmology and general relativity textbooks.[7,9,18,21,22,23,32,33]

First we will start with the Robertson-Walker metric describing homogeneous and isotropic space,

$$ds^2 = -dt^2 + S(t)^2\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right)$$

$$ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta$$

$$c = \hbar = 1.$$

The Minkowski metric has signature (–,+,+,+), k is the curvature, and Ω is the solid angle. S(t) is the scale factor given by

$$\mathbf{r}(t) = S(t)\boldsymbol{\zeta}$$

where ζ is a constant vector, a comoving coordinate. According to the Hubble law of the expansion of the universe the speed of a receding object is proportional to its distance from the observer,

$$\mathbf{v} = H\mathbf{r}$$

$$\left|\frac{\mathbf{v}}{\mathbf{r}}\right| = \frac{\dot{S}}{S} = H$$

where H is the Hubble parameter, which varies with time. To describe the dynamics of the universe we must use Einstein's equations that relate the energy density to curvature of spacetime. First we describe a model universe in which we have different forms of energy contributing to the stress energy tensor. Mass energy, radiation energy, and vacuum energy will contribute. The Einstein equations are given by:

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = 8\pi G T_{\alpha\beta}$$

$$R_{\alpha\beta} = R^{\mu}{}_{\alpha\mu\beta} = \partial_{\mu}\Gamma^{\mu}{}_{\beta\alpha} - \partial_{\beta}\Gamma^{\mu}{}_{\mu\alpha} + \Gamma^{\mu}{}_{\mu\lambda}\Gamma^{\lambda}{}_{\beta\alpha} - \Gamma^{\mu}{}_{\beta\lambda}\Gamma^{\lambda}{}_{\mu\alpha}$$

$$R = R^{\alpha}{}_{\alpha}$$

$$\Gamma^{\mu}{}_{\beta\alpha} = \frac{1}{2} g^{\mu\lambda}(\partial_{\beta} g_{\alpha\lambda} + \partial_{\alpha} g_{\beta\lambda} - \partial_{\lambda} g_{\beta\alpha})$$

where $R_{\alpha\beta}$ is the Ricci tensor, R is the Ricci scalar, Γ is a Christoffel symbol, G is the gravitational constant and $T_{\alpha\beta}$ is the stress-energy tensor. For an isotropic fluid we have a stress-energy tensor of the form,

$$T_{\alpha\beta} = (p+\rho)U_{\alpha}U_{\beta} + p g_{\alpha\beta}.$$

At rest with respect to fluid we have U_{α}=(1,0,0,0) and

$$T_{\alpha\beta} = \rho\delta_{00} + p g_{ij}$$

$$trT = T = T^{\alpha}_{\alpha} = g^{\alpha\beta}T_{\alpha\beta} = 3p - \rho.$$

So rewriting Einstein's equations as

$$R_{\alpha\beta} = 8\pi G(T_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} T)$$

and doing a fair bit a calculation one gets the Friedmann equations,

$$H^2 = \left(\frac{\dot{S}}{S}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{S^2}$$

$$\frac{\ddot{S}}{S} = -\frac{4\pi G}{3}(3p+\rho)$$

which describe the dynamics of the of the scale factor. The first equation is generally referred to as THE Friedmann equation and we will investigate it a bit further. The density is a sum of densities from matter, radiation, and the vacuum, and each varies differently with the scale factor.

$$\left(\frac{\dot{S}}{S}\right)^2 = \frac{8\pi G}{3}(\rho_m + \rho_{rad} + \rho_{vac}) - \frac{k}{S^2}$$

$$\rho_m = \frac{\rho_{m0}}{S^3}$$

$$\rho_{rad} = \frac{\rho_{rad\,0}}{S^4}$$

$$\rho_{vac} = \rho_{vac0}$$

The vacuum energy is associated with dark energy and theories are being developed to explain its existence and the reason it has taken its current value. Often we like to describe the dynamics of the universe in terms of a "critical" density that gives us a solution with zero curvature, k=0:

$$\rho_c = \frac{3H^2}{8\pi G}$$

We define the densities in terms of the critical density:

$$\Omega_i = \frac{\rho_i}{\rho_c}$$

$$H^2 = H_0^2(\Omega_m + \Omega_{rad} + \Omega_{vac}) - \frac{k}{S^2}$$

where Ω_i is the density parameter for each component and H_o is the current Hubble constant. WMAP and other measurements has put the value of Ω_{total}=1.02±0.02, amazingly close to unity!

III. COSMOLOGICAL CONSTANT

One of the models for the vacuum energy density is a cosmological constant associated with Einstein's equations. This term comes in as an add-on to his field equations: so Λ is a scalar that acts like a source in the stress-energy tensor, contributing to the curvature

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = 8\pi G T_{\alpha\beta} + g_{\alpha\beta}\Lambda.$$

of spacetime.

Overall the effect of the cosmological constant is to act like a negative pressure, increasing the expansion rate. We can write the pressure as p=wρ where w is –1 for vacuum energy, as we will show below. The universe currently has a value very close to –1. In general we can write the cosmological constant energy density as

$$\rho_{vac} = \frac{\Lambda}{8\pi G}$$

In the Friedmann equation the cosmological constant would look like:

$$H^2 = \frac{8\pi G}{3}(\rho_m + \rho_{rad}) - \frac{k}{S^2} + \frac{\Lambda}{3}$$

where the last term is the cosmological constant term. Now the question: how do we get this term to be repulsive? Let's look back at the second Friedmann equation:

$$\frac{\ddot{S}}{S} = -\frac{4\pi G}{3}(3p + \rho)$$

So we see if $3p+\rho>0$ then $d^2S/dt^2<0$ and the universe decelerates. If, on the other hand, $3p+\rho<0$ then $d^2S/dt^2>0$ and the universe accelerates. Let's look at some

$$\begin{aligned} dE &= -pdV \\ d\rho &= -(p+\rho)\frac{dV}{V} \\ &= -(p+\rho)d(\ln V) \\ &= -3(p+\rho)d(\ln S) \\ \dot{\rho} &= -3(p+\rho)H \end{aligned}$$

thermodynamics.

For a constant energy density the derivative is zero and we see $p= -\rho$. Plugging this back into the second Friedmann equation we get:

$$\frac{\ddot{S}}{S} = -\frac{4\pi G}{3}(-3\rho + \rho)$$

$$\frac{\ddot{S}}{S} = \frac{8\pi G}{3}\rho_{vac} = \frac{\Lambda}{3}$$

Thus the scale factor is accelerating in its expansion under these conditions.

IV. SCALAR FIELDS

Another source of vacuum energy is a scalar field given by the Lagrangian density

$$-\mathcal{L} = -\frac{R}{16\pi G} + \frac{1}{2}(\partial_\mu \phi)^2 + V(\phi)$$

R is the Ricci scalar and we have made the fields minimally coupled. There can be non minimal-coupling terms such as $\xi R\phi^2$ added to the Lagrangian, although the quantum theory of gravitation is not well defined. This will be neglected here and we will use minimal coupling. We will then write down the action

$$S = \int d^4x\sqrt{-g}\,\mathcal{L}$$

where g is the determinant of the metric. By minimizing the action, setting $\delta S=0$, we will find for our stress energy tensor

$$T_{\alpha\beta} = -\partial_\alpha \phi \frac{\partial \mathcal{L}}{\partial(\partial^\beta \phi)} + g_{\alpha\beta}\mathcal{L}$$
$$= \partial_\alpha \phi \partial_\beta \phi - g_{\alpha\beta}\left(\frac{1}{2} g^{\mu\nu}\partial_\mu \phi \partial_\nu \phi + V(\phi)\right)$$

The 00 component of the stress-energy tensor is the energy density and the ij components give the pressure, without anisotropies in this case. Again, we assume the field is spatially homogeneous.

$$T^{00} = \rho_{vac} = \frac{1}{2}\dot{\phi}^2 + V(\phi)$$
$$T_{ij} = p\delta_{ij} = \frac{1}{2}\dot{\phi}^2 - V(\phi)$$

Now we take the derivative of the 0-components of the stress energy tensor to find the dynamics of the energy density. We assume the scalar field is spatially homogeneous.

$$\partial_0 T^{0i} = 0 \qquad (conservation \quad of \quad energy)$$
$$\dot{\rho} = -3\frac{\dot{S}}{S}(\rho + p)$$

Inserting our equations for ρ and p we get

$$\ddot{\phi} + 3H\dot{\phi} + \frac{\partial V(\phi)}{\partial \phi} = 0.$$

This is our scalar field equation. This equation is similar to an oscillator equation with a damping term with H.

IV. A. INFLATION

To see the effect of the scalar field on the expansion rate we will look back at earlier times in the universe. There is strong evidence to indicate that at an early time in the history of the universe the scale factor underwent sudden exponential growth. This growth has been dubbed inflation and has since been blamed on a scalar field, the inflaton.[18,23] The inflaton has the effect of explaining the flatness of the curvature of space, the homogeneity of the CMB, and why the density is so close to the critical density.

Let's discus the causal relationship between spacetime events; a photon traveling through space will travel a proper distance dl=S(t)dx=dt in time dt, where c=1 and we have ignored curvature. In general we can solve for the coordinate displacement,

$$x = \int_0^t \frac{dt}{S(t)} = \int_{S(0)}^{S} \frac{dS}{\dot{S}S} = \sqrt{\frac{3}{8\pi G}} \int_{S(0)}^{S} \frac{dS}{S^2\sqrt{\rho}}.$$

If $\rho_{vac}=0$, $\rho \sim S^{-3}$ then as we go back in time x is finite. This means there are galaxies that we see now with which we have had no contact since the initial singularity. How is it then that the universe is so homogenous? We introduce energy density in the form of a scalar field, the inflaton, and it dominates the energy density of the universe at a certain epoch. This scalar field has a negative pressure given by $p<-\rho/3$, so the energy density goes like positive powers of the scale factor. We see that x goes to infinity as t goes back in time, thus all objects in the past will be in causal contact. Thus, $S(t) \propto e^{Ht}$, where H is a constant and the scale factor is increasing exponentially.

IV. B. SLOW ROLL

First we will deal with the dynamics of the early universe. We need to have the dynamics of the scalar potential such that it takes many e-folds of the Hubble time (75 or more) for the inflation to end and the field to reach a ground state. Thus we want a field that is suitably flat and we can use the "slow roll" approximation[18,21] where,

$$\left|\dot{\phi}\right|^2 << V$$

$$\left|\ddot{\phi}\right| << 3H\dot{\phi} \sim \left|\frac{\partial V(\phi)}{\partial \phi}\right|.$$

So that our equation of motion for the scalar field becomes

$$3H\dot{\phi} = -\frac{\partial V(\phi)}{\partial \phi}.$$

A general solution to the slow roll will be shown below:

$$H = \frac{d(\ln S)}{dt} = \dot{\phi}\frac{d(\ln S)}{d\phi} = -\frac{\partial_\phi V}{3H}\frac{d(\ln S)}{d\phi}$$

$$-\partial_\phi V\frac{d(\ln S)}{d\phi} = 8\pi G V(\phi)$$

Integrating we get:

$$\int_{S_0}^{S} d(\ln S) = -8\pi G\int_{\phi_0}^{\phi}\frac{V}{\partial_\phi V}d\phi$$

$$S = S_0 \exp\left[-8\pi G\int_{\phi_0}^{\phi}\frac{V}{\partial_\phi V}d\phi\right]$$

IV. C. QUINTESSENCE

Now we turn to the present. It is possible that at this time there is a scalar field, quintessence, which is slowly rolling towards a zero field state, or perhaps increasing.[10] In general we would like the scalar field to be decreasing with the scale factor, and time, at a smaller rate than the mass energy so that it will become dominant at redshifts less than one. The manifestation of this scalar field would be an inflation of the scale factor given by $S(t) \propto e^{Ht}$, rather like the inflation that is suspected to have occurred early in the universe. We have now detected this increase in expansion rate directly using Sn Ia data and indirectly from the CMB. [16,17,26,27,28,29,31]

V. QUINTESSENCE MODELS

Let's look back at our scalar field Lagrangian density:

$$-\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 + V(\phi)$$

Where our potential V needs to be suitably flat to give us our slow roll conditions. A massive scalar field will have a phi squared term and other forms including self-interacting terms:

$$V(\phi) = \frac{1}{2}m^2\phi^2 + \frac{1}{4!}\lambda\phi^4.$$

This form was given by Frieman et al (1995). Note that m^2 can be less than zero. Other forms of the potential with references:[28]

Table 1. Quintessence Potentials

$V(\phi)$	Author	Suitability
$K\phi^{-\alpha}$	Peebles & Ratra (1988)	Its use as an attractor or tracker at high redshifts seems useful.
$K \exp[-\phi\sqrt{(8\pi qG)}]$	Lucchin & Matarrese (1985)	Again, attractor properties but ruled out under certain conditions by WMAP. See below.
$K\,\phi^{-\alpha} \exp[\lambda\phi^2]$	Brax & Martin (1999)	Same
$K\,(e^{\alpha k\phi}+e^{\beta k\phi})$	Barreiro, Copeland, & Nunes (2000)	Same
$K\,(\exp[M_p/\phi]-1)$	Zlatev, Wang, & Steinhart (2000)	Same
$K \sinh^{-\alpha}(\lambda\phi)$	Sahni & Starobinsky(2000) Urena-Lopez & Matos(2000)	Same

"Tracker" models are useful in that for a wide range of initial conditions they follow a similar formative path. In these models the scalar field density remains near the background density throughout most of the cosmic past. For larger ϕ the potentials that are useful become flat to ensure slow roll and that they will dominate over the matter density during the present epoch.

One can divide quintessence models into two forms. Those that roll to large values of ϕ such that $\phi/m_p \gtrsim 1$, and those that roll to small values where $\phi/m_p << 1$ at present. An important concern for the first is quantum corrections that could significantly alter the shape of the potential.

Also important is the coupling of the scalar field with standard model fields. It is a given that the coupling must be small or it would have been detected already. It is interesting to note that even small couplings can give rise to changes in cosmology.

V. A. TRACKER MODELS

Models in which the potential is sufficiently steep to satisfy

$$\frac{V''V}{V'} \geq 1$$

are tracker solutions in that they approach a common evolutionary path from a large range of initial conditions. An example of a solution that tracks is given by

$$V(\phi) = \frac{1}{2} m^2 \phi^2$$

For our scalar field equation we get

$$\ddot{\phi} + 3H\dot{\phi} + \frac{\partial V(\phi)}{\partial \phi} = \ddot{\phi} + 3\dot{\phi}\sqrt{\frac{8\pi G\left(\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}m^2\phi^2\right)}{3}} + m^2\phi = 0.$$

Also use

$$\ddot{\phi} = \dot{\phi}\frac{\partial \dot{\phi}}{\partial \phi}$$

to get

$$\frac{\partial \dot{\phi}}{\partial \phi} = -\frac{\dot{\phi}\sqrt{12\pi G\left(\dot{\phi}^2 + m^2\phi^2\right)} + m^2\phi}{\dot{\phi}}.$$

Now we investigate two solutions: first we look at the inflationary solution, where the derivative of $\dot{\phi}$ with respect to ϕ is approximately zero and the $\dot{\phi}$ is small compared to the value of the field:

$$\dot{\phi}\sqrt{12\pi G\left(\dot{\phi}^2 + m^2\phi^2\right)} = -m^2\phi$$

$$\dot{\phi}\sqrt{12\pi G m^2\phi^2\left(1+\left(\frac{\dot{\phi}}{m\phi}\right)^2\right)} \cong \dot{\phi}\sqrt{12\pi G m^2\phi^2}$$

$$\dot{\phi} \cong -\frac{m}{\sqrt{12\pi G}}.$$

So we see the solutions all track to the same value, regardless of their initial conditions. This is the attractor for our model. Next we look at the ultra-hard equation of state, where the kinetic energy is small compared to the potential. In this case p≈ρ and we look in the region where ϕ>0 and $\dot{\phi}$<0.

$$\frac{\partial\dot{\phi}}{\partial\phi} = -\frac{\dot{\phi}\sqrt{12\pi G\left(\dot{\phi}^2 + m^2\phi^2\right)} + m^2\phi}{\dot{\phi}}$$

$$= -\frac{\dot{\phi}\sqrt{12\pi G\dot{\phi}^2\left(1+\frac{m^2\phi^2}{\dot{\phi}^2}\right)} + m^2\phi}{\dot{\phi}}$$

$$= -\sqrt{12\pi G\dot{\phi}^2} + O\left(\frac{\dot{\phi}}{\phi}\right)$$

$$\int_C^{\dot{\phi}} \frac{d\dot{\phi}}{\dot{\phi}} = \sqrt{12\pi G}\int_{\phi_o}^{\phi} d\phi$$

$$\dot{\phi} = Ce^{\sqrt{12\pi G}(\phi-\phi_0)}$$

Where C<0 and we have used the fact that $\dot{\phi}$<0. As ϕ decreases we see the time derivative decays exponentially more rapidly than the field itself. So large initial values of $|\dot{\phi}|$ are rapidly damped towards the attractor.

Our universe as an attractor. What is needed obviously is a model that gives as an attractor our present universe with ~70% dark energy (Ω_{vac}~0.7), ~25% dark matter, and ~4% matter. What is not known is whether it is the final state or if the universe will transition to some other attractor in the future, with different values for densities. Fig. 1 below shows an example of a tracker solution. In this plot quintessence energy density varies with the redshift, z+1, where at low redshifts–recent times–it overtakes matter and radiation. In this plot the scalar field is given by Q rather than ϕ as in our paper. Note the matter-radiation equivalence at $z+1 \cong 10^4$.

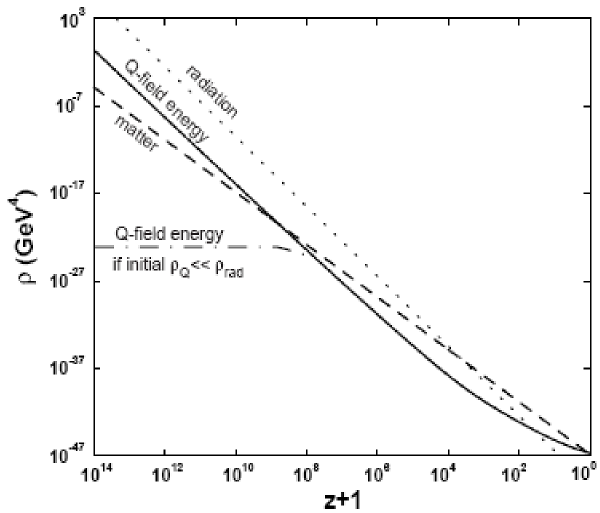


Fig. 1. The quintessence Q-field while rolling on an inverse power law potential tracks first radiation then matter, before coming to dominate the energy density of the universe at present. If the initial value of the Q-field density is small then ρ_Q remains constant until $\rho_Q \sim \rho_{rad}$, and then follows the tracker trajectory. From Zlatev, Wang and Steinhardt (1999).

V. B. GRAVITATION, FIELDS, AND TRACKING

Let us first start out discussing the relevant Lagrangians for our model. We need gravitation in the form of the Einstein-Hilbert Lagrangian, quintessence, cold dark matter, baryons, and radiation.[7,8,14,21,22,25,32]

$$\mathcal{L} = \mathcal{L}_{\mathcal{G}} + \mathcal{L}_{\phi} + \mathcal{L}_{cdm} + \mathcal{L}_{b} + \mathcal{L}_{rad}$$

$$-\mathcal{L}_{\mathcal{G}} = -\frac{R}{16\pi G} \quad \text{(Einstein - Hilbert)}$$

$$-\mathcal{L}_{\phi} = \frac{1}{2} g^{\mu\nu} \partial_{;\mu}\phi \partial_{;\nu}\phi + V(\phi)$$

$$-\mathcal{L}_{cdm} = ?$$

$$-\mathcal{L}_{b} = \bar{e}_i(\not{D} + m_i)e_i + \bar{q}_j(\not{D} + m_j)q_j + \ldots$$

$$-\mathcal{L}_{rad} = \frac{1}{4} tr\left(g_{\mu\alpha} g_{\nu\beta} F^{\mu\nu} F^{\alpha\beta}\right)$$

We also have interactions between the different components, of importance to us is the coupling between the scalar field, gravitation, and the other forms of matter. Examples of interaction terms are shown below:

$$
\begin{aligned}
\mathcal{L}_{\mathcal{G}\phi} &= -\xi f(\phi) R \\
\mathcal{L}_{m\phi} &= ? \\
\mathcal{L}_{\mathcal{A}\phi} &= -\frac{1}{4} B(\phi) tr(F^2)
\end{aligned}
$$

where the functions are generally described by polynomials in phi. By looking at the action for our Lagrangian:

$$S = \int d^4x \sqrt{-g}\mathcal{L} = \int d^4x \sqrt{-g}(\mathcal{L}_{\mathcal{G}} + \mathcal{L}_{\phi} + \mathcal{L}_{\mathcal{G}\phi} + \mathcal{L}_{m\phi} + \sum \mathcal{L}_i)$$

and then varying the action and setting it to zero we find the equations of motion and the stress-energy tensors for our different components.

$$
\begin{aligned}
\delta S &= \delta \int d^4x \sqrt{-g}\mathcal{L} = 0 \\
0 &= \delta S_{\mathcal{G}} + \delta S_{\phi} + \delta S_{\mathcal{G}\phi} + \delta S_{m\phi} + \sum_i \delta S_i. \\
\delta S_{\mathcal{G}} &= \int (\delta(\sqrt{-g}) g^{\mu\nu} R_{\mu\nu} + \sqrt{-g}\delta g^{\mu\nu} R_{\mu\nu} + \sqrt{-g} g^{\mu\nu} \delta R_{\mu\nu}) \\
&= \int \sqrt{-g}\delta g^{\mu\nu}(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R) + \text{Boundary Terms}
\end{aligned}
$$

If we let the boundary terms go to zero we get:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \mathrm{R}$$

which is the Einstein tensor. But we have other terms to deal with which makes the derivation less straightforward. Variation with respect to the field gets us a term:

$$
\begin{aligned}
\delta S_{\phi} &= \frac{1}{2} \int \sqrt{-g}(\delta g^{\mu\nu}) T_{\mu\nu} \\
T_{\mu\nu(\phi)} &= -2 \frac{\partial \mathcal{L}_{\phi}}{\partial g^{\mu\nu}} + g_{\mu\nu} \mathcal{L}_{\phi} \\
T_{\mu\nu(\phi)} &= \nabla_{\mu}\phi \nabla_{\nu}\phi - g_{\mu\nu}\left(\frac{1}{2} g^{\alpha\beta} \nabla_{\alpha}\phi \nabla_{\beta}\phi + V(\phi)\right), \\
T_{\mu\nu(m)} &= (\rho + p) u_{\mu} u_{\nu} - p g_{\mu\nu}.
\end{aligned}
$$

Where we assume the dark matter is a perfect fluid. Now we look at the interaction term with the field and gravity:

$$\delta S_{G\phi} = \int (\sqrt{-g}\frac{\partial \mathcal{L}_{G\phi}}{\partial g^{\mu\nu}}\delta g^{\mu\nu} + \mathcal{L}_{G\phi}\,\delta\sqrt{-g})d^4x$$

$$= \int (\sqrt{-g}\frac{\partial \mathcal{L}_{G\phi}}{\partial g^{\mu\nu}}\delta g^{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{L}_{G\phi}\,\delta g^{\mu\nu}\sqrt{-g})d^4x$$

$$= \int \left\{ (\frac{\partial(-\xi f(\phi)R)}{\partial g^{\mu\nu}} + \frac{1}{2}g_{\mu\nu}\xi f(\phi)R) \right\}\delta g^{\mu\nu}\sqrt{-g}\,d^4x$$

$$= \int \frac{\xi}{2}\left\{ (-2\frac{\partial f(\phi)}{\partial g^{\mu\nu}}R - 2f(\phi)\frac{\partial(g^{\mu\nu}R_{\mu\nu})}{\partial g^{\mu\nu}} + g_{\mu\nu}f(\phi)R) \right\}\delta g^{\mu\nu}\sqrt{-g}\,d^4x$$

$$= \int \xi\left\{ (-\frac{\partial f(\phi)}{\partial g^{\mu\nu}}R - f(\phi)g^{\mu\nu}\frac{\partial R_{\mu\nu}}{\partial g^{\mu\nu}} - f(\phi)R_{\mu\nu} + \frac{1}{2}g_{\mu\nu}f(\phi)R) \right\}\delta g^{\mu\nu}\sqrt{-g}\,d^4x$$

$$= \int \left\{ (-\xi f(\phi)G_{\mu\nu} - \xi\frac{\partial f(\phi)}{\partial g^{\mu\nu}}R - \xi f(\phi)g^{\mu\nu}\frac{\partial R_{\mu\nu}}{\partial g^{\mu\nu}} \right\}\delta g^{\mu\nu}\sqrt{-g}\,d^4x$$

and with our other components we get :

$$G_{\mu\nu}(1 + 8\pi G\xi f(\phi)) = 8\pi G(T_{\mu\nu(m)} + T_{\mu\nu(\phi)}) - \xi\frac{\partial f(\phi)}{\partial g^{\mu\nu}}R - \xi f(\phi)g^{\mu\nu}\frac{\partial R_{\mu\nu}}{\partial g^{\mu\nu}}$$

We see these modify our field equations with additional terms. As an example $f(\phi)=\phi^2$ for a Yukawa like potential. Now we want to deal with the important coupling of quintessence to cold dark matter (m). To do this we will turn to a linear coupling theory and re-evaluate our stress-energy tensors. The above calculation was useful in showing us the relationship between coupling and modifications to our field equations.

Linear Coupling and Brans-Dicke theory.

Next we will investigate the simplest scalar-tensor theory, the Jordan-Brans-Dicke theory. The action for this theory is given by:[1,2,6,12,13]

$$S = \int d^4x\sqrt{-g}\left(\tilde{\phi}\tilde{R} - \frac{\omega}{\tilde{\phi}}\tilde{g}^{\mu\nu}\partial_\mu\tilde{\phi}\partial_\nu\tilde{\phi}\right) + S_{(m)}$$

This is in the Jordan conformal frame. We now transform this to the Einstein conformal frame and we get:

$$S = \int d^4x\sqrt{-g}\left(\frac{R}{16\pi G} - \frac{1}{16\pi G}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right) + S_{(i)}[e^{2c_i\phi}g_{\mu\nu}]$$

By minimizing the action we get the field equations:

$$G_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} g^{\alpha\beta} \nabla_\alpha \phi \nabla_\beta \phi + \sum_i 8\pi G T^{(i)}_{\mu\nu}$$
$$g^{\mu\nu} \nabla_\mu \nabla_\nu \phi = -8\pi G \sum_i c_i T^{(i)}$$
$$T_i = g^{\mu\nu} T^{(i)}_{\mu\nu}$$

Finally we will get our continuity equation:

$$\nabla^\alpha T^{(i)}{}_{\alpha\mu} = c_i T^{(i)} \nabla_\mu \phi$$

We can generalize the above equations to include our different components, with the knowledge that the sum of terms in the continuity equation must be zero.[1,19]

$$T^{\beta}_{\alpha(\phi);\beta} = (c_m T_{(m)} + c_b T_{(b)}) \phi_{;\alpha}$$
$$T^{\beta}_{\alpha(m);\beta} = -c_m T_{(m)} \phi_{;\alpha}$$
$$T^{\beta}_{\alpha(b);\beta} = -c_b T_{(b)} \phi_{;\alpha}$$
$$T_{(r)} = 0$$

Assume $c_b=0$ and we get:

$$T^{\beta}_{\alpha(m);\beta} = -\sqrt{\frac{16\pi G}{3}} \beta T_{(m)} \phi_{;\alpha}$$
$$T^{\beta}_{\alpha(\phi);\beta} = \sqrt{\frac{16\pi G}{3}} \beta T_{(m)} \phi_{;\alpha}$$

Thus we have dark matter coupled to dark energy, or dark-dark coupling. Now we must explain the behavior seen in figure 1.

V. C. COUPLING OF QUITESSENCE AND MATTER

In these models dark energy in the form of quintessence is coupled to itself, cold dark matter, and baryons. Another coupling is the one discussed earlier; the coupling of quintessence with the gravitational field. With a conformal transformation we can get the following relations in terms of the stress energy tensors of the different components. Let's write down the stress-energy tensors of each component, and the coupling terms, where T is the trace of the tensor:[1,2,3,4,19]

$$T^{\beta}_{\alpha(\phi);\beta} = (c_m T_{(m)} + c_b T_{(b)}) \phi_{;\alpha}$$
$$T^{\beta}_{\alpha(m);\beta} = -c_m T_{(m)} \phi_{;\alpha}$$
$$T^{\beta}_{\alpha(b);\beta} = -c_b T_{(b)} \phi_{;\alpha}$$
$$T_{(r)} = 0$$

General covariance requires the sum of these terms to be zero and this is the continuity equation. Each individually can be zero, but this is not required. In this case the baryon term is zero, and the *sum* of the CDM and quintessence is also zero. Also, It turns out the above stress-energy equations are equivalent to the Lagrangian couplings under a conformal transformation. In particular we can get the following:[19]

$$T^{\beta}_{\alpha(cdm);\beta} = -\sqrt{\frac{16\pi G}{3}}\beta T_{(cdm)}\phi_{;\alpha}$$
$$T^{\beta}_{\alpha(\phi);\beta} = \sqrt{\frac{16\pi G}{3}}\beta T_{(cdm)}\phi_{;\alpha}$$

We have neglected the baryon coupling since experimental and observational constraints restrict the dark energy/baryon coupling to β_b<0.01. Radiation is not here because its trace vanishes from its equation of state, p=1/3 ρ. Notice that the sum is zero. In one model by Maccio et al, (2003)[19] they assume a flat background metric given by:

$$ds^2 = S^2(-dt^2 + \delta_{ij}dx^i dx^j)$$

and they get for the continuity equations:

$$\ddot{\phi} + 2H\dot{\phi} + S^2\frac{\partial V(\phi)}{\partial \phi} = \sqrt{\frac{16\pi G}{3}}\beta S^2 \rho_{cdm}$$
$$\dot{\rho}_{cdm} + 3H\rho_{cdm} = -\sqrt{\frac{16\pi G}{3}}\beta S^2 \rho_{cdm}\dot{\phi}$$
$$\dot{\rho}_b + 3H\rho_b = 0$$
$$\dot{\rho}_r + 4H\rho_r = 0.$$

The universe in this model starts out radiation dominated and then becomes DE/matter dominated (ϕ-M) after radiation and ϕ-M densities are equal. We include cold dark matter in the mass terms. The expansion is slightly decreased from a pure matter dominated expansion, as the kinetic term is greater than V. This large kinetic term increases perturbation growth. As ϕ-M dominates V increases and eventually the solutions lead to a tracking phase, whose details depend upon the potential. For most potentials they eventually lead to a global attractor, where DE overwhelms all other components.

One way to accomplish this we devise a β (coupling) which transforms from a zero—or small—coupling to a large coupling as ϕ rolls down its potential.[4] We get for the power law expansion of the universe:

$$S \sim t^{p}$$
$$p = \frac{2}{3}\left(1 + \frac{\beta}{\mu}\right)$$
$$\mu = k\sqrt{\frac{3}{16\pi G}}$$

where k is in the exponent of an exponential type potential $e^{k\phi}$. We have as an expression for β:

$$\beta(\phi) = \frac{1}{2}\beta_2\left(\tanh\left(8\pi G\sqrt{\xi}\{\phi - \phi\}\right) + 1\right)$$
$$1 << \mu << \beta_2$$

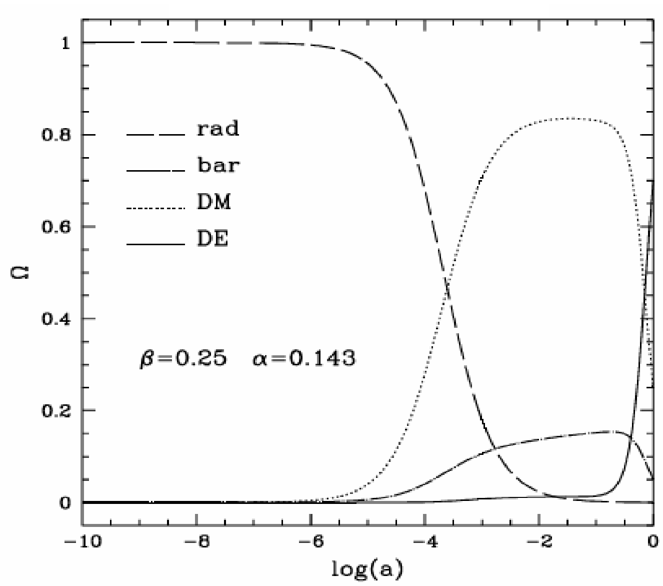
where ξ is the coupling from the gravitational-quintessence Lagrangian and β_2 is a constant. The exact form of the coupling function above is not important; any step-like function that switches on after structure formation will give qualitatively similar results.

Other models involve constant β, for which I show the results below.[19] For this model, after equivalence of rad-matter, the world goes through three stages: (i) ϕ-MDE (ii) tracking phase and (iii) final global attractor. During (i) the universe is expanding very much like a matter dominated universe (MDE) but has a small correction due to the scalar field. Solving the Friedmann equation gives:

$$S \sim t^{\frac{4}{(6+4\beta^2)}}$$

which is slower growth than pure MDE. V continues to grow during (i) and eventually the kinetic term is smaller than V. At this point the world goes to (ii) and tracks according to the choice of potential. Most of these finally reach (iii) and then end up in a global attractor and the DE density becomes larger than DM density. See figure 2 below.[19]

Fig. 2 Maccio et al (2003)



In the above figure we see Ω, the density parameter, as a function of the log of the scale factor. Both beta and alpha are given, where alpha is given in the potential, $V=k^{4+\alpha}/\phi^{\alpha}$. Note that if $\alpha=0$ then this is equivalent to the cosmological constant. The graph shown is fairly representative of other models found for the evolution of Ω.

V. D. TRACKING, COUPLING & WMAP

WMAP has put limits on our models and narrowed the range of uncertainty in the parameters. In general the constraints on CMB presented in the literature on the dark energy equation of state are somewhat model dependent. The angular-diameter distance to last scattering depends on w through two integrals and we have degeneracy among different parameters for a particular evolution curve. Initial conditions are chosen so that trajectories reach tracking solutions, and in general the power law $V\sim\phi^{-\alpha}$ is used for the quintessence, where as α gets large it will approximate an exponential potential. Below is an overview of numerical results from several groups who have done calculations using WMAP data as constraints.[3,11]

Constant equation of state. First we will look at a constant equation of state which is of the form $p=w\rho$. We learned above of the equivalence between the scalar field with potential $V(\phi)$ and w. In general we expect the $w<-1/3$ for an accelerating universe, and from Sn Ia and WMAP results we get a value of $-1.25<w<-0.8$, seen below in Fig. 3.[11]

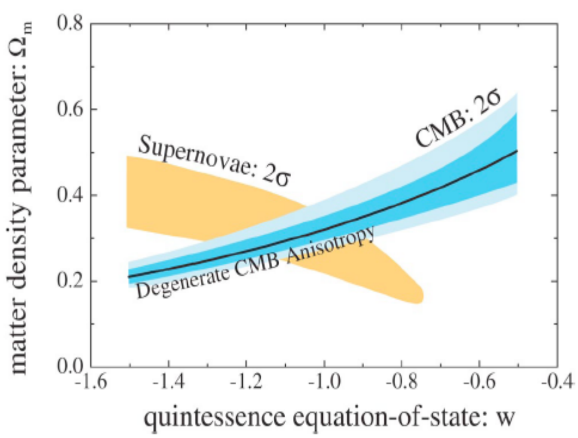


Fig. 3. The constraints on constant equation of state models due to CMB (WMAP, ACBAR, CBI) and type 1a supernovae (Hi-Z, SCP) are shown. The starting point for our parameter search, the family of CMB-degenerate models, is shown by the thick, black line. [Amendola & Quercellini, 2003]

Coupled quintessence. We now discuss coupled quintessence. Following Amendola and Quercellini (2003)[3] we assume a power law dependence $V\sim\phi^{-\alpha}$ which will give us a present equation of state due to tracking of:

$$w_{\phi}=\frac{p_{\phi}}{\rho_{\phi}}\approx-\frac{2}{2+\alpha}$$

Again we assume weak coupling with a Yukawa like term ~Rϕ^2 coupling DE to CDM, which we denote as ϕ-M. This term will modify the gravitational effects on CDM, and on scales well below the horizon distance this simply renormalizes Newton's constant for dark matter:

$$G = G_0 (1 + \frac{4\beta^2}{3})$$

where β is the coupling constant introduced earlier, and G_0 is the unrenormalized Gravitational constant. Coupling allows us after equivalence, but before current acceleration, to assume the universe enters a field-matter-dominated era (ϕMDE), in which the equation of state stays constant. This epoch ends when DE enters the tracking phase and the potential takes over and accelerates the expansion. In all of this M or mass always indicates CDM which is the dominant form of matter. Baryons are coupled very weakly to DE and aren't

$$\Omega_\phi = \frac{4\beta^2}{3}$$

$$w_{(\phi MDE)} = \frac{p_\phi + p_{cdm}}{\rho_\phi + \rho_{cdm}} = \frac{4\beta^2}{9}$$

included. During ϕMDE we get the following:

Again, this is during the ϕMDE epoch and will help in finding the current equation of state given by w_ϕ. From measurements there is a 95% C.L. (confidence level) for the following:

$$\alpha < 2.08$$
$$\beta < 0.13.$$

This certainly indicates that the ϕ-M coupling is weak, but it also disallows the exponential potentials (corresponding to $\alpha \to \infty$) for this particular model.

The tables below give bounds on coupled dark energy and associated cosmic parameters:

$$h = \frac{H}{100 \, km/s} MPc$$

$$\omega_c = \Omega_c h^2 \quad (CDM)$$

$$\omega_b = \Omega_b h^2 \quad (baryons)$$

$$n_s = \text{slope of primordial fluctuations.}$$

Table 2. Coupled Dark Energy

Parameter	WMAP	WMAP+HST	pre-WMAP
w_ϕ	$<-0.67(-0.49)$	$<-0.69(-0.52)$	$<-0.50(-0.25)$
α	$<0.99(2.08)$	$<0.90(1.84)$	$<2.0(6.0)$
w_e	$<0.0025(0.0075)$	$<0.0023(0.0075)$	$<0.0075(0.016)$
β	$<0.075(0.13)$	$<0.072(0.13)$	$<0.13(0.19)$
h	0.73 ± 0.05	0.73 ± 0.04	$>0.62(0.55)$
n_s	1.019 ± 0.025	1.018 ± 0.025	0.97 ± 0.03
ω_b	0.0247 ± 0.0008	0.0250 ± 0.0008	0.021 ± 0.003
ω_c	0.123 ± 0.016	0.120 ± 0.016	0.12 ± 0.04

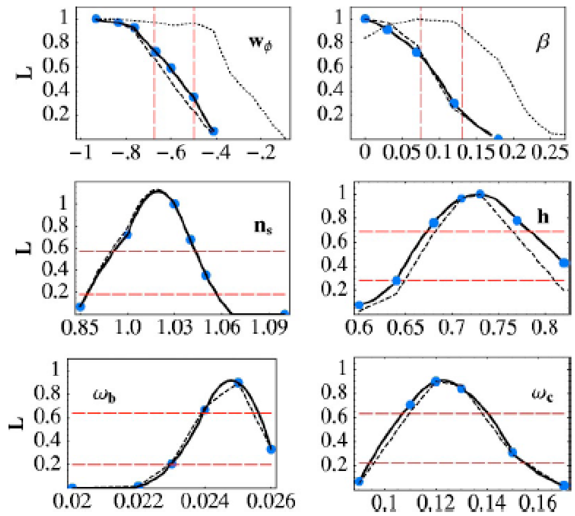


FIG. 4. Marginalized likelihood for tracking trajectories. The solid curves are for the WMAP data, the short-dashed curves are for the HST prior, and the dotted curves in the panels for w_ϕ and β for the pre-WMAP compilation. The horizontal long-dashed lines are the confidence levels at 68% and 95%. The vertical long-dashed lines in the panel for w_ϕ mark the upper bounds at 68% and 95% confidence levels. [From Amendola & Quercellini, 2003]

All of these models give proper tracking behavior and give results consistent with measurement, but the veracity of their early universe behavior by definition isn't certain. That is, since many of these models give us the same solution, their measure of reality is somewhat speculative. In the future we can hope more cosmological observations and experiments can help decide which model is the best description of reality.

VI. BRANE WORLD MODELS

As a final note we will discuss briefly strings, braneworld cosmologies, and dark energy in those models.[5,6,15,20,28] In trying to develop unified theories with gravity it is often useful to introduce higher spatial dimensions beyond the three we experience on a day to day basis. Also, string theory has been employed with some success in this regard, also with the introduction of higher dimensions. Note there is also a time dimension, which is often referred to as the fourth dimension. In this universe the four dimensional world in which we live is a 3-brane, since it has three spatial dimensions, and extra dimensions, say an additional fourth spatial dimension (fifth overall dimension), are called the bulk. Branes can be imagined as "surfaces" in which matter and energy are localized.[15]

In theses models matter fields are confined to the 3-brane but the quanta of gravity, the graviton, is free to move about the bulk and the brane. In the Randall-Sundrum scenario the equation of motion of a scalar field is:

$$\ddot{\phi} + 3H\dot{\phi} + \frac{\partial V(\phi)}{\partial \phi} = 0$$

$$H^2 = \frac{8\pi}{3m_4^2}\rho\left(1 + \frac{\rho}{2\sigma}\right) + \frac{\Lambda_4}{3} + \frac{\varepsilon}{S^2}$$

$$\rho = \frac{1}{2}\dot{\phi}^2 + V$$

$$m_4 = G^{-\frac{1}{2}}$$

where ε is an integration constant transmitting bulk graviton influence onto the 3-brane. σ is the brane tension, which gives us a relationship between the bulk (five dimensional) Planck mass, m_5, and the four dimensional (brane) Planck mass, m_4. It also relates the bulk cosmological constant with the four dimensional brane cosmological constant:

$$m_4 = \sqrt{\frac{3}{4\pi}}\frac{m_5^3}{\sqrt{\sigma}}$$

$$\Lambda_4 = \frac{4\pi}{m_5^3}\left(\Lambda_5 + \frac{4\pi}{3m_5^3}\sigma^2\right)$$

In this model we see that the Friedmann equation contains an extra term given by ρ^2/σ which is associated with the junction conditions imposed on the bulk-brane boundary. Because of this term, the damping of the scalar field in the slow roll is increased dramatically. This allows for a greater range of potentials, which would normally be too steep. This leaves open the possibility that the same scalar field may source quintessence and inflation. This is called "quintessential inflation."[28]

VII. CONCLUSION.

Current astronomical measurements have turned cosmology into a precise science. It has become well established the expansion rate of the universe is increasing and the responsible agent has been dubbed "dark energy." Models for this dark energy include a background cosmological constant and a scalar field called "quintessence." Quintessence has the property of being very weakly coupled to baryons (us) but contributing a negative pressure to the equation of state. In the past it had a small contribution but with time it has decreased less quickly with the scale factor than the matter and radiation densities and is dominant now. Now that it is dominant it is "inflating" the scale factor at an exponential rate.

"Tracker" models are used regularly to describe quintessence because for a wide range of initial conditions they give results consistent with modern observations. WMAP and other experiments have limited some of the models, but more refinement in observation is needed to differentiate more acutely. Brane world quintessence is interesting in that it allows even wider range of potentials due to increased damping during slow roll.

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