Recurrence Relations and Generating Functions

Ngày 3 tháng 12 năm 2011

Recursive Problem Solving

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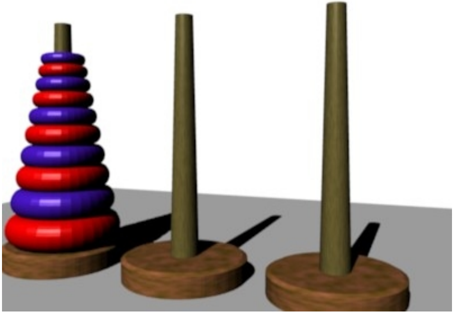
If we denote the number of bacteria at second number k by b_k then we have: $b_{k+1} = 2b_k,\ b_1 = 1$.

This is a recurrence relation.

The Towers of Hanoi

Another example of a problem that lends itself to a recurrence relation is a famous puzzle: **The towers of Hanoi**



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Clearly, before we move the large disk from the left to the right, all but the bottom disk, have to be on the middle tower. So if we denote the smallest number of moves by h_n then we have:

$$h_{n+1} = 2h_n + 1$$

A simple technique for solving recurrence relation is called *telescoping*.

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A simple technique for solving recurrence relation is called *telescoping*.

Start from the first term and sequntially produce the next terms until a clear pattern emerges. If you want to be mathematically rigoruous you may use induction.

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$$h_1 = 1,\ h_2 = 3,\ h_3 = 7,\ h_4 = 15, \ldots\ h_n = 2^n - 1$$

Proof by induction:

1. $h_1 = 1 = 2^1 - 1$

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5. *Solve:* $a_n = \frac{1}{1+a_{n-1}},\ a_1 = 1.$
6. *Telescoping yields:* $1, \frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \frac{5}{8}, \frac{8}{13}$

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3. $$a_{n+1} = \frac{1}{1+a_n} = \frac{1}{1+\frac{f_{n-1}}{f_n}} = \frac{f_n}{f_n + f_{n-1}} = \frac{f_n}{f_{n+1}}$$

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Recurrence Relations Terminology

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A recurrence relation for a sequence a_n is a relation of the form $a_{n+1} = f(a_1, a_2, \ldots, a_n)$.

We do not expect to have a useful method to solve all recurrence relations. This definition actually applies to any sequence! We shall break down the functions for which we do have effective methods to "solve" the recurrence relation. By solving we mean obtaining an explicit expression of the form $a_n = g(n)$. To accomplish this we need some terminology.

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A recurrence relation is **linear** *if:*

$$f(a_1, a_2, \ldots, a_n) = \sum_{i=1}^{n} h_i \cdot a_i + h(n) \text{ Where } h(n) \text{ is a function of } n.$$

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4. $a_n = a_{n-1} + 2a_{n-2} + 4a_{n-5} + 2^n$ is a non-homogeneous, linear recurrence relation with constant coefficients of order 5.

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3. This is a linear, homogeneous recurrence relation with constant coefficients, but not of finite order.

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If $f(n)$ and $g(n)$ are solutions to a non homgeneous recurrence relation then $f(n) - g(n)$ is a solution to the associated homogeneous recurrence relation.

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This means that in order to solve a non homogeneous linear recurrence relation all we need to do is find the general solution $g(n)$ to the homogeneous part and a particular solution $p(n)$ to the non homogenesous equation.

The general solution will be: $g(n) + p(n)$.

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5. *Solving for c and d we get: $a_n = \alpha 2^n - 3n - 5$*

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Theorem (observation)

Let $a_n = b \cdot a_{n-1} + c \cdot a_{n-2} + g(n), \quad a_1 = \alpha,\ a_2 = \beta$.
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Corollary

Any solution that satisfies the recurrence relation and initial conditions is THE ONLY solution.

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2. *Let* r_1, r_2 *be the roots of the characteristic equation.*
3. *Then the general solution of this recurrence relation is* $a_n = \alpha r_1^n + \beta r_2^n$*.*

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Let $a_n = ba_{n-1} + ca_{n-2}$.
The quadratic equation $x^2 - bx - c = 0$ is called the **characteritic equation** *of the recurrence relation.*

Theorem (Solving Linear Homogeneous RR with Constant Coefficients)

1. *Let $a_n = b \cdot a_{n-1} + c \cdot a_{n-2}$.*
2. *Let r_1, r_2 be the roots of the characteristic equation.*
3. *Then the general solution of this recurrence relation is $a_n = \alpha r_1^n + \beta r_2^n$.*
4. *If $r_1 = r_2$ then the general solution is $a_n = \alpha r^n + \beta n r^n$*

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4. Thus $\alpha r_1^n + \beta r_2^n$ solves the recurrence relation.
5. As previously proved, $r^n = br^{n-1} + cr^{n-2}$. Taking the derivative we get: $nr^{n-1} = b(n-1)r^{n-2} + c(n-2)r^{n-3}$ and if we multiply both sides by r we get: $nr^n = b(n-1)r^{n-1} + c(n-2)r^{n-2}$

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1. Let $a_0 = m,\ a_1 = k$. We need to show that we can choose α *and* β so that $\alpha r_1^0 + \beta r_2^0 = m$ *and* $\alpha r_1 + \beta r_2 = k$.

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3. In the second case we have: $\alpha = m$ *and* $\alpha + \beta = k$ which obviously has a solution.

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- In general, try a function “similar” to $f(n)$. The following examples will demonstrate the general approach.

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   - *So the general solution is:* $a_n = \alpha + \beta n + n^2 + \frac{1}{3}n^3$.

Generating Functions

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$$f(x) = \sum_{i=0}^{\infty} a_n x^n$$

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In this section we shall explore the interaction among polynomials, power series and counting.

Definition

The function $f(x) = \sum_{n=0}^{\infty} a_n x^n$ *is the* **genrating function** *of the sequence* a_n*.*

The funciton $f(x) = \sum_{k=0}^{\infty} \frac{a_k x^k}{k!}$ *is the* **exponential generating function** *of the sequence* a_k*.*

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If $f(x) = \sum_{n=0}^{\infty} a_n x^n, \;\; g(x) = \sum_{n=0}^{\infty} b_n x^n$ then:

$$f(x)g(x) = \sum_{n=0}^{\infty} c_n x^n \;\text{ where }\; c_n = \sum_{k=0}^{n} a_k b_{n-k}$$

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8. So the answer will be the coefficient of x^{27} in the expansion of:
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the generalized binomial coefficients

Recall: $(1+x)^n = \sum_{k=1}^{\infty} \binom{n}{k} x^k$.

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Recall: $(1+x)^n = \sum_{k=1}^{\infty} \binom{n}{k} x^k$.

Does this formula hold for all numbers n, like negative numbers, fractions, irrationals?

Would it be nice if we could use an extended Binomial Coefficient and write the answer:

$$\binom{-4}{27} \text{ or in general } \binom{-m}{k}$$

. Can we write:

$$(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n$$

How can we use it for solving counting problems?

Example

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2. *The coefficient of* x^{70} *in the product* $(1+x+\ldots+x^{30})(1+x+\ldots+x^{40})(1+x+\ldots+x^{50})$ *is the answer.*
3. *Note that:*

$$(1+x+\ldots+x^{30})(1+x+\ldots+x^{40})(1+x+\ldots+x^{50}) =$$

$$\frac{1-x^{31}}{1-x}\frac{1-x^{41}}{1-x}\frac{1-x^{51}}{1-x} = (1-x)^{-3}(1-x^{31})(1-x^{41})(1-x^{51})$$

.

Examples

All we need is to find the coefficient of x^{70} in:

$$\left(\sum_{i=0}^{\infty}\binom{-3}{i}x^i\right)(1-x^{31}-x^{41}-x^{51}+\dots)$$

which turns out to be 1061 once we understand the meaning of

$$\binom{-3}{i}$$

.

Drill

Use this technique to find the number of distinct solution to:

$$x_1+x_2+x_3+x_4=50$$

$$10\leq x_1\leq 25,\ 15\leq x_2\leq 30,\ 10\leq x_3,\ 15\leq x_4\leq 25.$$

The Generalized Binomial Theorem

Theorem (The generalized binomial theorem)

$$(1+x)^r = \sum_{i=0}^{\infty} \binom{r}{i} x^i \quad \binom{r}{i} = \frac{r(r-1)\dots(r-i+1)}{i!}$$

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For negative integers we get:

$$\binom{r}{i} = \frac{r(r-1)\dots(r-i+1)}{i!} = (-1)^i \binom{-r+i-1}{-r-1}$$

Drill

Show that:

$$\binom{\frac{1}{2}}{k} = \frac{(-1)^k}{4^k} \binom{2k}{k}$$

Derangements

Recall: an n-derangement is an n-permutation $\pi = a_1 a_2 \dots a_n$ in which $\forall i: \ a_i \neq i$. If we denote the number of n-derangments by D_n then:

$$D_1 = 0, \ D_2 = 1 \ and \ D_{n+1} = n(D_n + D_{n-1}).$$

Let: $D(x) = \sum_{n=0}^{\infty} D_n \frac{x^n}{n!}$ (the exponential generating function for D_n).

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Or: $\frac{D_n}{n!} = \sum_{k=0}^{n} \frac{(-1)^k}{k!} \Longrightarrow D_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$

Question

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Why does it matter?

Drill

Let $A[m,n]$ *denote an* $m \times n$ *matrix (m rows and n columns). For each possible multiplication of the following product calculate the number of multiplications of real numbers needed to calculate the product.*

$$A[10,20]A[20,40]A[40,50]A[50,10]$$

Catalan Numbers

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6. $$\textit{Hence}: \qquad m_{n+1} = \sum_{i=0}^{n} m_i \cdot m_{n-i}$$

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7. Since $2xA(x) = 0$ *when* $x = 0$ we have:
$$A(x) = \frac{1}{2x}(1 - \sqrt{1-4x}).$$

Catalan Numbers

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m_n is the coefficient of x^n in the expansion of: $(1 - \sqrt{1-4x})/(1/2x)$

A simple calculation yields:

$$m_n = \frac{1}{n+1}\binom{2n}{n}$$

Lattice walks

Question

Given a lattice. In how many ways can you walk from $(0,0)$ to (n,n) if you can only move to the right or up?

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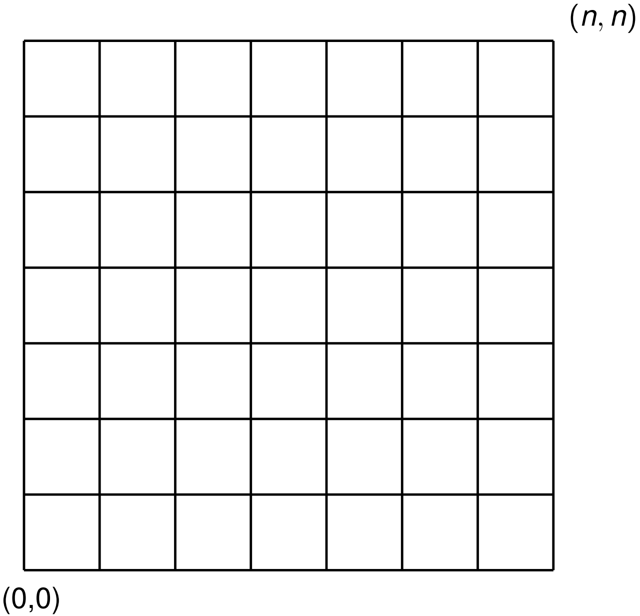
So the answer is:

$$\binom{2n}{n}$$

Question

The same question but this time your walk is restricted to stay below the diagonal.



(n, n)
(0,0)

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A minor change: We want to count the number of moves that stay below the diagonal.

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- *To stay below the diagonal, for each k the subsequence $x_1, x_2, \ldots x_k$ must have at least as many right-moves as up-moves.*

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- *But we already counted such sequences!*
- *Balanced parenthesis $(()((()())),\ (: \rightarrow) : \uparrow$.*
  So the number of walks is the Catalan number m_{2n}.

Other counting problems can be solvde by “mapping” them to solve problems.

- How many binary sequences $b_1 b_2 b_3 \ldots b_{2n}$ consisting of n 1’s and n 0’s are there in which $\sum_{i=1}^{k} b_i \geq \lceil \frac{k}{2} \rceil \ \forall k \geq 1$?

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- n Persons line up to buy tickets to the theater. The cost of a ticket is 50,000 VND. Each person has a 50,000 VND or a 100,000 VND. The cashier opens the box office with no money. So if the first person has a 100,000 VND the line will get stuck as the cashier will not be able to give him change. In how many ways can n persons arrange the line so all of them will be able to buy tickets with no delays?

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- We need to assume that at least $\lceil \frac{n}{2} \rceil$ have a 50,000 VND note.