Linear Algebra and Finite Sets

September 18, 2011

A curious example

Question (Even teams)

How many different teams can be formed from students in a class with $2n$ students subject to the following two conditions:

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Question (Odd teams)

Let us modify this question slightly:

1. *Each team must have an odd number of students.*
2. *Each two teams must have an even number of students in common.*

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1. *We can form n pairs of students. Each subset of the n pairs can form a team. Clearly, each team will have an even number of students and each two teams will have an even number of students in common. The total number of teams is* 2^n, *so if for instance, there are only 40 students in the class, we can form* 2^{20} *teams which is more than* $1,000,000$ *teams.*

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2. *For the "odd" case, we can form 2n teams (each team will have 1 student). Another way, each team has* $2n-1$ *students, again we can form 2n teams. In case we have 40 students in class, we can form "only" 40 teams subject to the "odd" condition.*

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4. *Is there an explanation for the discrepancy between the "even" and "odd" class?*

Linear Algebra to the recsue

In this lecture we shall learn how linear algebra can be used to solve problems on finite sets. The basic tool is actually the computer representation of sets.

Definition

Let $A = \{a_1, a_2, \ldots a_n\}$ *be a finite set. The* **incidence vector** *of a finite subset* $S \subset A$ *is the vector* $(e_1, e_2, \ldots, e_n)$ *where* $e_i = 1$ *if* $a_i \in S$ *and* $e_i = 0$ *otherwise.*

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Here is a short list of Linear Algebra objects that we shall use:

1. Groups
2. Fields, finite fields
3. Vector spaces over fields.
4. Linear independence, dimension
5. Matrices, determinants

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3. $rank(M \times N) \leq \; min\{rank(M), rank(N)\}$
4. If M is an $n \times n$ matrix (a square matrix) then $rank(M) = n$ if and only if $Det(M) \neq 0$.

The Proof

Proof.

- Let $T_1, T_2, \ldots, T_k$ be k teams each with an odd number of students. Let t_i be the incidence vector coresponding to team T_i that is $t_i \in R^{2n}$.

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- But this means that $Det(M \times M^{tr}) \pmod 2 = 1$ a contradiction. **Conclusion:** $k \leq 2n$.

Fields

Definition

A **field** $\{F, +, \cdot\}$ *is a set together with two operations, usually called addition and multiplication, and denoted by* $+$ *and* $\cdot$ *respectively, such that the following axioms hold:*

1. $\{F, +\}$ *is a commutative group,* 0 *is the additive identity.*
2. $\{F \setminus \{0\}, \cdot\}$ *is a commutative group,* 1 *is the multiplicative identity.*
3. *The ditributive law holds:* $a \cdot (b + c) = a \cdot b + a \cdot c$.

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- $GF(2^2) = \{0, 1, \alpha, 1 + \alpha\}$, *where* $\alpha + \alpha = 0, 1 + 1 = 0, \alpha \cdot \alpha = \alpha + 1$.

Vector spaces over fields

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A vector space of dimension k over the field F, denoted by F^k is the set: $\{(x_1, x_2, \ldots, x_k)\}$ where $x_i \in F$ together with the following two operations:

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It is easy to verify that:

$$< u, \sum_{i=1}^{n} v_i >= \sum_{i=1}^{n} < u, v_i >.$$

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Every line in $GF^2(5)$ *contains* 5 *points.*

Some basic facts about vector spaces

- A set of vectors $\{v_1, v_2, \ldots v_m\} \subset F^k$ is **linearly independent** if: $\sum_{i=1}^m \alpha_i v_i = 0 \rightarrow \alpha_i = 0$.
- A set of vectors $\{v_1, v_2, \ldots v_m\} \subset F^k$ is a **basis** if every vector $u \in F^k$ can be expressed **uniquely** as a linear combination of $\{v_1, v_2, \ldots v_m\} \subset F^k : \quad u = \sum_{i=1}^m \alpha_i v_i$

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- If $W_0 = \{w_1, w_2, \ldots w_m\} \subset U \subset F^k$ is a linearly independent set and $m < dim(U)$ then we can add $dim(U) - m$ vectors to W_0 to form a basis of U.

The odd teams, revisited

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But $< v_j, v_j > = 1$ so $\alpha_j = 0$ or $k \leq n$. □

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For instance, how to add more teams if possible (see exercise).

Some more set problems...

Theorem

Assume you formed 23 teams in our class, each team having an odd number of students and any two teams have an even number of students in common. Prove that you can add 3 more teams each with an odd number of students such that any two different teams will have an even number of students in common.

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Left to you... □

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Question

How many days are needed?

Answer

Each girl walks with two other girls every day. So to walk with 8 other girls we need at least four days.

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Now do the rest.

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Should be easy now!