Chapter 8

1. List all the binary relations on the set {0,1}.
   Ans: There are 16 binary relations:

{}	{(0,0)}	{(0,1)}	{(1,0)}
{(1,1)}	{(0,0),(0,1)}	{(0,0),(1,0)}	{(0,0),(1,1)}
{(0,1),(1,0)}	{(0,1),(1,1)}	{(1,0),(1,1)}	{(0,0),(0,1),(1,0)}
{(0,0),(0,1),(1,1)}	{(0,0),(1,0),(1,1)}	{(0,1),(1,0),(1,1)}	{(0,0),(0,1),(1,0),(1,1)}

2. List the reflexive relations on the set {0,1}.
   Ans: 8, 13, 14, 16.

3. List the irreflexive relations on the set {0,1}.
   Ans: 1, 3, 4, 9.

4. List the symmetric relations on the set {0,1}.
   Ans: 1, 2, 5, 8, 9, 12, 15, 16.

5. List the transitive relations on the set {0,1}.
   Ans: 1, 2, 3, 4, 5, 6, 7, 8, 10, 11, 13, 14, 16.

6. List the antisymmetric relations on the set {0,1}.
   Ans: 1, 2, 3, 4, 5, 6, 7, 8, 10, 11, 13, 14.

7. List the asymmetric relations on the set {0,1}.
   Ans: 1, 3, 4.

8. List the relations on the set {0,1} that are reflexive and symmetric.
   Ans: 8, 16.

9. List the relations on the set {0,1} that are neither reflexive nor irreflexive.
   Ans: 2, 5, 6, 7, 10, 11, 12, 15.

Use the following to answer questions 10-23:

In the questions below determine whether the binary relation is: (1) reflexive, (2) symmetric, (3) antisymmetric, (4) transitive.

10. The relation R on {1,2,3,...} where aRb means $a \mid b$.
    Ans: 1, 3, 4.

11. The relation R on $\{w,x,y,z\}$ where $R = \{(w,w),(w,x),(x,w),(x,x),(x,z),(y,y),(z,y),(z,z)\}$.
    Ans: 1.

12. The relation R on $\boldsymbol{Z}$ where aRb means $|a - b| \le 1$.
    Ans: 1, 2.

13. The relation R on $\boldsymbol{Z}$ where aRb means $a^2 = b^2$.
    Ans: 1, 2, 4.

14. The relation R on $\{a,b,c\}$ where $R = \{(a,a),(b,b),(c,c),(a,b),(a,c),(c,b)\}$.
    Ans: 1, 3, 4.

15. The relation R on $A = \{x,y,z\}$ where $R = \{(x,x),(y,z),(z,y)\}$.
    Ans: 2.

16. The relation R on $\boldsymbol{Z}$ where aRb means $a \ne b$.
    Ans: 2.

17. The relation R on Z where aRb means that the units digit of a is equal to the units digit of b.
    Ans: 1, 2, 4.

18. The relation R on N where aRb means that a has the same number of digits as b.
    Ans: 1, 2, 4.

19. The relation R on the set of all subsets of $\{1,2,3,4\}$ where SRT means $S \subseteq T$.
    Ans: 1, 3, 4.

20. The relation R on the set of all people where aRb means that a is at least as tall as b.
    Ans: 1, 4.

21. The relation R on the set of all people where aRb means that a is younger than b.
    Ans: 3, 4

22. The relation R on the set $\{(a,b) \mid a,b \in \boldsymbol{Z}\}$ where $(a,b)R(c,d)$ means $a = c$ or $b = d$.
    Ans: 1, 2.

23. The relation R on R where aRb means $a - b \in \boldsymbol{Z}$.
    Ans: 1, 2, 4.

24. A company makes four kinds of products. Each product has a size code, a weight code, and a shape code. The following table shows these codes:

	Size Code	Weight Code	Shape Code
1	42	27	42
2	27	38	13
3	13	12	27
4	42	38	38

Find which of the three codes is a primary key. If none of the three codes is a primary key, explain why.

Ans: Shape code.

25. If X =(Fran Williams, 617885197, MTH 202, 248B West), find the projections $P_{1,3}(X)$ and $P_{1,2,4}(X)$.

Ans: $P_{1,3}(X)$ = (FranWilliams,MTH 202) $P_{1,2,4}(X)$ = (FranWilliams,617885197, 248BWest).

Use the following to answer questions 26-31:

In the questions below suppose R and S are relations on $\{a,b,c,d\}$, where $R = \{(a,b),(a,d),(b,c),(c,c),(d,a)\}$ and $S = \{(a,c),(b,d),(d,a)\}$.

26. Construct R^2.

Ans: $\{(a,a),(a,c),(b,c),(c,c),(d,b),(d,d)\}$.

27. Construct R^3.

Ans: $\{(a,b),(a,c),(a,d),(b,c),(c,c),(d,a),(d,c)\}$.

28. Construct S^2.

Ans: $\{(b,a),(d,c)\}$.

29. Construct S^3.

Ans: $\{(b,c)\}$.

30. Construct $R \circ S$.

Ans: $\{(a,c),(b,a),(d,b),(d,d)\}$.

31. Construct $S \circ R$.

Ans: $\{(a,a),(a,d),(d,c)\}$.

Use the following to answer questions 32-41:

In the questions below find the matrix that represents the given relation. Use elements in the order given to determine rows and columns of the matrix.

32. R on $\{1,2,3,4\}$ where aRb means $|a - b| \le 1$.

Ans: $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$.

33. R on $\{w,x,y,z\}$ where $R = \{(w,w),(w,x),(x,w),(x,x),(x,z),(y,y),(z,y),(z,z)\}$.

Ans: $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$.

34. R on $\{-2,-1,0,1,2,\}$ where aRb means $a^2 = b^2$.

Ans: $\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}$.

35. R on $\{1,2,3,4,6,12\}$ where aRb means $a \mid b$.

Ans: $\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$.

36. R on $\{1,2,4,8,16\}$ where aRb means $a \mid b$.
Ans:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

37. R on $\{1,2,4,8,16\}$ where aRb means $a \le b$.

Ans: $$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

38. R^2, where R is the relation on on $\{1,2,3,4\}$ such that aRb means $|a-b| \le 1$.

Ans: $$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}.$$

39. R^2, where R is the relation on $\{w,x,y,z\}$ such that $R = \{(w,w),(w,x),(x,w),(x,x),(x,z),(y,y),(z,y),(z,z)\}$.

Ans: $$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

40. R^{-1}, where R is the relation on $\{1,2,3,4\}$ such that aRb means $|a-b| \le 1$.
Ans:

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

41. $\overline{R}$, where R is the relation on $\{w,x,y,z\}$ such that $R = \{(w,w),(w,x),(x,w),(x,x),(x,z),(y,y),(z,y),(z,z)\}$.

Ans: $\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$.

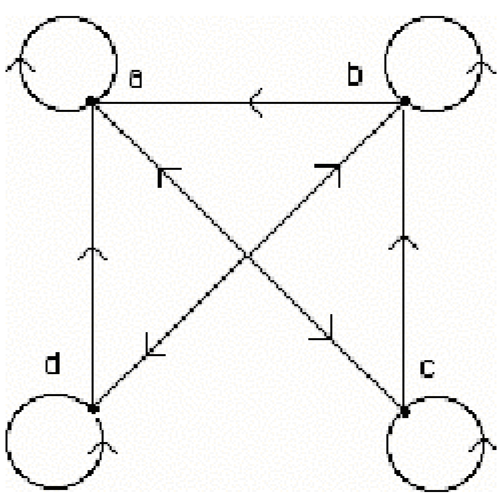
42. If $\mathbf{M}_R = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$, determine if R is: (a) reflexive (b) symmetric (c) antisymmetric (d) transitive.

Ans: (a) Yes. (b) No. (c) No. (d) No.

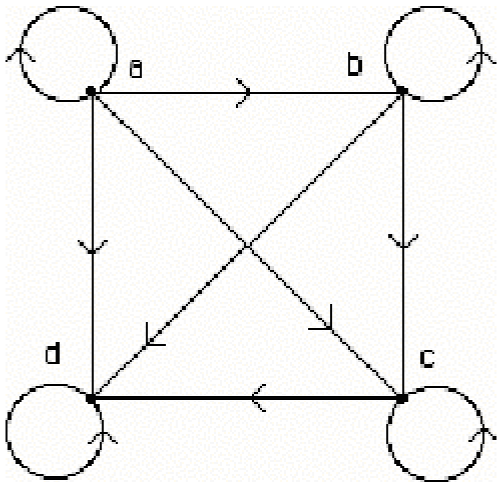
43. If $\mathbf{M}_R = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, determine if R is: (a) reflexive (b) symmetric (c) antisymmetric (d) transitive.

Ans: (a) Yes. (b) No. (c) Yes. (d) Yes.

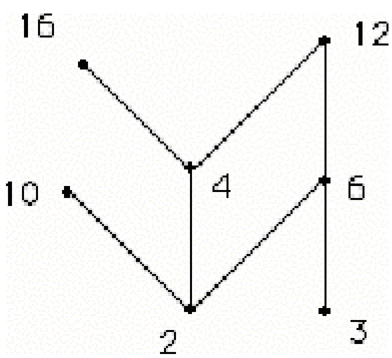
44. Draw the directed graph for the relation defined by the matrix $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$,

Ans: 

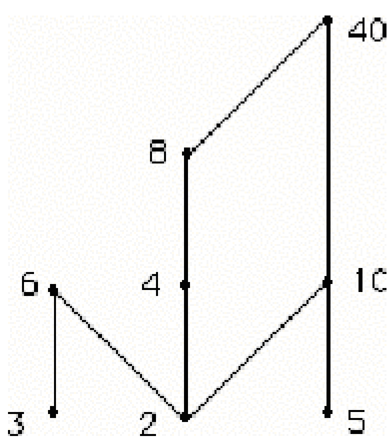
45. Draw the directed graph for the relation defined by the matrix $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Ans: 

46. Draw the Hasse diagram for the relation R on $A = \{2,3,4,6,10,12,16\}$ where aRb means $a \mid b$.

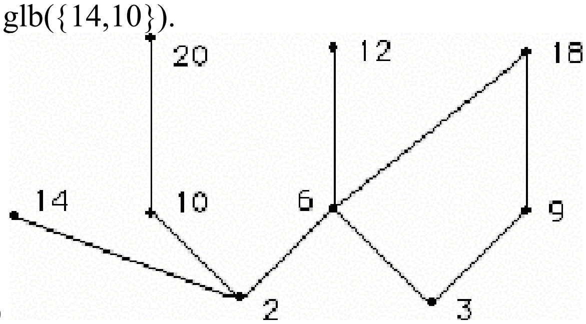
Ans: 

47. Draw the Hasse diagram for the relation R on $A = \{2,3,4,5,6,8,10,40\}$ where aRb means $a \mid b$.

Ans: 

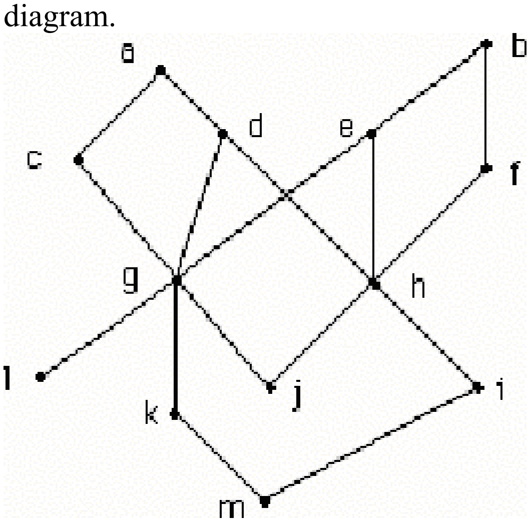
48. Suppose $A = \{2,3,6,9,10,12,14,18,20\}$ and R is the partial order relation defined on A where xRy means x is a divisor of y.
(a) Draw the Hasse diagram for R.
(b) Find all maximal elements.
(c) Find all minimal elements.
(d) Find lub($\{2,9\}$).
(e) Find lub($\{3,10\}$).
(f) Find glb($\{14,10\}$).

Ans: (a)



(b) 12,14,18,20. (c) 2,3. (d) 18.
(e) Does not exist. (f) 2.

49. The diagram shown is the Hasse diagram for a partially ordered set. Referring to this diagram:
(a) List the maximal elements
(b) List the minimal elements
(c) Find all upper bounds for f,g
(d) Find all lower bounds for d,f
(e) Find lub($\{g,j,m\}$)
(f) Find glb($\{d,e\}$)
(g) Find the greatest element
(h) Find the least element
(i) Use a topological sort to order the elements of the poset represented by this Hasse diagram.



Ans: (a) a,b. (b) l,m. (c) b. (d) h,i,j,m. (e) g. (f) None. (g) None. (h) None.
(i) For example: $m,k,i,j,l,h,g,f,e,c,d,b,a$.

50. Find the transitive closure of R if $\mathbf{M}_R$ is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$.

Ans: . $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$.

51. Find the transitive closure of R if $\mathbf{M}_R$ is $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$.

Ans: $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$.

52. If $R = \{(1,2),(1,4),(2,3),(3,1),(4,2)\}$, find the reflexive closure of R.
Ans: $\{(1,1),(1,2),(1,4),(2,2),(2,3),(3,1),(3,3),(4,2),(4,4)\}$.

53. If $R = \{(1,2),(1,4),(2,3),(3,1),(4,2)\}$, find the symmetric closure of R.
Ans: $\{(1,2),(1,3),(1,4),(2,1),(2,3),(2,4),(3,1),(3,2),(4,1),(4,2)\}$.

54. If $R = \{(x,y) \mid x$ and y are bit strings containing the same number of 0s$\}$, find the equivalence classes of
(a) 1.
(b) 00.
(c) 101.
Ans: (a) All strings that contain no 0s (including the empty string). (b) All strings with exactly two 0s. (c) All strings with exactly one 0.

55. Find the smallest equivalence relation on $\{1,2,3\}$ that contains (1,2) and (2,3).
Ans: $\{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3)\}$.

56. Find the smallest partial order relation on $\{1,2,3\}$ that contains (1,1), (3,2), (1,3).
Ans: $\{(1,1),(2,2),(3,3),(3,2),(1,3),(1,2)\}$.

57. What is the covering relation of the partial ordering $\{(a,b) \mid a \text{ divides } b\}$ on the set $\{1,2,3,4,6,8,12,24\}$?
Ans: $\{(1,2),(1,3),(2,4),(2,6),(3,6),(4,8),(4,12),(6,12),(8,24),(12,24)\}$.

58. What is the covering relation of the partial ordering $\{(a,b) \mid a \text{ divides } b\}$ on the set $\{2,4,6,8,10,12\}$?
Ans: $\{(2,4),(2,6),(2,10),(4,8),(4,12),(6,12)\}$.

59. Find the join of the 3-ary relation

$$\left\{\begin{array}{l}\text{(Wages,MS410,N507),}\\ \text{(Rosen,CS540,N525),}\\ \text{(Michaels,CS518,N504),}\\ \text{(Michaels,MS410,N510)}\end{array}\right\}$$

and the 4-ary relation

$$\left\{\begin{array}{l}\text{(MS410,N507,Monday,6:00),}\\ \text{(MS410,N507,Wednesday,6:00),}\\ \text{(CS540,N525,Monday,7:30),}\\ \text{(CS518,N504,Tuesday,6:00),}\\ \text{(CS518,N504,Thursday,6:00)}\end{array}\right\}$$

with respect to the last two fields of the first relation and the first two fields of the second relation.
Ans:

$$\left\{\begin{array}{l}\text{(Wages,MS410,N507,Monday,6:00),}\\ \text{(Wages,MS410,N507,Wednesday,6:00),}\\ \text{(Rosen,CS540,N525,Monday,7:30),}\\ \text{(Michaels,CS518,N504,Tuesday,6:00),}\\ \text{(Michaels,CS518,N504,Thursday,6:00)}\end{array}\right\}$$

60. Find the transitive closure of R on $\{a,b,c,d\}$ where $R = \{(a,a),(b,a),(b,c),(c,a),(c,c),(c,d),(d,a),(d,c)\}$.
Ans: $\{(a,a),(b,a),(b,c),(b,d),(c,a),(c,c),(c,d),(d,a),(d,c),(d,d)\}$.

61. Which of the following are partitions of $\{1,2,3,\ldots,10\}$?
(a) $\{2,4,6,8\}$, $\{1,3,5,9\}$, $\{7,10\}$.
(b) $\{1,2,4,8\}$, $\{2,5,7,10\}$, $\{3,6,9\}$.
(c) $\{3,8,10\}$, $\{1,2,5,9\}$, $\{4,7,8\}$.
(d) $\{1\},\{2\},\ldots,\{10\}$.
(e) $\{1,2,\ldots,10\}$.
Ans: a, d, e.

62. Suppose R is the relation on N where aRb means that a ends in the same digit in which b ends. Determine whether R is an equivalence relation on N.
Ans: Yes.

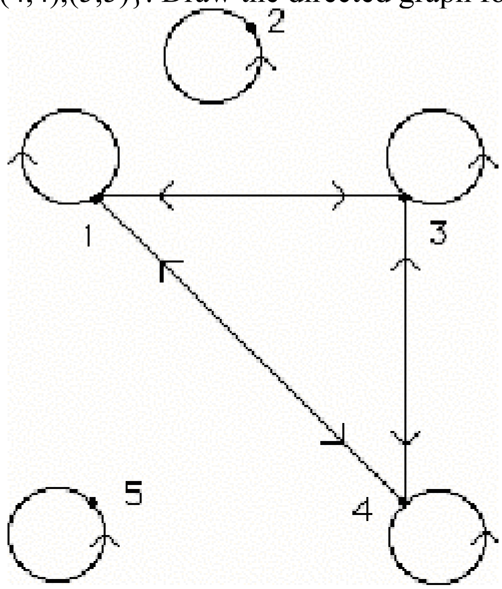
63. Suppose the relation R is defined on the set $\boldsymbol{Z}$ where aRb means that $ab \le 0$. Determine whether R is an equivalence relation on $\boldsymbol{Z}$.
Ans: No (not reflexive, not transitive).

64. Suppose A is the set composed of all ordered pairs of positive integers. Let R be the relation defined on A where $(a,b)R(c,d)$ means that $a + d = b + c$.
(a) Prove that R is an equivalence relation.
(b) Find $[(2,4)]$.
Ans: (a) Reflexive: $a + b = b + a$; Symmetric: if $a + d = b + c$, then $c + b = d + a$; Transitive: if $a + d = b + c$ and $c + f = d + e$, then $a + d - (d + e) = (b + c) - (c + f)$, therefore $a - e = b - f$, or $a + f = b + e$. (b) $[(2,4)] = \{(a,b) \mid b = a + 2\}$.

65. Suppose that R and S are equivalence relations on a set A. Prove that the relation $R \cap S$ is also an equivalence relation on A.
Ans: Reflexive: for all $a \in A$, aRa and aSa; hence for all $a \in A$, $a(R \cap S)a$. Symmetric: suppose $a(R \cap S)b$; then aRb and aSb; by symmetry of R and S, bRa and bSa; therefore $b(R \cap S)a$. Transitive: suppose $a(R \cap S)b$ and $b(R \cap S)c$; then aRb, aSb, bRc, and bSc; by transitivity of R and S, aRc and aSc; therefore $a(R \cap S)c$.

66. Let R be the relation on $A = \{1,2,3,4,5\}$ where $R = \{(1,1),(1,3),(1,4),(2,2),(3,1),(3,3),(3,4),(4,1),(4,3),(4,4),(5,5)\}$. Write the matrix for R.
Ans: $\begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$.

67. Let R be the relation on $A = \{1,2,3,4,5\}$ where $R = \{(1,1),(1,3),(1,4),(2,2),(3,1),(3,3),(3,4),(4,1),$ $(4,3),(4,4),(5,5)\}$. Draw the directed graph for R.

Ans: 

68. Let R be the relation on $A = \{1,2,3,4,5\}$ where $R = \{(1,1),(1,3),(1,4),(2,2),(3,1),(3,3),(3,4),(4,1),$ $(4,3),(4,4),(5,5)\}$. Find the equivalence classes for the partition of A given by R.
Ans: $\{1,3,4\},\{2\},\{5\}$.

Use the following to answer questions 69-71:

In the questions below give an example or else prove that there are none.

69. A relation on $\{a,b,c\}$ that is reflexive and transitive, but not antisymmetric.
Ans: $\{(a,a),(b,b),(c,c),(a,b),(b,a)\}$.

70. A relation on $\{1,2\}$ that is symmetric and transitive, but not reflexive.
Ans: $\{(1,1)\}$.

71. A relation on $\{1,2,3\}$ that is reflexive and transitive, but not symmetric.
Ans: $\{(1,1),(2,2),(3,3),(1,2)\}$.

72. Suppose $|A| = n$. Find the number of binary relations on A.
Ans: 2^{n^2}.

73. Suppose $|A| = n$. Find the number of symmetric binary relations on A.
Ans: $2^{n(n+1)/2}$.

74. Suppose $|A| = n$. Find the number of reflexive, symmetric binary relations on A.
Ans: $2^{n(n-1)/2}$.