A Letter to vua Le

Ngày 1 tháng 11 năm 2010

The honorable vua Le,

The students of the math honors program at the Vietnam University of Science in Hanoi are humbly submitting their solution to your fake coin problem.
Actually we did even more, we offer a solution you can use in the future for identifying a fake coin among 12 coins.
It is a very general solution, the same weighing scheme will identify the fake coin through the same sequence of three weighings.
We also can prove that if you have more than 12 coins you will need more than three weighings.

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Here are the details:

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5. There are 24 possible inputs: coin number i is Heavy, coin number i is Light, $i = 1, 2, \ldots 12$.
6. We can associate with every coin a pair of complementary triples.
7. For instance, if coin number i is heavy, if we place it on the left side of the scale, the left side will be heavy and if we place it on the right side the right side will be heavy.

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You can see our solution in the next slide

The universal algorithm for identifying the fake coin

Coin	Heavy 1	Heavy 2	Heavy 3	Light 1	Light 2	Light 3
1	1	0	1	-1	0	-1
2	0	0	-1	0	0	1
3	-1	1	1	1	-1	-1
4	1	-1	0	-1	1	0
5	0	-1	1	0	1	-1
6	-1	-1	-1	1	1	1
7	1	1	-1	-1	-1	1
8	-1	0	1	1	0	-1
9	-1	0	0	1	0	0
10	1	1	0	-1	-1	0
11	0	-1	-1	0	1	1
12	0	1	0	0	-1	0
13						

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3. So ± 1 appears in the remaining 18 triples in the first coordinate.
4. They will have to be divided equally among the two tables.
5. But this means that in the first column we cannot have an equal number of $1's$ and $-1's$.