Discrete Mathematics and Applications

Moshe Rosenfeld

Hanoi 2010
moishe@u.washington.edu

1 Assignment No. 5: Integers, modular arithmetic, sets

Due: Friday, 15 Oct.

Please submit your answer in a neat, readable properly organized format. This assignment includes some forward preparations simple exercises in number theory.

1. Simple calculation, do by hand or a simple hand held calculator.

   a. Find integers m, n such that $GCD(6743, 273) = 6743m + 273n$

   b. Calculate 2^{-1} mod 751. Verify your answer.

   c. Calculate 200^{-1} mod 751. Verify your answer.

   d. Does $sqrt(2)$ mod 751 exist? If yes, find it and verify your answer.

   e. Let $m = 10\sum_{i=1}^{10} d_i$ where d_i are the digits of your phone number.
   Verify that : $m = (385 \times m \bmod 3 + 231 \times m \bmod 5 + 330 \times m \bmod 7 + 210 \times m \bmod 11) \bmod 1155$
   Explain why this is true. (Some of us know that this is the Chinese Remainder Theorem, I am looking for a direct explanation that will convince a person who does not know the Chienese Remainder Theorem).

2. Hint: $385 \bmod 3 = 1, 385 = 5 \cdot 7 \cdot 11$.

3. Clearly, $\sqrt(49) \bmod 3869 = 7$. Find other integers $8 \leq x \leq 3868$ such that $x = \sqrt(49) \bmod 3869$.

4. Prove that there are infinitely many primes $p = 3 \bmod 4$.

5. Hint: assume that there is a finite number. Consider the number $4q_1q_2 \dots q_n - 1$.

6. For any given positive integer n prove that there are n consecutive composite integers.

7. Answer: $(n+1)! + 2, (n+1)! + 3, \dots, (n+1)! + (n+1)$ are n consecutive composite integers.

2 Primes related sequences

1. Find a formula or rule for the sequence: $1, 2, 3, 2, 5, 2, 7, 2, 2, 2, 11, 2, 13, 2, \dots$. What are the next two entries?

2. Find a formula or rule for the sequence: $1, 1, 2, 4, 4, 6, 6, 10, 10, 10, 10, 12, 12, 16, 16 \dots$. What are the next two entries?

A section on sets will be added later.

2.1 Sets

1. Let $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13\}$. Construct a set of 13 quadruples (subsets of size four) such that:
   - each two sets have exactly one number in common.
   - each number appears in exactly four subsets.
   - each pair of numbers appears in one set.
   - Hint: Construct the 13 lines of the projective plane $PG(3)$.

2. You are given a list of 23 teams, each with an odd number of students such that any two teams have an even number of students in common. There are 26 students in the class. Prove that you can add a new team with an odd number of students that has an even number of students in common with each of the other teams.

3. Answer (sketch).
   - Let $u_1, \dots, u_{23}$ be the characteristic vectors of the 23 teams.
   - Let $j = (1, 1, \dots, 1)$.
   - Let $u_{24} = j - \sum_{i=1}^{23} < j, u_i > u_i$.
   - Prove that $< u_{24}, u_i >= 0$ mod 2. (what does it mean?)
   - Prove that $< u_{24}, j >= 1$ mod 2. (what does this mean?)