Rosen, Discrete Mathematics and Its Applications, 6th edition
Extra Examples

Section 7.1—Recurrence Relations

Extra Examples — *Page references correspond to locations of Extra Examples icons in the textbook.*

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#1. Find a recurrence relation (and initial condition) for each of the following:

(a) the number of strings of length n of letters of the alphabet.

(b) the number of strings of length n of letters of the alphabet, if no adjacent letters can be the same.

(c) the number of strings of length n of letters of the alphabet with no repeated letters.

Solution:

(a) Let a_n equal the number of strings of length n of letters of the alphabet. We can obtain any such string by taking a string s of length $n-1$ and appending a letter to the end of s. This can be done in 26 ways. Therefore, $a_n = 26a_{n-1}$. The initial condition is $a_1 = 26$.

(b) Let b_n equal the number of strings of length n of letters of the alphabet with no adjacent letters identical. Each such string can be obtained from a string s of length $n-1$ by taking s and appending to it a letter that is different from the last letter of s. Because there are 25 letters that can be appended, there are 25 ways to extend s to a string of length n. Therefore, $b_n = 25b_{n-1}$. The initial condition is $b_1 = 26$.

(c) Let c_n equal the number of strings of length n of letters of the alphabet with no repeated letters. Each such string can be obtained from a string s of length $n-1$ by taking s and appending to it a letter that is different from each of the letters of s. Because there are $n-1$ letters used in s, there are $26-(n-1) = 27-n$ letters available to be appended to s. Therefore, $c_n = (27-n)c_{n-1}$. The initial condition is $c_1 = 26$. Note that $c_{27} = (27-27)c_{26} = 0$ because there are only 26 letters in the alphabet. Likewise, the recurrence relation yields $c_{28} = c_{29} = \ldots = 0$.

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#2. Find a recurrence relation for the sequence $1, \dfrac{1}{3}, \dfrac{1}{5}, \dfrac{1}{7}, \dfrac{1}{9}, \ldots$, which is given by the formula $a_n = \dfrac{1}{2n+1}$ for $n = 0, 1, 2, 3, \ldots$.

Solution:

We will try to relate $a_n = \dfrac{1}{2n+1}$ and $a_{n-1} = \dfrac{1}{2(n-1)+1} = \dfrac{1}{2n-1}$ to each other:

$$a_n = \frac{1}{2n+1} = \frac{1}{(2n-1)+2}.$$

But we can rewrite $a_{n-1} = \dfrac{1}{2n-1}$ as $2n-1 = \dfrac{1}{a_{n-1}}$.

Therefore,

$$a_n = \frac{1}{2(n-1)+2} = \frac{1}{\dfrac{1}{a_{n-1}}+2} = \frac{1}{\dfrac{1+2a_{n-1}}{a_{n-1}}} = \frac{a_{n-1}}{1+2a_{n-1}}.$$

Thus, a recurrence relation for the given sequence is

$$a_n = \frac{a_{n-1}}{1 + 2a_{n-1}},$$

with initial condition $a_0 = 1$.

Alternately, we could write $a_n = \dfrac{1}{2n+1} = \dfrac{1}{2n+1} \cdot \dfrac{2n-1}{2n+1} = \dfrac{2n-1}{2n+1} \cdot \dfrac{1}{2n+1} = \dfrac{2n-1}{2n+1} \cdot a_{n-1}$, obtaining a second recurrence relation

$$a_n = \frac{2n-1}{2n+1} \cdot a_{n-1},$$

with initial condition $a_0 = 1$.

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#3. Suppose $b_n = 2b_{n-1} + n - 2^n$ and $b_0 = 5$.

(a) Find b_{n-1} in terms of b_{n-2}.
(b) Find b_n in terms of b_{n-2}.
(c) Find b_n in terms of b_{n-3}.
(d) Use parts (b) and (c) to conjecture a formula for b_n.

Solution:
(a) The recurrence relation for b_n doubles the previous term (which is $2b_{n-1}$), adds the subscript number of b_n (which is n), and subtracts 2 raised to the power of the subscript of b_n (which is 2^n).

The term b_{n-1} is obtained in the same way: double the previous term (which is $2b_{n-2}$), add the subscript number of b_{n-1} (which is $n-1$), and subtract 2 raised to the power of the subscript of b_{n-1} (which is 2^{n-1}). Therefore $b_{n-1} = 2b_{n-2} + (n-1) - 2^{n-1}$.

(b) To obtain b_n in terms of b_{n-2}, we first use the recurrence relation to obtain b_n in terms of b_{n-1} and then use part (a) to obtain b_{n-1} in terms of b_{n-2}:

$$
\begin{aligned}
b_n &= 2b_{n-1} + n - 2^n \\
&= 2[2b_{n-2} + (n-1) - 2^{n-1}] + n - 2^n \\
&= 2^2 b_{n-2} + (n-1) + n - 2^{n-1} - 2^n.
\end{aligned}
$$

(c) To obtain b_n in terms of b_{n-3}, we can use the recurrence equation for b_{n-2} and part (b). The recurrence relation for b_{n-2} is $b_{n-2} = 2b_{n-3} + (n-2) - 2^{n-2}$. Substituting for b_{n-2} into the result in part (b), we have

$$
\begin{aligned}
b_n &= 2^2 b_{n-2} + (n-1) + n - 2^{n-1} - 2^n \\
&= 2^2[2b_{n-3} + (n-2) - 2^{n-2}] + (n-1) + n - 2^{n-1} - 2^n \\
&= 2^3 b_{n-3} + (n-2) + (n-1) + n - 2^{n-2} - 2^{n-1} - 2^n.
\end{aligned}
$$

(d) Mentally continuing the pattern in part (c), it seems reasonable to guess that the first term becomes $2^n b_0$, the sum $(n-2) + (n-1) + n$ becomes $1 + 2 + 3 + \cdots + (n-2) + (n-1) + n$, and the sum of the powers of 2 being subtracted becomes $-2^1 - 2^2 - 2^3 - \cdots - 2^{n-1} - 2^{n-1} - 2^n$. Thus, it is reasonable to conjecture that

$$
\begin{aligned}
b_n &= 2^n b_0 + [1 + 2 + 3 + \cdots + (n-2) + (n-1) + n] - [2^1 + 2^2 + 2^3 + \cdots + 2^{n-1} + 2^{n-1} + 2^n] \\
&= 5 \cdot 2^n + \frac{n(n+1)}{2} - 2^{n+1},
\end{aligned}
$$

where summation formulas were used at the last step to replace $1 + 2 + 3 + \cdots + (n-2) + (n-1) + n$ and $2^1 + 2^2 + 2^3 + \cdots + 2^{n-1} + 2^{n-1} + 2^n$.

Note: You can check that this is correct by substituting the formula for b_n and b_{n-1} into the given recurrence relation and showing that both sides of the recurrence relation are equal.

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#4. Solve: $a_n = 3a_{n-1} + 1$, $a_0 = 4$, by substituting for a_{n-1}, then a_{n-2}, etc.

Solution:
Beginning with $a_n = 3a_{n-1} + 1$ and substituting for a_{n-1}, then a_{n-2}, then a_{n-3}, etc., yields:

$$
\begin{aligned}
a_n &= 3a_{n-1} + 1 \\
&= 3(3a_{n-2} + 1) + 1 \\
&= 3^2 a_{n-2} + 3 \cdot 1 + 1 \\
&= 3^2(3a_{n-3} + 1) + 3 \cdot 1 + 1 \\
&= 3^3 a_{n-3} + 3^2 \cdot 1 + 3 \cdot 1 + 1 \\
&\ \ \vdots \\
&= 3^n a_0 + (3^{n-1} + 3^{n-2} + \cdots + 3^2 + 3 + 1) \\
&= 4 \cdot 3^n + \frac{3^n - 1}{2} \\
&= \frac{9}{2} \cdot 3^n - \frac{1}{2} \\
&= \frac{3^{n+2}}{2} - \frac{1}{2}.
\end{aligned}
$$

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#5. Find a formula for the recurrence relation $a_n = 2a_{n-1} + 2^n$, $a_0 = 1$, using a recursive method.

Solution:

$$
\begin{aligned}
a_n &= 2a_{n-1} + 2^n \\
&= 2(2a_{n-2} + 2^{n-1}) + 2^n = (2^2 a_{n-2} + 2 \cdot 2^{n-1}) + 2^n = (2^2 a_{n-2} + 2^n) + 2^n = 2^2 a_{n-2} + 2 \cdot 2^n \\
&= 2^2(2a_{n-3} + 2^{n-2}) + 2 \cdot 2^n = (2^3 a_{n-3} + 2^2 \cdot 2^{n-2}) + 2 \cdot 2^n = (2^3 a_{n-3} + 2^n) + 2 \cdot 2^n = 2^3 a_{n-2} + 3 \cdot 2^n.
\end{aligned}
$$

At this stage it would be reasonable to guess that if we continue this process we will obtain:

$$a_n = 2^n a_0 + n \cdot 2^n = 2^n \cdot 1 + n \cdot 2^n = (n+1)2^n.$$

Therefore, $a_n = (n+1)2^n$ is a formula for the given sequence.

We can verify that this formula is correct by substituting $a_n = (n+1)2^n$ and $a_{n-1} = n2^{n-1}$ into the original recurrence relation and checking that an equality results.:

$$2a_{n-1} + 2^n = 2(n2^{n-1}) + 2^n = n2^n + 2^n = (n+1)2^n = a_n.$$

#6. You begin with \$1000. You invest it at 5% compounded annually, but at the end of each year you withdraw \$100 immediately after the interest is paid.

    (a) Set up a recurrence relation and initial condition for the amount you have after n years.

    (b) How much is left in the account after you have withdrawn \$100 at the end of the third year?

    (c) Find a formula for a_n.

    (d) Use the formula to determine how long it takes before the last withdrawal reduces the balance in the account to \$0.

Solution:

(a) For $n > 0$, let a_n be the amount in the account at the end of year n; i.e., just after the interest has been added to the account and the \$100 has been withdrawn. Then a_n is equal to the amount from the previous year (a_{n-1}) plus the interest earned ($0.05a_{n-1}$) minus the \$100 withdrawal. That is,

$$a_n \;=\; a_{n-1} + 0.05a_{n-1} - 100 \;=\; 1.05a_{n-1} - 100, \;\; \text{if } n > 0, \;\; a_0 = 1000.$$

(b) Using the recurrence relation yields

$$a_1 = 1050 - 100 = 950$$
$$a_2 = 1.05a_1 - 100 = 1.05(950) - 100 = 997.50 - 100 = 897.50$$
$$a_3 = 1.05a_2 - 100 = 1.05(897.50) - 100 = 942.38 - 100 = 842.38$$

Thus, the answer is \$842.38.

(c) To develop a formula, note that

$$a_n = 1.05a_{n-1} - 100,$$
$$a_{n-1} = 1.05a_{n-2} - 100,$$
$$a_{n-2} = 1.05a_{n-3} - 100,$$
$$\vdots$$

Therefore

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$$\begin{aligned}
a_n &= 1.05a_{n-1} - 100 \\
&= 1.05(1.05a_{n-2} - 100) - 100 \\
&= 1.05^2 a_{n-2} - (1.05 \cdot 100) - 100 \\
&= 1.05^2(1.05a_{n-3} - 100) - (1.05 \cdot 100) - 100 \\
&= 1.05^3 a_{n-3} - (1.05^2 \cdot 100) - (1.05 \cdot 100) - 100 \\
&\;\;\vdots \\
&= 1.05^n a_0 - (1.05^{n-1} \cdot 100) - (1.05^{n-2} \cdot 100) - \cdots - (1.05^2 \cdot 100) - (1.05 \cdot 100) - 100 \\
&= 1.05^n \cdot 1000 - 100(1.05^{n-1} + 1.05^{n-2} + \cdots + 1.05^2 + 1.05 + 1) \\
&= 1.05^n \cdot 1000 - 100 \cdot \frac{1.05^n - 1}{1.05 - 1} \\
&= 1.05^n \cdot 1000 - 2000(1.05^n - 1) \\
&= 2000 - 1.05^n \cdot 1000
\end{aligned}$$

and hence a formula for a_n is $a_n = 2000 - 1.05^n \cdot 1000$.

(d) Using various values of n in the formula for a_n yields $a_{14} = 20.07$ and $a_{15} = -78.93$. Hence, at the end of the 15th year the balance will be $1.05 \cdot 20.07 = 21.07$ before a withdrawal is made; if this amount is withdrawn, the balance will become \$0. (Alternately, we could solve the equation $2000 - 1.05^n \cdot 1000 = 0$ for n and obtain $\lfloor n \rfloor$ as the solution.)

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#7. Find a recurrence relation for the number of strings of letters of the ordinary alphabet that do not have adjacent vowels.

Solution:
Let us call a string of letters of the alphabet "good" if it has no adjacent vowels. Let a_n be the number of strings of length n of letters of the alphabet that do not have adjacent vowels.

The set of all good strings of length n is the union of the following two disjoint sets —

A: the set of good strings of length n that end with a consonant, and
B: the set of good strings of length n that end with a vowel.

Each string in A can be obtained from a good string of length $n-1$ by adding any consonant at the end of the string. Thus, $|A| = 21 \cdot a_{n-1}$.

Each string in B ends with a vowel. In this case we know that that second letter from the end must be a consonant (otherwise the string would have adjacent vowels). Thus, each string in B is a good string of length $n-2$ followed by a consonant (in the second last position) and a vowel at the end. Therefore, $|B| = 5 \cdot 21 \cdot a_{n-2}$.

Because A and B are disjoint, the set of all good strings of length n is their sum, $|A| + |B|$. That is,

$$a_n = 21\,a_{n-1} + 105\,a_{n-2}.$$

The two initial conditions are obtained by counting the number of good strings of lengths 1 and 2: $a_1 = 26$ because any string of one of the 26 letters of the alphabet cannot have adjacent vowels; $a_2 = 26^2 - 5^2 = 651$ (we take all strings of length 2 and subtract those with two vowels).

#8. You have two distinct parallel lines L_1 and L_2. You keep adding additional lines, $L_3, L_4, \ldots$, with none parallel to L_1 or L_2 or to each other, and no three passing through the same point.

(a) Find a recurrence relation and initial condition(s) for r_n, which is defined to be the number of regions into which the plane is divided by the lines $L_1, L_2, \ldots, L_n$.

(b) Find a formula for the number of regions into which the plane is divided by $L_1, L_2, \ldots, L_n$.

Solution:
(a) The recurrence relation is $r_n = r_{n-1} + n$ $(n > 2)$, with initial condition $r_2 = 3$. To see this, note that line L_n must cut each of the previous $n - 1$ lines in exactly one point. This in effect divides L_n into n segments, each of which divides an existing region into two parts. Therefore, $r_n = r_{n-1} + n$.

(b) Proceeding inductively:

$$r_2 = 3,$$
$$r_3 = r_2 + 3 = 3 + 3,$$
$$r_4 = r_3 + 4 = 3 + 3 + 4,$$
$$r_5 = r_4 + 5 = 3 + 3 + 4 + 5,$$
$$r_6 = r_5 + 6 = 3 + 3 + 4 + 5 + 6.$$

This suggests that

$$r_n = 3 + (3 + 4 + \cdots + n)$$
$$= (1 + 2) + (3 + 4 + \cdots + n)$$
$$= 1 + 2 + 3 + \cdots + n$$
$$= \frac{n(n+1)}{2}.$$

To verify that this guess is correct, take $r_n = r_{n-1} + n$ and substitute $\dfrac{n(n+1)}{2}$ for r_n and $\dfrac{(n-1)n}{2}$ for r_{n-1}, obtaining $\dfrac{n(n+1)}{2} = \dfrac{(n-1)n}{2} + n$, which is true because the right side simplifies to give the left side: $\dfrac{(n-1)n}{2} + n = \dfrac{(n-1)n + 2n}{2} = \dfrac{n^2 + n}{2} = \dfrac{n(n+1)}{2}$.

#9. This is a variation on Fibonacci's rabbit sequence. We begin with one pair of newborn rabbits. Once a pair is two months old, the pair has two pairs of offspring, and continues to have two pairs of offspring each month thereafter. Give a recurrence relation and initial condition(s) for the sequence f_n, where f_n is equal to the number of pairs of rabbits alive at the end of the nth month (after the offspring are born). Assume that the rabbits never die during the period being considered.

Solution:
The number of pairs at the end of n months (f_n) is equal to the number of pairs alive one month earlier (f_{n-1}) plus the number of newborn pairs. But each pair of rabbits alive two months earlier gives birth to two pairs of newborn rabbits. Therefore, there are $2f_{n-2}$ pairs of newborn rabbits. Hence

$$f_n = f_{n-1} + 2f_{n-2}, \quad f(1) = 1, \ f(2) = 3.$$

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#10. Here is another variation on Fibonacci's rabbit sequence. We begin with one pair of newborn rabbits. At the end of each month a new pair of newborn rabbits is added to the population. Once any pair is two months old, the pair has one pair of offspring and continues to have one pair of offspring each month thereafter. Give a recurrence relation and initial condition(s) for the sequence f_n, where f_n is equal to the number of pairs of rabbits alive at the end of the nth month (after the rabbits have given birth and the newborn pair has been introduced). Assume that the rabbits never die during the period being considered.

Solution:
The number of pairs at the end of n months (f_n) is equal to 1 (the newborn pair added) plus the number of pairs alive one month earlier (f_{n-1}) plus the number of newborn pairs (which is equal to the number of pairs alive two months earlier, f_{n-2}. Thus

$$f_n = 1 + f_{n-1} + f_{n-2}, \quad f(1) = 2, \ f(2) = 4.$$

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#11. Here is a third variation on Fibonacci's rabbit sequence. We begin with one pair of newborn rabbits. Once the pair is three months old, the pair has one pair of offspring, and continues to have one pair of offspring every other month thereafter. Give a recurrence relation and initial condition(s) for the sequence f_n, where f_n is equal to the number of pairs of rabbits alive at the end of the nth month (just after any offspring are born). Assume that the rabbits never die during the period being considered.

Solution:
The number of pairs alive at the end of n months (f_n) is equal to the number of pairs alive one month earlier (f_{n-1}) plus the number of pairs of newborn rabbits. To determine the number of newborn pairs, we need to know the number of pairs born three months earlier, five months earlier, seven months earlier, etc. But, for example, the number of pairs born three months before month n is equal to $f_{n-3} - f_{n-4}$ and the number of pairs born five months before month n is equal to $f_{n-5} - f_{n-6}$.
Thus,
$$f_n = f_{n-1} + \text{number of pairs born in month } n$$
$$= f_{n-1} + (f_{n-3} - f_{n-4}) + (f_{n-5} - f_{n-6}) + (f_{n-7} - f_{n-8}) + \cdots$$

where the sum continues as long as the subscripts are positive. Similarly,

$$f_{n-1} = f_{n-2} + \text{number of pairs born in month } n-1$$
$$= f_{n-2} + (f_{n-4} - f_{n-5}) + (f_{n-6} - f_{n-7}) + (f_{n-8} - f_{n-9}) + \cdots.$$

Substitute f_{n-1} from the second equation into the first equation. Almost all terms cancel and we obtain

$$f_n = f_{n-2} + f_{n-3},$$

with $f(1) = 1$, $f(2) = 1$, $f(3) = 2$.

#12. (Problem A1 from the 1990 William Lowell Putnam Mathematics Competition)

Here are the first ten terms of an infinite sequence:

$$2, 3, 6, 14, 40, 152, 784, 5168, 40567, 363392.$$

(a) Find a formula for an infinite sequence $a_0, a_1, a_2, a_3, \ldots$ such that the first ten terms of the sequence are the ones given here. (Hint: consider the sum of two rapidly increasing sequences.)

(b) Show that the sequence in (a) satisfies the recurrence relation

$$a_n = (n+4)a_{n-1} - 4na_{n-2} + (4n-8)a_{n-3}.$$

Solution:
(a) The sequence increases rapidly, which suggests the possibility that the formula may involve an exponential function c^n. If we look for 2^n in each term, we find the the first few terms are:

$$2^0 + 1, \ 2^1 + 1, \ 2^2 + 2, \ 2^3 + 6, \ 2^4 + 24, \ 2^5 + 120, \ 2^6 + 720, \ 2^7 + 5040 \,.$$

The second term in each sum is a factorial, yielding

$$2^0 + 0!, \ 2^1 + 1!, \ 2^2 + 2!, \ 2^3 + 3!, \ 2^4 + 4!, \ 2^5 + 5!, \ 2^6 + 6!, \ 2^7 + 7! \,.$$

Thus, $a_n = 2^n + n!$ is one such formula.

(b) We need to show that $a_n = (n+4)a_{n-1} - 4na_{n-2} + (4n-8)a_{n-3}$ is satisfied by the sequence $a_n = 2^n + n!$. That is,

$$2^n + n! = (n+4)[\, 2^{n-1} + (n-1)!\,] - 4n[\, 2^{n-2} + (n-2)!\,] + (4n-8)[\, 2^{n-3} + (n-3)!\,].$$

The right side can be simplified as follows:

$$
\begin{aligned}
(n+4)&[\, 2^{n-1} + (n-1)!\,] - 4n[\, 2^{n-2} + (n-2)!\,] + (4n-8)[\, 2^{n-3} + (n-3)!\,] \\
&= n2^{n-1} + 4 \cdot 2^{n-1} - 4n2^{n-2} + 4n2^{n-3} - 8 \cdot 2^{n-3} + \\
&\qquad n(n-1)! + 4(n-1)! - 4n(n-2)! + (4n-8)(n-3)! \\
&= n2^{n-1} + 2^{n+1} - n2^n + n2^{n-1} - 2^n + n! + 4(n-1)! - 4n(n-2)! + (4n-8)(n-3)! \\
&= n2^n + 2^{n+1} - n2^n - 2^n + n! + 4(n-1)! - 4n(n-2)! + (4n-8)(n-3)! \\
&= 2^{n+1} - 2^n + n! + 4(n-1)! - 4n(n-2)! + (4n-8)(n-3)! \\
&= 2^n + n! + 4(n-1)! - 4n(n-2)! + (4n-8)(n-3)! \\
&= 2^n + n! + 4(n-1)! - 4n(n-2)! + 4(n-2)! \\
&= 2^n + n! + 4(n-1)! + (4-4n)(n-2)! \\
&= 2^n + n! + 4(n-1)! - 4(n-1)(n-2)! \\
&= 2^n + n! + 4(n-1)! - 4(n-1)! \\
&= 2^n + n!.
\end{aligned}
$$

#13. Suppose a chess king is placed on the lower left square of an $m \times n$ chessboard (that is, a rectangular board with m rows and n columns). Let $M(m, n)$ be equal to the number of paths that a king can use moving from the lower left corner to the upper right corner of an $m \times n$ board, with the restriction that each move is either up, to the right, or diagonally up and to the right.

    (a) Find a recurrence relation and initial condition(s) for $M(m, n)$.

(b) Find the number of ways in which the king can move from the lower left square to the upper right square on a 5×5 chessboard.

Solution:

(a) In order for the king to reach the upper right square, the king's last move must be from one of the following three squares: the square immediately below the corner square, the square immediately to the left of the corner square, or the square diagonally down and to the left of the corner square. The number of ways in which the king could have arrived at each of these three squares is $M(m, n-1)$, $M(m-1, n)$, and $M(m-1, n-1)$, respectively (assuming that $m > 1$ and $n > 1$). Therefore,

$$M(m, n) = M(m, n-1) + M(m-1, n) + M(m-1, n-1),$$

with initial conditions $M(1, 1) = M(2, 1) = M(1, 2) = 1$.

(b) First note that $M(k, 1) = M(1, k) = 1$ for all $k > 1$ and $M(j, k) = M(k, j)$ for all j and k (by symmetry).

The following steps use the recurrence relation to find $M(5, 5)$:

$$M(2, 2) = M(2, 1) + M(1, 2) + M(1, 1) = 1 + 1 + 1 = 3$$
$$M(2, 3) = M(3, 2) = M(1, 3) + M(2, 2) + M(1, 2) = 1 + 3 + 1 = 5$$
$$M(3, 3) = M(3, 2) + M(2, 3) + M(2, 2) = 5 + 5 + 3 = 13$$
$$M(2, 4) = M(4, 2) = M(1, 4) + M(2, 3) + M(1, 3) = 1 + 5 + 1 = 7$$
$$M(2, 5) = M(5, 2) = M(1, 5) + M(2, 4) + M(1, 4) = 1 + 7 + 1 = 9$$
$$M(3, 4) = M(4, 3) = M(3, 3) + M(2, 4) + M(3, 2) = 13 + 7 + 5 = 25$$
$$M(3, 5) = M(5, 3) = M(4, 3) + M(5, 2) + M(2, 4) = 25 + 9 + 7 = 41$$
$$M(4, 4) = M(4, 3) + M(3, 4) + M(3, 3) = 25 + 25 + 13 = 63$$
$$M(4, 5) = M(5, 4) = M(4, 4) + M(3, 5) + M(3, 4) = 63 + 41 + 25 = 129$$
$$M(5, 5) = M(5, 4) + M(4, 5) + M(4, 4) = 129 + 129 + 63 = 321.$$

Therefore, $M(5, 5) = 321$.

This is shown in the following table, where, beginning from the lower left corner, each square has as its value the sum of the numbers in the squares directly below, to the left, and diagonally below on the left.

1	9	41	129	321
1	7	25	63	129
1	5	13	25	41
1	3	5	7	9
1	1	1	1	1