Recurrence Relations and Generating Functions

Ngày 27 tháng 10 năm 2011

Recursive Problem Solving

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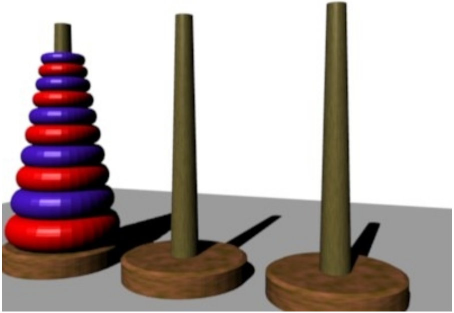
If we denote the number of bacteria at second number k by b_k then we have: $b_{k+1} = 2b_k,\ b_1 = 1$.

This is a recurrence relation.

The Towers of Hanoi

Another example of a problem that lends itself to a recurrence relation is a famous puzzle: **The towers of Hanoi**



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Clearly, before we move the large disk from the left to the right, all but the bottom disk, have to be on the middle tower. So if we denote the smallest number of moves by h_n then we have:

$$h_{n+1} = 2h_n + 1$$

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A simple technic for solving recurrence relation is called *telescoping*.

Start from the first term and sequntially produce the next terms until a clear pattern emerges. If you want to be mathematically rigoruous you may use induction.

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$$h_1 = 1,\ h_2 = 3,\ h_3 = 7,\ h_4 = 15, \ldots\ h_n = 2^n - 1$$

Proof by induction:

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6. *Telescoping yields:* $1, \frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \frac{5}{8}, \frac{8}{13}$

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Recurrence Relations Terminology

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A recurrence relation for a sequence a_n is a relation of the form $a_{n+1} = f(a_1, a_2, \ldots, a_n)$.

We do not expect to have a useful method to solve all recurrence relations. This definition actually applies to any sequence! We shall break down the functions for which we do have effective methods to "solve" the recurrence relation. By solving we mean obtaining an explicit expression of the form $a_n = g(n)$. To accomplish this we need some terminology.

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A recurrence relation is **linear** *if:*

$$f(a_1, a_2, \ldots, a_n) = \sum_{i=1}^{n} h_i \cdot a_i + h(n) \text{ Where } h(n) \text{ is a function of } n.$$

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If $f(n)$ and $g(n)$ are solutions to a non homgeneous recurrence relation then $f(n) - g(n)$ is a solution to the associated homogeneous recurrence relation.

Remark

This means that in order to solve a non homogeneous linear recurrence relation all we need to do is find the general solution $g(n)$ to the homogeneous part and a particular solution $p(n)$ to the non homogenesous equation.

The general solution will be: $g(n) + p(n)$.

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5. *Solving for c and d we get:* $a_n = \alpha 2^n - 3n - 5$

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Let $a_n = b \cdot a_{n-1} + c \cdot a_{n-2} + g(n), \quad a_1 = \alpha, \; a_2 = \beta$.
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Corolary

Any solution that satisfies the recurrence relation and initial conditions is THE ONLY solution.

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The quadratic equation $x^2 - bx - c = 0$ *is called the* **characteritic equation** *of the recurrence relation.*

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5. As previously proved, $r^n = br^{n-1} + cr^{n-2}$. Taking the derivative we get: $nr^{n-1} = b(n-1)r^{n-2} + c(n-2)r^{n-3}$ and if we multiply both sides by r we get: $nr^n = b(n-1)r^{n-1} + c(n-2)r^{n-2}$

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3. In the second case we have: $\alpha = m$ *and* $\alpha + \beta = k$ which obviously has a solution.

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- In general, try a function “similar” to $f(n)$. The following examples will demonstrate the general approach.

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Generating Functions

With every sequence a_n we can associate a power series:

$$f(x) = \sum_{i=0}^{\infty} a_n x^n$$

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8. So the answer will be the coefficient of x^{27} in the expansion of $(1 - x)^{-4}$.

Examples, continued

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3. *Note that:*
   $(1+x+\ldots+x^{30})(1+x+\ldots+x^{40})(1+x+\ldots+x^{50}\ldots) =$
   $\frac{1-x^{31}}{1-x}\frac{1-x^{41}}{1-x}\frac{1-x^{51}}{1-x} = (1-x)^{-3}(1-x^{31})(1-x^{41})(1-x^{51})$.

Examples

All we need is to find the coefficient of x^{70} in:

$$\left(\sum_{i=0}^{\infty}\binom{-3}{i}x^i\right)(1 - x^{31} - x^{41} - x^{51} + \ldots)$$

which turns out to be 1061 once we understand the meaning of

$$\binom{-3}{i}$$

.

Drill

Use this technique to find the number of distinct solution to:

$$x_1 + x_2 + x_3 + x_4 = 50$$

$$10 \leq x_1 \leq 25,\ 15 \leq x_2 \leq 30,\ 10 \leq x_3,\ 15 \leq x_4 \leq 25.$$

The Generalized Binomial Theorem

Theorem (The generalized binomial theorem)

$$(1+x)^r = \sum_{i=0}^{\infty} \binom{r}{i} x^i \quad \binom{r}{i} = \frac{r(r-1)\dots(r-i+1)}{i!}$$

For negative integers we get:

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Drill

Show that:

$$\binom{\frac{1}{2}}{k} = \frac{(-1)^k}{4^k}\binom{2k}{k}$$

Catalan Numbers

Question

You need to calculate the product of n matrices $A_1 \times A_2 \times \ldots \times A_n$*. How do we parenthesize the expression to do it in the most economical way?*

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Why does it matter?

Drill

Let $A[m, n]$ denote an $m \times n$ matrix (m rows and n columns). For each possible multiplication of the following product calculate the number of multiplications of real numbers needed to calculate the product.

$$A[10, 20]A[20, 40]A[40, 50]A[50, 10]$$

Catalan Numbers

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4. $m_1 = 1,\ m_2 = 2,\ m_3 = 5,\ m_n = ?$ *(for convenience, we set $m_0 = 0$).*

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5. $$m_n = \sum_{i=0}^{n} m_i \cdot m_{n-i}$$

Catalan Numbers

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3. For $n = 0, 1$ $\sum_{i=0}^{n} m_i \cdot m_{n-i} = 0$. Since $m_1 = 1$ this means that:

$$A^2(x) = \sum_{i=0}^{\infty} b_i x^i = \sum_{i=0}^{\infty} m_i x^i - x = A(x) - x$$

Catalan Numbers

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Substituting the initial condition $m_0 = A(0) = 0$ we get:

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$$(1-4x)^{\frac{1}{2}} = \sum_{k=0}^{\infty} \binom{1/2}{k} (-4)^k x^k = \sum_{k=0}^{\infty} \binom{2k}{k} x^k$$

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m_n is the coefficient of x^n in the expansion of: $(1 - \sqrt{1-4x})/(1/2x)$

A simple calculation yields:

$$m_n = \frac{1}{n+1}\binom{2n}{n}$$

Summary

These are the **Catalan Numbers.** They count many other objects, for instance the number of binary trees, the number bof grid paths from $(0, 0)$ to $(0, 2n)$ that stay above the x-axis, the number of binary sequences of length $2n$ with n $1's$ such that when scanning from left to right the number of 1's is never less than the number of 0's and more.

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In most of these cases, we show that these sequences satisfy the same recurrence relation and initial conditions.

Recurrence relations are a powerful tool for solving many problems. There are many types of generating function, we only scratched the surface of this beautiful theory.

Some more challenging problems will be posted in our assignments folder.