An Application of Number Theory, the RSA Cryptosystem

Ngày 10 tháng 12 năm 2010

Securing Transactions

Question

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How can we accomplish this?

This is exactly how business transactions are being conducted on the internet today, except that the boxes are virtual boxes. Closing a box is accomplished by encrypting the message. So while the message is traveling on the internet, being exposed to hackers and others, it is encrypted using a "key". Only the owner of the key knows how to open the box and retrieve its content.

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- 3 Can messages be "signed"?.

RSA Public Key System

Discussion

Until the mid-70's encryptions were done using private keys. Two persons or institutions that needed to establish secure communications shared a private key they used for encryption.

The system worked quite well, except for one problem: how to share keys.

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In 1976 Rivest, Shamir and Adelman proposed the public key cryptosystem: RSA.

Each key consisted of two parts, a public part used for encryption and a private part used for decryption.

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- Observation: The key owner first finds $d = e^{-1} \mod (p-1)(q-1)$.
- **5** To retrieve M, the owner of the key (K, e) calculates: $EM^d \mod K = M$.



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- **3** T3: If p(x) is a polynomial with coefficients in GF(p) and $f(\beta) = 0$ then $p(x) = (x \beta)p_1(x)$ where $p_1(x)$ is a polynomial with coefficients in GF(p).

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- 4 T4: A finite field has p^n (p prime) elements and is unique upto isomorphism.



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Since GF(p) is a field for any

$$a \in GF(p) \quad \{a, 2a, 3a, \dots, (p-1)a\} = \{1, 2, 3, \dots, p-1\}.$$

So
$$a \cdot 2a \cdot 3a \cdots (p-1)a = 1 \cdot 2 \cdot 3 \cdots (p-1)$$

$$a^{p-1} \cdot \prod_{i=1}^{p-1} i = \prod_{i=1}^{p-1} i \mod p \Rightarrow a^{p-1} = 1 \mod p$$

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Given an integer n, if $a^{n-1} \mod n \neq 1$, a < n then n is composite.

But what if $a^{n-1} = 1$?

For example: $2^{340} \mod 341 = 1$ but $341 = 11 \cdot 31$ $a^{1728} \mod 1729 = 1 \ \forall a \text{ relatively prime to } 1729$.

Question

Can you prove it? It is not difficult, give it a try.

Question (Challenge)

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 2^{Key-1} mod $Key \neq 1$ Can we find its prime factors?

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Are there any other ways to factor integers besides trying the GCD of the integer with smaller integers?

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Theorem (Wallis)

p is prime if and only if $(p-1)! \mod p = -1$.

Let N be an integer. By Fermat's theorem if N is prime then $a^{N-1} \mod N = 1$. This calculation can be executed very fast on integers with a few thousand digits. This means that if for some 1 < a < N-1, $a^{N-1} \mod N \ne 1$ then N is definitely not a prime number.

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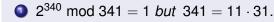
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- **3** On the other hand, if gcd(a, 1729) = 1 then $a^{1728} mod 1729 = 1$.
- Since $\phi(1729) = 1729(1 \frac{1}{7})(1 \frac{1}{13})(1 \frac{1}{19}) = 1296$ if we select a randomly we do not have a good chance to find an integer that will prove that 1729 is not a prime number .

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Theorem (Miller-Rabin Test)

Let N be an integer, $N-1=2^m \cdot (2k+1)$.

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If p is prime then by Fermat's theorem $a^{p-1} \mod p = 1$. So $a^{(p-1)/2} \mod p = \sqrt{1} = \pm 1$.

If $a^{(p-1)/2} \mod p = -1$ then the test stops. In other words, it will not say that p is composite.

If $a^{(p-1)/2} \mod p = 1$ then we calculate $a^{(p-1)/4} \mod p = \pm 1$ We continue until we reach $a^{2k+1} \mod p$

We skip the important part of the proof. They proved that if N is composite then more than 50% of the integers a < N will be composite-witnesses. In other words, to test whether an integer p is prime, we randomly select say 100 integers a < p and apply to them the Miller-Rabin test. If the test fails, we assume that p is a prime number. The probabilty that we made a mistake, that is decided that p is prime while in fact it is not, is less then $\left(\frac{1}{2}\right)^{100}$.

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- 3^{9746347772160/4} mod 9746347772161 = 4485448662696 proving that c is composite.

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- The Miller-Rabin test is commonly used for this purpose.
- There are also efficient algorithms to manufacture "certified" primes.
- Are all large primes safe?

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We shall start by learning how to find $\sqrt{n} \mod p$.

\sqrt{n} mod p

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- 3 *n* is a quadratic residue mod p if and only if $n^{\frac{p-1}{2}} \mod p = 1$.
- **3 Claim:** If n is a quadratice residue mod p then we can find an integer β such that $n^{2m+1}\beta^{2s}$ mod p=1

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- ② n is a quadratic residue mod p if and only if $n = \alpha^{2k} \mod p$.
- 3 *n* is a quadratic residue mod p if and only if $n^{\frac{p-1}{2}} \mod p = 1$.
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- 8 Repeat the same process ubtil you reach $n^{2k+1}\beta^{2s} \mod p = 1$



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 $234428^2 \mod 337639 = 71.$

So $\sqrt{71} \mod 337639 = 234428$.

• **1** p = 2701297, $p - 1 = 2^4 \cdot 3^3 \cdot 13^2 \cdot 37$.

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- **9** Verify: $2309891^2 \mod p = 75$.
- **10** We can verify it in yet another way. $75 = 25 \cdot 3$. This means that $\sqrt{3} \mod p = 2309891/5$. Indeed $5^{-1} \mod p = 1080519$ and $1080519 \cdot 2309891 \mod p = 1542497$ and $1542497^2 \mod p = 3$.