Discrete Mathematics and Applications

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1 Assignment No. 4: mathematical induction and permutations

Due: Friday, 8 Oct.

Please submit your answer in a neat, readable properly organized format.

1. Express each of the following cycles as a product of transpositions.
   a. (1 2 3 4)
   b. $(1\ 2\ 3\ \ldots n)$
   c. $(1\ 2\ 3\ 4) \circ (2\ 3\ 4\ 5)$

2. Let $\pi = 7\ 3\ 1\ 6\ 5\ 2\ 4\ 8 \quad \mu = 7\ 3\ 1\ 4\ 5\ 8\ 6\ 2$
   a. Which permutation is lexicography "smaller?"
   b. Which permutation immediately follows π in the lexicographic order, which precedes μ?
   c. How many permutations lie between π *and* μ in the lexicographic order?

3. * 30 marbles numbered $1, 2, \ldots, 30$ are randomly arranged in a circular track. They can be move d around the track in clockwise or counter clockwise direction (operation 1). At the bottom of the track is a circular disk that can hold exactly four marbles. The circular disk can rotate 180^0 (reversing the orientation of the four marbles it holds, operation 2). Prove that you can sort the marbles using these two operations.

2 Mathematical Induction

1. Prove that if k is odd then 2^{n+2} *divides* $k^{2^n} - 1 \quad \forall n \in Z^+$.

2. a. Prove that $(1+\sqrt{2})^{2n} + (1-\sqrt{2})^{2n}$ is an even integer.
   b. Prove that $(1+\sqrt{2})^{2n} - (1-\sqrt{2})^{2n} = k\sqrt{2}$ for some positive integer k.

3. Prove that $\sum_{i=1}^{n} i \cdot i! = (n+1)! - 1$

4. In a country called Oz they have only two types of coins: 7 gnods and 30 gnods. The cheapest item you can buy in Oz costs 1000 gnods. Can you use exact change to pay for any purchase greater than 1000 gnods?

5. Prove that $\sqrt{n} \leq \sum_{k=1}^{n} \frac{1}{\sqrt{k}} \leq 2\sqrt{n}$

6. Recall the Fibonacci sequence $1, 1, 2, 3, 5, \ldots$ or $f_1 = f_2 = 1,\ f_i = f_i - 1 + f_i - 2\ for\ i \geq 3$ Prove that $f_k f_n + f_{k+1} f_{n+1} = f_{n+k+1} \quad \forall n, k \in Z^+$.

7. * Prove the *means inequality:* $\frac{1}{n}(\sum_{i=1}^{n} a_i) \geq (a_1 a_2 \ldots a_n)^{\frac{1}{n}}$ using the following steps:
a. Prove it by induction for $n = 2^k$.
b. Assume that it is true for $n = m$ and prove it for $n = m - 1$.
(Note that now you have the freedom to choose the value of a_m).