Finite sets Review

October 9, 2010

Question 1

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Press PgDn to see the answer

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1. *Let* $A = \{\{x,y,z\} | \{x,y,z\} \subset \{1,2,3,4,5\}\}$.
2. *Clearly, if* $A_i, A_j \in A$, *then* $1 \leq |A_i \cap A_j| \leq 2$.
3. *This gives us a set of* $\binom{5}{3} = C(5,3) = 10$ *subsets.*

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Hint

1. *First note that such a set, if it exists, cannot contain subsets with less than 3 letters.*

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Hint

1. *First note that such a set, if it exists, cannot contain subsets with less than 3 letters.*
2. *Argue that it cannot contain a set with 4 letters.*

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$A \times B \times C$ *is a set of triples* $\{(x, y, z)|x \in A, y \in B, z \in C\}$.
$(A \times B) \times C$ *is a set of pairs* $\{(u, v)|u \in A \times B, v \in C\}$

Observation

There is a bijection between the two sets.

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$P(\emptyset)$ *has one element:* $\{\emptyset\}$.
$P(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\}$ *(it has two elements).*
At this point it might help to rewrite the previous line as follows:
let $a = \emptyset, \ b = \{\emptyset\}$.
With this notation we have: $P(\{\emptyset\}) = P(b) = \{a, b\}$
So $P(P(b)) = P(\{a, b\}) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$
So $P(P(P(\emptyset)))$ *has eight elements.*

Question (5)

Let $A = \{a, b, c, d, e\}$

1. *What is the characteristic (incidence) vector of* $\{a, b, e\}$*?*
2. *What is the characteritic vector of* $\{a, b, c, d\} \cap \{b, c, d, e\}$*?*
3. *List all characteristic vectors of all subsets of* A *with cardinality* 4*.*

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1. *The characteristic vector of* $\{a, b, e\}$ *is* $(1, 1, 0, 0, 1)$*.*

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1. *The characteristic vector of* $\{a, b, e\}$ *is* $(1, 1, 0, 0, 1)$*.*
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1. *The characteristic vector of* $\{a, b, e\}$ *is* $(1, 1, 0, 0, 1)$*.*
2. $\{a, b, c, d\} \cap \{b, c, d, e\} = \{b, c, d\}$ *so its characteristic vector is* $(0, 1, 1, 1, 0)$*.*
3. *The characteritic vectors are:*
   - $(1, 1, 1, 1, 0)$
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Prove that $|\overline{A} \cap \overline{B}| = |U| - |A| - |B| + |A \cap B|$.

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Hint

One simple way to handle this question is to use DeMorgan's law:
$\overline{A} \cap \overline{B} = \overline{A \cup B}$ *and the simple fact:* $|\overline{C}| = |U| - |C|$.

Question (7)

Prove that the five characteristic vectors representing the five 4-subsets of a set with five elements (see question (5)) are linearly independent.

Press PgDn to see the answer (to Question 7)

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Proof.

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2. *We have:* $< v_i, v_j >= 3$ *if* $i \neq j$ *and* $< v_i, v_i >= 5$.

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3. *Assume that* $\sum_{i=1}^{5} \alpha_i v_i = 0$.

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   We need to prove that $\forall i \; \alpha_i = 0$.
4. $< v_1, \sum_{i=1}^{5} \alpha_i v_i >= \sum_{i=1}^{5} \alpha_i < v_1, v_i >= 0$.

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4. $< v_1, \sum_{i=1}^{5} \alpha_i v_i >= \sum_{i=1}^{5} \alpha_i < v_1, v_i >= 0$.
5. $\sum_{i=1}^{5} \alpha_i < v_1, v_i >= 2\alpha_1 + 3 \sum_{i=1}^{5} \alpha_i = 0$.

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6. *If* $\sum_{i=1}^{5} \alpha_i = 0$ *then* $\alpha_1 = 0$ *and we are done.*

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6. *If* $\sum_{i=1}^{5} \alpha_i = 0$ *then* $\alpha_1 = 0$ *and we are done.*
7. *If not, we obtain forall other indices* i, $2\alpha_i + 3 \sum_{i=1}^{5} \alpha_i = 0$.

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   We need to prove that $\forall i \; \alpha_i = 0$.
4. $< v_1, \sum_{i=1}^{5} \alpha_i v_i >= \sum_{i=1}^{5} \alpha_i < v_1, v_i >= 0$.
5. $\sum_{i=1}^{5} \alpha_i < v_1, v_i >= 2\alpha_1 + 3 \sum_{i=1}^{5} \alpha_i = 0$.
6. *If* $\sum_{i=1}^{5} \alpha_i = 0$ *then* $\alpha_1 = 0$ *and we are done.*
7. *If not, we obtain forall other indices* i, $2\alpha_i + 3 \sum_{i=1}^{5} \alpha_i = 0$.
8. *If we sum all five equations we get:*
   $2 \sum_{i=1}^{5} \alpha_i + 15 \sum_{i=1}^{5} \alpha_i = 17 \sum_{i=1}^{5} \alpha_i = 0$.
9. *But this contradicts the assumption that* $\sum_{i=1}^{5} \alpha_i \neq 0$.

□

Question (8)

a. Find the equation of a line through the origin in $GF^2(7)$ *parallel to the line that includes the points* $(4, 5), (3, 6)$.
b. Find the intersection of the original line and the line through the points $(2, 5), (4, 1)$.

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(Recall: we are working in $GF(7)$*).*
a. The equation of the line through $(4, 5), (3, 6)$ *is:*
$x - 4 = 5 - y$ *or* $x + y = 2$*.*
Therefore the line through the origin parallel to this line is
$x + y = 0$*.*

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a. Find the equation of a line through the origin in $GF^2(7)$ *parallel to the line that includes the points* $(4,5),(3,6)$.
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b. The equation of the line through $(2,5),(4,1)$ *is:* $\frac{x-2}{2} = \frac{y-5}{1-5}$.
Simplifying: $y + 2x = 2$.

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b. The equation of the line through $(2,5),(4,1)$ is: $\frac{x-2}{2}=\frac{y-5}{1-5}$.
Simplifying: $y+2x=2$.
The lines intersect at: $(0,2)$.

Question (9)

a. Find the equation of the line through the points $(4,3,0),(1,0,2)$ *in the projective plane* $PG(5)$.
b. Find the eqauation of the line through $(0,1,0),(1,2,3)$ *in* $PG(5)$.
c. Find the intersection point of these lines.

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a. We are looking for a triple $(a,b,c) \neq (0,0,0)$ *such that*
$4a+3b=a+c=0$
We may choose $a=1$ *(do you see why this is justified?) So* $(a,b,c)=(1,2,4)$ *and the equation of the line is:* $x+2y+4z=0$.
b. Similarly, the equation of the line is $2x+z=0$.
c. The intersection point is: $(1,1,3)$.

Question (10)

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b. Prove that every line in this projective plane conatins $q + 1$ points.

Answer Question 10

Press PgDn to see the answer

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Proof.

a. The number of triples
$\{(x,y,z)\}|x,y,z \in GF(q) \setminus \{(0,0,0)\}$ *is* $q^3 - 1$.
Every equivalence class contains $q-1$ *triples.*
Since the equivalence classes are pairwise disjoint the number of equivalence classes is $\frac{q^3-1}{q-1} = q^2 + q + 1.$.
b. The equation of a line is: $ax + by + cz = 0,\ (a,b,c) \neq (0,0,0)$
Note that if (x_0, y_0, z_0) *satisfies this equation then so does* $(\alpha x, \alpha y, \alpha z)\ \forall \alpha \in GF(q)$. *But* $(x,y,z) \equiv (\alpha x, \alpha y, \alpha z)$ *or they are the same point.*
The total number of triples that satisfy the linear equation is q^2.
Since $(0,0,0)$ *is excluded, the total number of points on this line in* $PG(q)$ *is* $\frac{q^2-1}{q-1} = q+1$.

□