Discrete Optimization
Lecture-13

Ngày 21 tháng 11 năm 2011

The Max-Flow Min-Cut problem

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The max-flow min-cut problem is a central problem in discrete optimization with many applications.

We shall study this problem, look at some applications, learn how it interacts with linear programming and explore the Ford Fulkerson algorithm.

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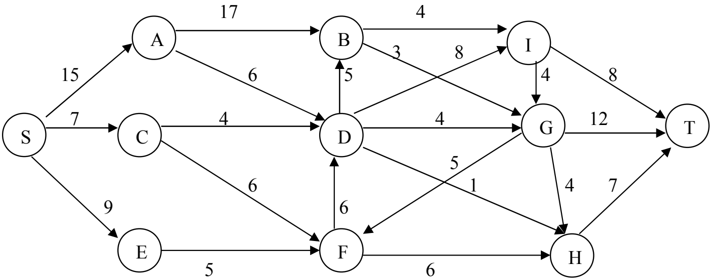
We denote by $N^+(v)$ the edges in G of the form $v \to x$ and by $N^-(v)$ the edges $x \to v$.

Example-1

A network with integer capacities.



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Suppose all capacities are integers and the flows cannot be broken into fractions. Will the optimal solution be integers?

Or do we need to add a constraint that $x(e)$ must be integers?

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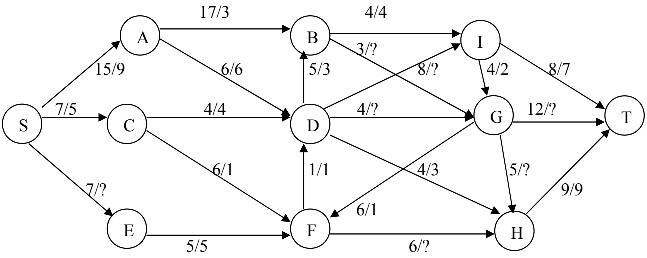
Comment

As network flows in applications can be quite large, we seek an alternative, more effcicent algorithm for solving netwrok flow problems: preferably a combinatorial algorithm.

An example



In this network an entry of the form n/m represents a flow of size m through an edge with capacity n.

Let us try to fill in the "?" so that the flow represented by this diagram will be a proper flow.

Ford-Fulkerson's Max-Flow Min-Cut Algorithm

We shall first go through the algorithm, execute it on a sample and then identify its components analyse it and prove the min-cut max-flow theorem.

Definition (f-Augmenting Path)

An $S-T$ path P in a network N with a flow f is $f-augmenting$ if replacing the flow on every edge $e \in P$ by the amount on this edge in P increases the total flow.

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  - Augment the flow and execute the algorithm on the new flow.
- If T is not labelled the flow is a maximum flow.

Example

Let N be the network:

$S \to A: \; 3 \quad S \to C: \; 7$

$\mathrm{A} \to B: \; 5 \quad A \to D: \; 4 \quad A \to C: \; 2$

$\mathrm{B} \to D: \; 2 \quad B \to T: \; 8$

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Using backflow: an example

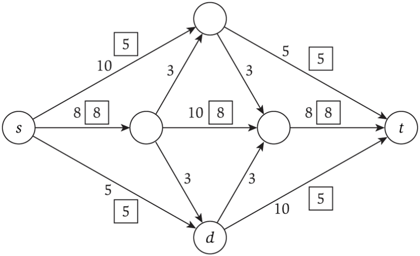

Figure 7.26 What is the value of the depicted flow? Is it a maximum flow? What is the minimum cut?

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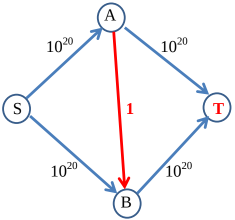

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- *Let $\alpha(N)$ be the length of the shortest $S - T$ path in the network N.*
- *Let $\mathcal{A}(N) = \{e \in E \mid e \in$ some shortest $S - T$ path$\}$.*

Analysis, continued

We define a network $N_f = (V, E_f)$ as follows:

- If $f((x,y)) < c((x,y))$ then $(x,y) \in E_f$.
- If $f((x,y)) > 0$ then $(y,x) \in E_f$
- Note that if $0 < f((x,y)) < c((x,y))$ then both (x,y) *and* $(y,x) \in E_f$.

Observation

- *Let $\alpha(N)$ be the length of the shortest $S-T$ path in the network N.*
- *Let $\mathcal{A}(N) = \{e \in E \mid e \in$ some shortest $S-T$ path$\}$.*
- *If $N' = (V, E \cup \{(y,x)\})$ $(x,y) \in E$ then $\alpha(N') = \alpha(N)$ and $\mathcal{A}(N') = \mathcal{A}(N)$.*

Final remarks

If f_1 is an augmentation of the flow f along a shortest path in the network N then N_{f_1} is a subgraph of the netwrok $N_f \cup \{(y,x) \mid (x,y) \in N_f$.

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So at each augmentation the algorithm will construct first all shortest paths (Dijkstra's algorithm) then scan all edges. So the running time of this algorithm will be $c \cdot |V| \cdot |E|^2$.

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Corollary (Min Cut Max Flow Theorem)

In a netwrok $N(V, E)$ the largest flow and the minimum cut have the same size.

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A weighted directed graph $D(V, E)$ with a weight function $f(u, v) \rightarrow R$ in which for every vertex $v \in V$:

$$\sum_{u \in N^+(v)} f(v, u) = \sum_{u \in N^-(v)} f(u, v)$$

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- This contradicts the choice of g.

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Netwrokflows have numerous applications in discrete optimization. We shall attempt to see some in the exercises.