# How "big" can a set be?

September 29, 2011

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**Corollary**

*There are functions* $f : N \to \{0,1\}$ *(decision problems) that are not programmable.*

**Theorem (4)**

*If* $\mid A \mid \leq \mid B \mid$ *and* $\mid B \mid \leq \mid A \mid$ *then* $\mid A \mid = \mid B \mid$

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*We will prove that there is no onto function* $f : A \to P(A)$ .
*Indeed given any function* $f : A \to P(A)$. *Let* $S = \{a \in A \mid a \notin f(a)\}$.
*(Recall that* $f(a) \subset A$, *or* $f(a) \in P(A)$*).*

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*(Recall that* $f(a) \subset A$*, or* $f(a) \in P(A)$*).*
*Assume that* $S = f(s)$ *for some* $s \in A$*.*
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*Whether* $s \in f(s)$ *or* $s \notin f(s)$ *we reach a contradicion.*
*Fill in the details.*
*Conclusion: since there is an injection* $g : A \to P(A)$ *and there is no onto function* $f : A \to P(A)$ *we conclude that* $\mid A \mid < \mid P(A) \mid$*.*

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*Let* $\{x_1, x_2, \ldots, x_n, \ldots\}$ *be a countable subset of* $\mathbb{U}$*. Let* $x_n = 0.x_{n,1}x_{n,2}\ldots x_{n,n}x_{n,n+1}\ldots$ *be the decimal expansion of* $x_n$*.*

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**Remark**

*This proof technique is called the Diagonal Method. It is used on many occaisons. For instance Theorem 1 is an abstract form of this method.*

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**Proof (Theorem 3, proof sketch)**

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**Proof (of the corollary)**

*Each program that implements a decision problem is stored in memory as a finite binary sequence. There are only countably many finite binary sequences. Hence there are non computable functions.*

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**Proof ( of theorem 4)**

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The mapping $F(a) = b$ where $a \to b$, if $a$ belongs to chains in (1), (2) or (3) and $F(a) = b$ where $b \to a$ if $a$ is in a chain of (4) is a bijection between A and B.

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**Verify this assertion.**

In Set Theory this is known as bernstein's Lemma.

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**Remark**

*There is a surprising consequence of this famous lemma. If you take two sets of points $A$ and $B$ in the plane, and if each set contains a disk, then each set can be disected into two sets $A_1, A_2, B_1, B_2$ such that $A_i$ and $B_i$ are similar.*

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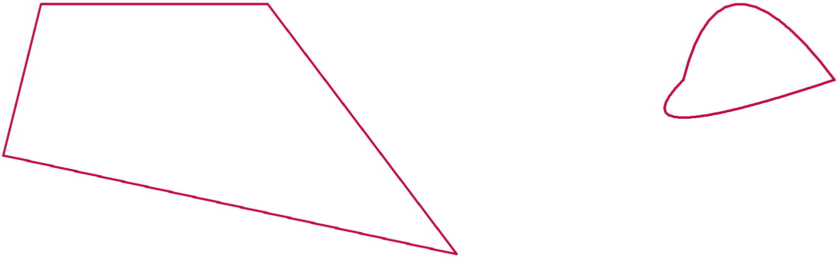

For example: these two sets can be disected into a pair of similar sets!