How ”big” can a set be?

September 18, 2011

The cardinality of sets

Question

What is the "size" of a set?

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Can we "compare" any two sets?

Observation

In this section we shall develop the tools that will enable us to compare sets. We will also prove that there "unlimited" sizes of sets and that there are many non computable functions.

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Observation

The function $f(n) = 2n$ is a bijection between the integers and the even integers.
This means that there is a bijection between a set and "half" its size!

The inverse function

We need a few more definitions to be ready for our goal.

Definition

A set B is finite if there is a bijection between B and N_k.

Observation

If $f : A \to B$ is a bijection then we can define a new function $f^{-1} : B \to A$, the inverse of f, as follows: to find how f^{-1} maps the element $b \in B$ find the unique $a \in A$ such that: $f(a) = b$ and define $f^{-1}(b) = a$.

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Example

$f(x) = 3x + 1, \; x \in R$.
$f^{-1}(x) = ?$

Definition

Let $g : A \to$B and $f : B \to C$. The **composition** of the functions f and g, denoted by $f \circ g$ is a function $f \circ g : A \to C$ defined by $f \circ g(a) = f(g(a))$.

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If f is a function on the set A, then $f \circ I(a) = I \circ f(a) = f(a)$.

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2. Let $h(x) = x^2 + 1$.

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$f \circ g(x)$ and $g \circ f(x)$ can be distinct functions, or the composition is not commutative.

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Theorem

If f, g, h are bijections on the set A then $(f \circ g) \circ h = f \circ (g \circ h)$

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Follows easily from the definitions. $\square$

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Infinities...

Definition

If there is a bijection between A and B we say that they have the same **cardinality** denoted by $\mid A \mid = \mid B \mid$

Remark

The relation $\mid A \mid = \mid B \mid$ *is an equivalence relation among sets.*

Question

1. *Naturally, we would like to say that* $\mid A \mid > \mid B \mid$ *if there is an injection* $f : B \rightarrow A$.
2. *Is this a proper comparison function? Can any two sets be compared? Can we decide which is "bigger?" Easy for finite sets, but what about infinite sets?*
3. *In particular, if* $\mid A \mid \geq \mid B \mid \wedge \mid B \mid \geq \mid A \mid$ *does it imply that* $\mid A \mid = \mid B \mid$?

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Countable sets play a central role in discrete mathematics.

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A set B is **countable** if $|B| = |N|$. We say that the cardinality of B is $\aleph_0$.

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- *If $A \subset N, \ A \neq \emptyset$ then A has a smallest member.*
- *(The axiom of mathematical induction).*
  If $1 \in A, \ \wedge \ ((n \in A) \rightarrow n+1 \in A)$ then $A = Z^+$.

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Observation

There are other equivalent forms of the principle of mathematical induction:

1. $1 \in A,\ (\forall k < n,\ k \in A \rightarrow n \in A)$ then $A = Z^+$.

2. If $(\exists a_n \in A,\ a_n \rightarrow \infty) \rightarrow (a_n - 1) \in A$ then $A = Z^+$.

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Theorem

If $A_i, \; i = 1, 2, \ldots$ *are countable sets then so is* $\cup_{i=1}^{\infty} A_i$.

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Corollary

There are functions $f : N \rightarrow \{0, 1\}$ *(decision problems) that are not programmable.*

Theorem (4)

If $\mid A \mid \leq \mid B \mid$ *and* $\mid B \mid \leq \mid A \mid$ *then* $\mid A \mid = \mid B \mid$

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We will prove that there is no onto function $f : A \to P(A)$ *.*
Indeed given any function $f : A \to P(A)$*. Let* $S = \{a \in A \mid a \notin f(a)\}$*.*
(Recall that $f(a) \subset A$*, or* $f(a) \in P(A)$*).*

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Assume that $S = f(s)$ *for some* $s \in A$*.*
Whether $s \in f(s)$ *or* $s \notin f(s)$ *we reach a contradicion.*

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Whether $s \in f(s)$ *or* $s \notin f(s)$ *we reach a contradicion.*
Fill in the details.
Conclusion: since there is an injection $g : A \to P(A)$ *and there is no onto function* $f : A \to P(A)$ *we conclude that* $\mid A \mid < \mid P(A) \mid$*.*

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Let $\{x_1, x_2, \ldots, x_n, \ldots\}$ *be a countable subset of* $\mathbb{U}$. *Let* $x_n = 0.x_{n,1}x_{n,2}\ldots x_{n,n}x_{n,n+1}\ldots$ *be the decimal expansion of* x_n.

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Let $y = 0.y_1y_2\ldots y_n\ldots$ *be defined as follows:*

Let $y_n = x_{n,n} + 5 \pmod{10}$. *We want to make sure that* $\forall n,\ y_n \neq x_{n,n}$.

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Remark

This proof technique is called the Diagonal Method. It is used on many occaisons. For instance Theorem 1 is an abstract form of this method.

Proofs

Here we go again.

Proof (Theorem 3, proof sketch)

It is enough to show that there is a bijection between the set of functions: $\{f : N \to \{0, 1\}\}$ *and* $P(N)$.

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Let $F(f) = \{i \mid f(i) = 1\}$.
Show that this is a bijection between $P(n)$ *and the functions.*

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Show that this is a bijection between $P(n)$ *and the functions.*

Proof (of the corollary)

Each program that implements a decision problem is stored in memory as a finite binary sequence. There are only countably many finite binary sequences. Hence there are non computable functions.

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Consider the following chains, (directed paths): $\ldots \rightarrow a \rightarrow f(a) \rightarrow g(f(a) \ldots$
Verify: Each chain is one of the following four types:

1. *A finite cycle with* $2n$ *"nodes"* n*, members of* A *interlaced with* n *members of .*

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Verify this assertion.

In Set Theory this is known as bernstein's Lemma.

Surprise

Remark

There is a surprising consequence of this famous lemma. If you take two sets of points A and B in the plane, and if each set contains a disk, then each set can be disected into two sets A_1, A_2, B_1, B_2 such that A_i and B_i are similar.

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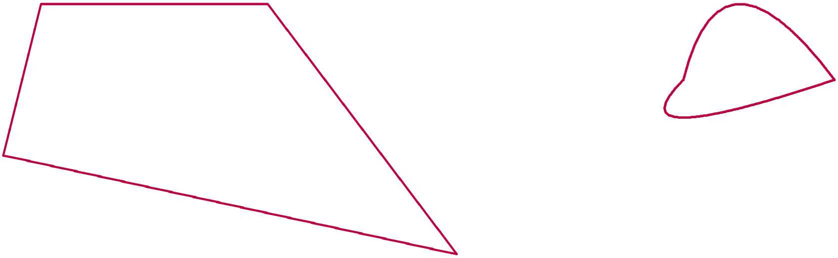

For example: these two sets can be disected into a pair of similar sets!