Stirling's Formula:

$$n! \sim \sqrt{2\pi}\, n^{(n+1/2)}\, e^{-n}$$

Proof of Stirling's Formula

First take the log of $n!$ to get

$$\log(n!) = \log(1) + \log(2) + \cdots + \log(n)\ .$$

Since the log function is increasing on the interval $(0, \infty)$, we get

$$\int_{n-1}^{n} \log(x)dx < \log(n) < \int_{n}^{n+1} \log(x)dx$$

for $n \geq 1$. Add the above inequalities, with $n = 1, 2, \cdots, N$, we get

$$\int_{0}^{N} \log(x)dx < \log(N!) < \int_{1}^{N+1} \log(x)dx$$

Though the first integral is improper, it is easy to show that in fact it is convergent. Using the anti-derivative of $\log(x)$ (being $x\log(x) - x$), we get

$$n\log(n) - n < \log(n!) < (n+1)\log(n+1) - n\ .$$

Next, set

$$d_n = \log(n!) - \left(n + \frac{1}{2}\right)\log(n) + n\ .$$

We have

$$d_n - d_{n+1} = \left(n + \frac{1}{2}\right)\log\left(\frac{n+1}{n}\right) - 1\ .$$

Easy algebraic manipulation gives

$$\frac{n+1}{n} = \frac{1 + \dfrac{1}{2n+1}}{1 - \dfrac{1}{2n+1}}\ .$$

Using the Taylor expansion

$$\frac{1}{2}\log\left(\frac{1+t}{1-t}\right) = t + \frac{1}{3}t^3 + \frac{1}{5}t^5 + \cdots$$

for -1 < t < 1, we get

$$d_n - d_{n+1} = \frac{1}{3}\frac{1}{(2n+1)^2} + \frac{1}{5}\frac{1}{(2n+1)^4} + \cdots$$

This implies

$$0 < d_n - d_{n+1} < \frac{1}{3}\left(\frac{1}{(2n+1)^2} + \frac{1}{(2n+1)^4} + \cdots\right)$$

We recognize a geometric series. Therefore we have

$$0 < d_n - d_{n+1} < \frac{1}{3}\frac{1}{(2n+1)^2 - 1} = \frac{1}{12}\left(\frac{1}{n} - \frac{1}{n+1}\right) .$$

From this we get

1. the sequence $\{d_n\}$ is decreasing;
2. the sequence $\left\{d_n - \frac{1}{12n}\right\}$ is increasing.

This will imply that $\{d_n\}$ converges to a number C with

$$\lim_{n\to\infty} d_n = \lim_{n\to\infty} d_n - \frac{1}{12n} = C$$

and that $C > d_1$ - 1/12 = 1 - 1/12 = 11/12. Taking the exponential of d_n, we get

$$\lim_{n\to\infty} \frac{n!}{n^{(n+1/2)}e^{-n}} = e^C .$$

The final step in the proof is to show that $e^C = \sqrt{2\pi}$. This will be done via Wallis formula (and Wallis integrals). Indeed, recall the limit

$$\lim_{n\to\infty} \frac{2.2.4.4.6.6\ldots(2n)(2n)}{1.1.3.3.5.5.\ldots(2n-1)(2n-1)(2n+1)} = \frac{\pi}{2}\ .$$

Rewriting this formula, we get

$$\frac{2.4.6\ldots(2n)}{1.3.5.\ldots(2n-1)\sqrt{2n}} \sim \sqrt{\frac{\pi}{2}}$$

Playing with the numbers, we get

$$\frac{(2^n n!)^2}{(2n)!}\frac{1}{\sqrt{2n}} \sim \sqrt{\frac{\pi}{2}}$$

Using the approximation

$$n! \sim n^{(n+1/2)}e^{-n}e^{C}\ .$$

we get

$$\frac{2^{2n}\left(n^{2n+1}e^{-2n}e^{2C}\right)}{(2n)^{(2n+1/2)}e^{-2n}e^{C}}\frac{1}{\sqrt{2n}} \sim \sqrt{\frac{\pi}{2}}$$

Easy algebra gives

$$e^{C} \sim \sqrt{2\pi}$$

since we are dealing with constants, we get in fact $e^{C} = \sqrt{2\pi}$. This completes the proof of Stirling's formula.