AA450: Control in Aerospace Systems
Linearization

1 Theory

The control methods we are developing in this class are for use with linear systems. (Recall that a linear system is one for which the dynamics have the form of the equation of a line: $\dot{x} = mx + b$.) In reality all systems are nonlinear, but we can often use our linear control methods in a restricted set of operating conditions. The analysis we use for setting up the appropriate linear approximation of a given nonlinear system is *local linearization* (sometimes referred to as a small signal linear model).

To begin, consider a scalar nonlinear function $f(x(t))$ and a desired system value $x_0(t)$ (which, for example, may be an equilibrium point or a trajectory). The Taylor expansion of $f(x(t))$ about $x_0(t)$ is (this is given in any introductory calculus text)

$$f(x(t)) \quad = \quad f(x_0(t)) + \left(\left. \frac{\partial f(x(t))}{\partial x(t)} \right|_{x(t)=x_0(t)} \right) (x(t) - x_0(t)) + \frac{1}{2!} \left(\left. \frac{\partial^2 f(x(t))}{\partial x(t)^2} \right|_{x(t)=x_0(t)} \right) (x(t) - x_0(t))^2 + \ldots$$

If the actual trajectory is close to the nominal trajectory (meaning $x(t) - x_0(t) << 1$), then we can neglect terms larger than first order in $x(t) - x_0(t)$. The linear system approximation is then

$$f(x(t)) \approx f(x_0(t)) + \left(\left. \frac{\partial f(x(t))}{\partial x(t)} \right|_{x(t)=x_0} \right) (x(t) - x_0).$$

Furthermore, if the condition about which we are evaluating the Taylor expansion is an equilibrium point, we know that $x_0(t) = x_0$ (a constant) and $f(x_0) = 0$. The linearization then simplifies to

$$f(x(t)) \quad \approx \quad \left(\left. \frac{\partial f(x(t))}{\partial x(t)} \right|_{x(t)=x_0} \right) (x(t) - x_0)$$

which we can write as

$$f(x(t)) \approx a(t)(x(t) - x_0).$$

If the equilibrium point is $x_0 = 0$, then we have

$$f(x(t)) \approx a(t)x(t).$$

If $f(x)$ is a function of more than one variable, then this expansion generalizes to

$$\begin{aligned}
f(x(t), u(t)) \quad = \quad & f(x_0(t), u_0(t)) \\
& + \left(\left. \frac{\partial f(x(t), u(t))}{\partial x(t)} \right|_{\substack{x(t)=x_0(t) \\ u(t)=u_0(t)}} \right) (x(t) - x_0(t)) + \left(\left. \frac{\partial f(x(t), u(t))}{\partial u(t)} \right|_{\substack{x(t)=x_0(t) \\ u(t)=u_0(t)}} \right) (u(t) - u_0(t)) \\
& + \frac{1}{2!} \left(\left. \frac{\partial^2 f(x(t), u(t))}{\partial x(t)^2} \right|_{\substack{x(t)=x_0(t) \\ u(t)=u_0(t)}} \right) (x(t) - x_0(t))^2 + \frac{1}{2!} \left(\left. \frac{\partial^2 f(x(t), u(t))}{\partial u(t)^2} \right|_{\substack{x(t)=x_0(t) \\ u(t)=u_0(t)}} \right) (u(t) - u_0(t))^2 + \\
& + \left(\left. \frac{\partial^2 f(x(t), u(t))}{\partial u(t) \partial x(t)} \right|_{\substack{x(t)=x_0(t) \\ u(t)=u_0(t)}} \right) (x(t) - x_0(t))(u(t) - u_0(t)) + \ldots
\end{aligned}$$

This idea generalizes directly to the case where the state, control and function are vectors: $\mathbf{x}, \mathbf{u}, \mathbf{f}(\mathbf{x}, \mathbf{u})$. We can then write the Taylor expansion in the standard form

$$\mathbf{f}(\mathbf{x}, \mathbf{u}) \quad = \quad \mathbf{f}(\mathbf{x}_0, \mathbf{u}_0) + \left(D_{\mathbf{x}} \mathbf{f}(\mathbf{x}, \mathbf{u}) |_{\mathbf{x}=\mathbf{x}_0, \mathbf{u}=\mathbf{u}_0} \right) (\mathbf{x} - \mathbf{x}_0) + \left(D_{\mathbf{u}} \mathbf{f}(\mathbf{x}, \mathbf{u}) |_{\mathbf{x}=\mathbf{x}_0, \mathbf{u}=\mathbf{u}_0} \right) (\mathbf{u} - \mathbf{u}_0) + \ldots$$

where $D_{\mathbf{x}}\mathbf{f}(\mathbf{x},\mathbf{u})$ is the set of partial derivatives of $\mathbf{f}$ with respect to $\mathbf{x}$ and similarly for $D_{\mathbf{u}}\mathbf{f}(\mathbf{x},\mathbf{u})$ and $\mathbf{u}$. Again, if we choose $\mathbf{x}_0, \mathbf{u}_0$ to be an equilibrium point, the equations reduce to

$$\begin{aligned}\mathbf{f}(\mathbf{x},\mathbf{u}) &= \left(D_{\mathbf{x}}\mathbf{f}(\mathbf{x},\mathbf{u})|_{\mathbf{x}=\mathbf{x}_0,\mathbf{u}=\mathbf{u}_0}\right)(\mathbf{x}-\mathbf{x}_0) + \left(D_{\mathbf{u}}\mathbf{f}(\mathbf{x},\mathbf{u})|_{\mathbf{x}=\mathbf{x}_0,\mathbf{u}=\mathbf{u}_0}\right)(\mathbf{u}-\mathbf{u}_0)\\ &= A(t)(\mathbf{x}-\mathbf{x}_0) + B(\mathbf{u}-\mathbf{u}_0)\end{aligned}$$

where we have assumed here that higher order terms are neglible. For reference note that

$$\mathbf{f} = \begin{bmatrix} f_1(\mathbf{x},\mathbf{u}) \\ \vdots \\ f_n(\mathbf{x},\mathbf{u}) \end{bmatrix},\ \mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix},\ \mathbf{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}$$

assuming that there are n states and m controls. The matrices in the linear approximation then work out to be

$$A(t) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ & \vdots & \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}_{\mathbf{x}=\mathbf{x}_0,\mathbf{u}=\mathbf{u}_0},\ B(t) = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \cdots & \frac{\partial f_1}{\partial u_m} \\ & \vdots & \\ \frac{\partial f_n}{\partial u_1} & \cdots & \frac{\partial f_n}{\partial u_m} \end{bmatrix}_{\mathbf{x}=\mathbf{x}_0,\mathbf{u}=\mathbf{u}_0}$$

To create a linearization of a set of state equations, we start from the set of equations

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x},\mathbf{u})$$

We will always be taking our Taylor expansion about an equilibrium point, so our first step is to solve for the equilibrium points $\mathbf{x}_0, \mathbf{u}_0$:

$$\mathbf{f}(\mathbf{x}_0,\mathbf{u}_0) = \mathbf{0}.$$

The expansion for our system is then

$$\begin{aligned}\dot{\mathbf{x}} &= \left(D_{\mathbf{x}}\mathbf{f}(\mathbf{x},\mathbf{u})|_{\mathbf{x}=\mathbf{x}_0,\mathbf{u}=\mathbf{u}_0}\right)(\mathbf{x}-\mathbf{x}_0) + \left(D_{\mathbf{u}}\mathbf{f}(\mathbf{x},\mathbf{u})|_{\mathbf{x}=\mathbf{x}_0,\mathbf{u}=\mathbf{u}_0}\right)(\mathbf{u}-\mathbf{u}_0)\\ &= A(t)(\mathbf{x}-\mathbf{x}_0) + B(t)(\mathbf{u}-\mathbf{u}_0)\end{aligned}$$

Now note that if we define a new pair of vectors $\mathbf{z} = \mathbf{x} - \mathbf{x}_0$ and $\mathbf{v} = \mathbf{u} - \mathbf{u}_0$ then we have the equations

$$\dot{\mathbf{z}} = \dot{\mathbf{x}} - \dot{\mathbf{x}_0} = \dot{\mathbf{x}} = A(t)\mathbf{z} + B(t)\mathbf{v}$$

We will generally employ an abuse of notation and simply state that our linearized equations are

$$\dot{\mathbf{x}} = A(t)\mathbf{x} + B(t)\mathbf{u}.$$

Remark: A nonlinear system can also be linearized about a trajectory as long as a nominal trajectory and nominal set of inputs are specified. The process above remains the same under the assumption that the error between the actual and nominal trajectories and inputs will be small.

2 Example - Cart and Pendulum

Given a state space description of a system where we would like to linearize the system, perform the following steps

- Transform the system equations to the state space form $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x},\mathbf{u})$.
- Find all equilibria by solving $\mathbf{f}(\mathbf{x},\mathbf{u}) = 0$ for all values of $\mathbf{x}$ and $\mathbf{u}$ that satisfy the equation.
- For each equilibria point, linearize the system to get $\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}$.

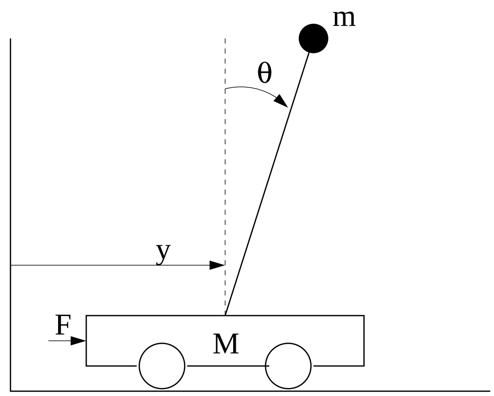

Figure 1: Force diagram of pendulum and cart.

As an example of linearization, consider the cart and pendulum shown in Fig. 1. The equations of motion for this system (see Friedland p. 31 for details) are given by

$$\begin{aligned}(M+m)\ddot{y}+ml\cos(\theta)\ddot{\theta}-ml\dot{\theta}^2\sin(\theta) &= F\\ ml\cos(\theta)\ddot{y}+ml^2\ddot{\theta}-mgl\sin(\theta) &= 0\end{aligned}$$

First, transform to first order state space:

$$\frac{d}{dt}\begin{bmatrix} y \\ \theta \\ \dot{y} \\ \dot{\theta}\end{bmatrix} = \begin{bmatrix} \dot{y} \\ \dot{\theta} \\ \begin{bmatrix} M+m & ml\cos(\theta) \\ ml\cos(\theta) & ml^2\end{bmatrix}^{-1}\begin{bmatrix} ml\dot{\theta}^2\sin(\theta)+F \\ mgl\sin(\theta)\end{bmatrix}\end{bmatrix}$$

where our state vector is $\mathbf{x}=[y,\theta,\dot{y},\dot{\theta}]^T$ and $u=F$. Note that in these equations we simply solve the second order equations for $[\ddot{y},\ddot{\theta}]$ and then convert to first order form.

Next, we solve the system equations for the equilibrium points:

$$\begin{bmatrix} \dot{y} \\ \dot{\theta} \\ \begin{bmatrix} M+m & ml\cos(\theta) \\ ml\cos(\theta) & ml^2\end{bmatrix}^{-1}\begin{bmatrix} ml\dot{\theta}^2\sin(\theta)+F \\ mgl\sin(\theta)\end{bmatrix}\end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0\end{bmatrix}$$

We can see immediately from this equality that all equilibria have $\dot{y}_e=0$ and $\dot{\theta}_e=0$. Knowing this, the last two equations reduce to

$$\begin{bmatrix} M+m & ml\cos(\theta) \\ ml\cos(\theta) & ml^2\end{bmatrix}^{-1}\begin{bmatrix} F \\ mgl\sin(\theta)\end{bmatrix} = \begin{bmatrix} 0 \\ 0\end{bmatrix}$$

Solving these equations in Mathematica gives

$$\{F=0,\theta=\pm n\pi\}\,,\ \left\{F=\pm g\sqrt{-M(m+M)},\theta=\pm\cos^{-1}\left(\pm\sqrt{\frac{m+M}{m}}\right)\right\}$$

The masses will be positive constants, therefore the second set of equilibria would require imaginary-valued forcing. We will ignore those equilibria. As a comment, allowing the system to have a nonzero velocity admits stable nonzero angles of the pendulum (but these are not equilibria because the velocities are nonzero).

Now we linearize about each of the equilibria. Computing the system Jacobian with respect to $\mathbf{x}$ gives:

$$\frac{\partial f_i}{\partial x_j} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{m(2gm-\dot{\theta}^2 l(m-4M)\cos(\theta)-2g(m+2M)\cos(2\theta)+\dot{\theta}^2 lm\cos(3\theta)-4f\sin(2\theta))}{(m+2M-m\cos 2\theta)^2} & 0 & \frac{4\dot{\theta}lm\sin(\theta)}{m+2M-m\cos(2\theta)} \\ 0 & \frac{\begin{array}{c}(2\dot{\theta}^2 lm^2 - g(m-4M)(m+M)\cos(\theta) - 2\dot{\theta}^2 lm(m+2M)\cos(2\theta)+ \\ gm^2\cos(3\theta) + gmM\cos(3\theta) + 5fm\sin(\theta) + 4fM\sin(\theta) + fm\sin(3\theta)\end{array}}{l(m+2M-m\cos(2\theta))^2} & 0 & -\frac{\dot{\theta}m\sin 2\theta}{m+M-m\cos(\theta)^2} \end{bmatrix}$$

And for u we have

$$\frac{\partial f_i}{\partial u} = \begin{bmatrix} 0 \\ 0 \\ \frac{-2}{-m-2M+m\cos(2\theta)} \\ \frac{-cos(\theta)}{l(m+M-m\cos^2(\theta)} \end{bmatrix}$$

Note that we will end up with only two unique linearizations because $\sin(n\pi) = 0$, $\cos(2n\pi) = 1$ and $\cos((2n+1)\pi) = -1$. Evaluating the Jacobian for the two sets of unique equilibria gives

$$F = 0, n \text{ even}, \quad A_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{gm}{M} & 0 & 0 \\ 0 & \frac{g(m+M)}{lM} & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{M} \\ -\frac{1}{lM} \end{bmatrix}$$

$$F = 0, n \text{ odd}, \quad A_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{gm}{M} & 0 & 0 \\ 0 & -\frac{g(m+M)}{lM} & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{M} \\ \frac{1}{lM} \end{bmatrix}$$