

Rosen, Discrete Mathematics and Its Applications, 6th edition
Extra Examples

Section 6.4—Expected Value and Variance



— Page references correspond to locations of Extra Examples icons in the textbook.

p.427, icon at Example 2

#1. You roll a die. If a 5 or 6 shows, you win three points. If a 1, 2, 3, or 4 shows, you lose one point. Set up a random variable X that measures the number of points you win, and find the expected value of X .

Solution:

$$X(1) = X(2) = X(3) = X(4) = -1, \quad X(5) = X(6) = 3.$$

$$E(X) = (-1) \cdot \frac{4}{6} + 3 \cdot \frac{2}{6} = \frac{1}{3}.$$

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#2. You roll a die four times. Let X be the random variable that counts the sum of the numbers rolled. Find $E(X)$.

Solution:

Let X = the sum of the four numbers rolled. For $i = 1, 2, 3, 4$, let X_i = the number rolled on the i th toss. Then $E(X) = E(X_1) + E(X_2) + E(X_3) + E(X_4)$. But for each i we have $E(X_i) = 7/2$. Therefore $E(X) = 7/2 + 7/2 + 7/2 + 7/2 = 14$.

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#3. A 6-sided die has its sides labeled 1, 1, 2, 2, 2, 3. If you roll the die once, what is the expected value of the number that shows?

Solution:

Here $S = \{1, 2, 3\}$ and $p(1) = 2/6$, $p(2) = 3/6$ and $p(3) = 1/6$. For $s \in S$, we let $X(s)$ = the number rolled. Therefore

$$E(X) = \sum_{s \in S} p(s)X(s) = \frac{2}{6} \cdot 1 + \frac{3}{6} \cdot 2 + \frac{1}{6} \cdot 3 = \frac{11}{6}.$$

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#4. A multiple-choice exam consists of a series of questions, each with four possible responses. If you

answer a question correctly, you receive 1 point. If you answer a question incorrectly, you lose $1/3$ point. If you do not answer a question, you neither lose nor gain any points. What is the expected value of the number of points you receive on a question

(a) if you randomly choose an answer?

(b) if you can eliminate one of the four choices and randomly choose one of the other three choices?

Solution:

(a) Let X be the random variable that measures the number of points you receive on the question. Then

$$E(X) = \frac{1}{4} \cdot 1 + \frac{3}{4} \cdot \left(-\frac{1}{3}\right) = 0.$$

This says that guessing will neither raise nor lower your test score.

(b) Let X be the random variable that measures the number of points you receive on the question. Then

$$E(X) = \frac{1}{3} \cdot 1 + \frac{2}{3} \cdot \left(-\frac{1}{3}\right) = \frac{1}{9}.$$

Thus, it is to your advantage to guess if you can eliminate one of the responses.

p.434, icon at Example 11

#1. A fair coin is flipped three times. Let X be the random variable that counts the number of heads and let Y be the random variable that counts the number of tails. Then

$$\begin{aligned} p(X=0) &= \frac{1}{8}, p(X=1) = \frac{3}{8}, p(X=2) = \frac{3}{8}, p(X=3) = \frac{1}{8} \\ p(Y=0) &= \frac{1}{8}, p(Y=1) = \frac{3}{8}, p(Y=2) = \frac{3}{8}, p(Y=3) = \frac{1}{8}. \end{aligned}$$

Determine whether X and Y are independent random variables.

Solution:

It is quickly seen that the random variables are not independent. For example,

$$p(X=0 \text{ and } Y=0) = 0$$

because it is impossible to have 0 heads and 0 tails in three tosses of the coin. But

$$p(X=0) \cdot p(Y=0) = \frac{1}{8} \cdot \frac{1}{8} \neq 0.$$

p.434, icon at Example 11

#2. A coin is flipped and a die rolled. The sample space

$$S = \{(H, 1), (H, 2), (H, 3), (H, 4), (H, 5), (H, 6), (T, 1), (T, 2), (T, 3), (T, 4), (T, 5), (T, 6)\}.$$

Let X be the random variable defined as follows: $X = 1$ if H is obtained on the coin and 0 if T is obtained on the coin. Let Y be the random variable that counts the number of spots showing on the die. Determine whether X and Y are independent random variables.

Solution:

We need to check that

$$p(X(s)=i \text{ and } Y(s)=j) = (p(X(s)=i) \cdot (p(Y(s)=j)$$

for $i = 0, 1$ and $j = 1, 2, 3, 4, 5, 6$.

If $i = 0$, we need to check

$$p(X=0 \text{ and } Y=j) = p(X=0) \cdot p(Y=j)$$

for each j . But this is true because the left side is equal to $\frac{1}{12}$ and the right side is equal to $\frac{1}{2} \cdot \frac{1}{6}$. If $i = 1$, we need to check

$$p(X=1 \text{ and } Y=j) = p(X=1) \cdot p(Y=j)$$

for each j . But this is true because the left side is equal to $\frac{1}{12}$ and the right side is equal to $\frac{1}{2} \cdot \frac{1}{6}$.

p.436, icon at Example 14

#1. Two tetrahedral dice are rolled. (A tetrahedral die is a die with four faces, which are numbered 1, 2, 3, 4. Let $X(i, j) = i + j$, where the first die shows i and the second die shows j . Find $E(X)$ and $V(X)$.

Solution:

The sample space S consists of 16 outcomes: $S = \{(i, j) \mid i, j = 1, 2, 3, 4\}$. We have the following probabilities:

$$p(2) = 1/16, \quad p(3) = 2/16, \quad p(4) = 3/16, \quad p(5) = 4/16, \quad p(6) = 3/16, \quad p(7) = 2/16, \quad p(8) = 1/16.$$

Therefore,

$$E(X) = 2 \cdot \frac{1}{16} + 3 \cdot \frac{2}{16} + 4 \cdot \frac{3}{16} + 5 \cdot \frac{4}{16} + 6 \cdot \frac{3}{16} + 7 \cdot \frac{2}{16} + 8 \cdot \frac{1}{16} = \frac{80}{16} = 5.$$

To find $V(X)$, we use the equality $V(X) = E(X^2) - E(X)^2$:

$$\begin{aligned} V(X) &= E(X^2) - E(X)^2 \\ &= \left(4 \cdot \frac{1}{16} + 9 \cdot \frac{2}{16} + 16 \cdot \frac{3}{16} + 25 \cdot \frac{4}{16} + 36 \cdot \frac{3}{16} + 49 \cdot \frac{2}{16} + 64 \cdot \frac{1}{16} \right) - 5^2 \\ &= \frac{1}{16}(440) - 5^2 = 27.5 - 25 = 2.5. \end{aligned}$$