Uniform columns

by Branko Grünbaum
University of Washington, Box 354350, Seattle, WA 98195
e-mail: grunbaum@math.washington.edu

A **uniform column** is

(i) a **locally finite polyhedron** in the 3-dimensional Euclidean space, infinite in one direction and finite in two other independent directions. We shall assume the infinite direction to be vertical;

(ii) **uniform**, that is, all vertices are equivalent under symmetries of the polyhedron, and all faces are regular polygons.

For the polyhedra considered here we allow selfintersections, as well as collinearity of edges and coplanarity of faces; however, we require that the elements (that is, faces, edges and vertices) be distinct.

The definition implies that all vertices of a uniform column are situated on a right circular cylinder. Hence no such polyhedron can have a face with five of more edges, and the only squares admissible must have edges that are horizontal or vertical. This makes feasible a complete enumeration of types of uniform columns.

The topic found very scant mention in literature, under the name cylindrical polyhedra. The single example of some generality is in [5, page viii], where only the six acoptic (that is, selfintersection-free) types are shown. The type we call helical is described in [3, page 117], where also references to earlier literature may be found. Nanotubes are often mentioned as cylindrical polyhedra. However, they are **not** uniform columns, since their hexagons are not planar. The same applies to the "cylindrical polyhedra" mentioned in [1] and [2], where non-planar "polygons" are used to form "polyhedra". The polyhedra described by Tarnai [4] have planar faces but two orbits of vertices.

The first collection of types consists of *stacked* uniform columns. These result by modular stacking of appropriate prisms and/or antiprisms. The kinds of modules that can be used are indicated in Figure 1; in all cases we use only the mantles of these modules, discarding their bases. The *unary* stacked columns use a single kind of modules, the *binary* ones use two kinds of modules in alternation. Examples of the resulting types of columns are shown in Figure 2. The modules have as bases regular polygons; specifically, the polygons usually denoted in the literature by $\{n/d\}$, where $1 \leq d < n/2$ and $n \geq 3$. The requirement that the elements are distinct implies that n and d are relatively prime. Obviously, for the binary stacked columns the two modules have to have the same bases. If $d = 1$, the polygon is the convex regular n-gon, usually denoted $\{n\}$.

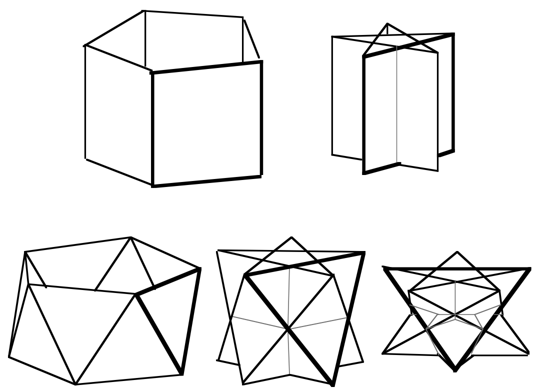

Figure 1. The modules (prisms and antiprisms) based on regular pentagons. The polygons are $\{5\}$ and $\{5/2\}$; as is well known (see, for example, [CLM]), if $n/3 < d < n/2$, there are two distinct antiprisms.

The three *acoptic* types of stacked columns are adequately illustrated in Figure 2. However, the *non-acoptic* types are more varied and more interesting. The selfintersections arise in two ways. The bases $\{n/d\}$ may be selfintersecting polygons (that is, $d > 1$), and/or the modules of the column may intersect. The first possibility is illustrated in Figure 3, by the three unary stacked columns.

The second possibility happens with binary stacked columns. As is shown in Figure 4, the uniform stacking can happen in two distinct ways: The two modules that share a base can either be on different sides of the plane of the shared base, or on the same side.

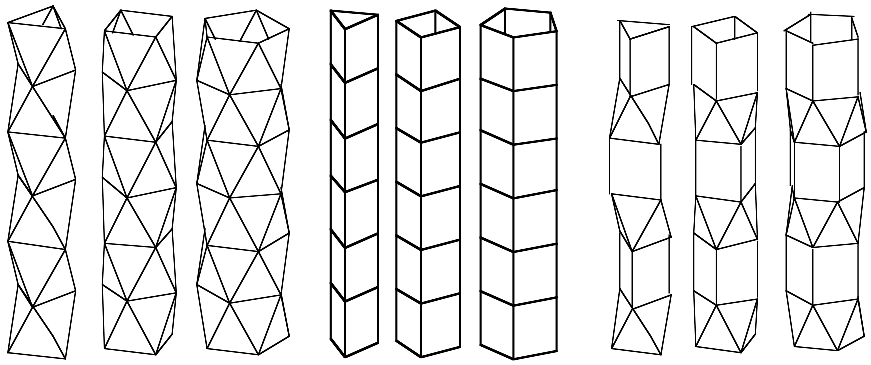

Figure 2. Examples of the three types of stacked uniform acoptic columns with n-gonal bases {n/1}, n = 3, 4, 5. The two unary and the only binary type are shown. These are the only possible types of acoptic uniform columns.

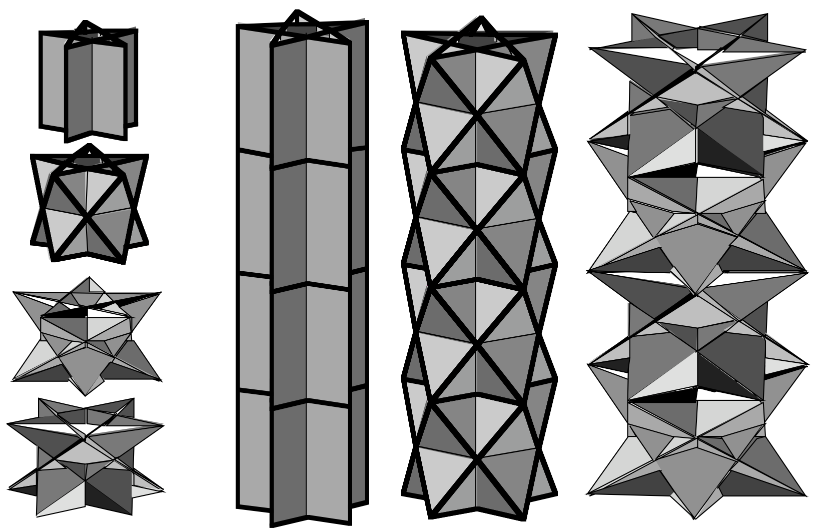

Figure 3. The three selfintersecting modules based on {5/2}, and the unary stacked columns they generate. The last module is shown in two views (from different sides), as needed in the column.

The former alternative leads to columns that are analogous to the binary acoptic stacks, and even the unary ones. In contrast, the second alternative happens only for binaries, and is possible since the altitudes of the different modules are distinct. However, this makes it very hard to show intelligible diagrams. For that reason, in Figure 4 we show only the union of the two modules that participate in the binary stack. The columns should be imagined as the repetition of the polyhedra shown.

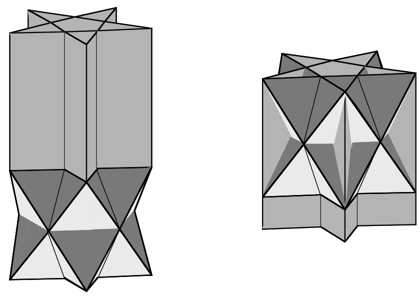

Figure 4. The two ways of constructing a binary stacked column from a {5/2} prism and one of the {5/2} antiprisms. In the second variant, each face of the prism is overlapping with faces of eight other prisms, thus making a drawing of the extended column essentially unintelligible.

The next collection of types are *ribboned* columns. These consist of infinite ribbons (strips) of either squares or equilateral triangles. The only unary columns we need consider are formed by ribbons of triangles, since the ones with squares coincide with those already listed as stacked columns. The triangular ribbons are arranged along cross-sections that are polygons $\{n/d\}$, where $1 \leq d < n/2$, n is even and $n \geq 3$, and either d and n are coprime or else $d = 2$ and $n = 2(2k+1)$, for an integer $k \geq 1$. The restrictions on n arise from the fact that in a circuit of an odd number of ribbons of triangles the vertical edges are not aligned. Figure 5 shows examples of all unary acoptic ribboned columns with at most ten ribbons. The only unary non-acoptic ribboned columns with at most ten ribbons are shown in Figure 6. In these columns pairs of ribbons coincide, but are displaced by one half the translational repeat of the column; hence the faces are distinct. However, this makes it exceedingly hard to depict these columns in a visually intelligible way.

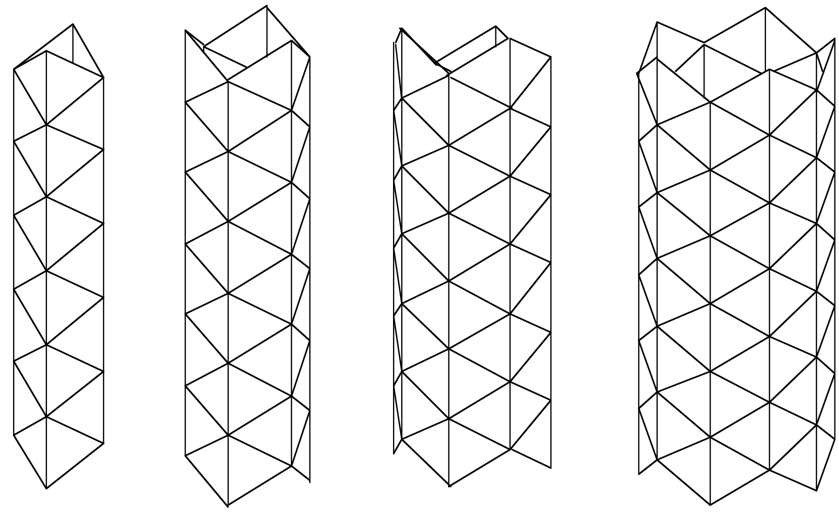

Figure 5. Unary acoptic columns of 4, 6, 8 and 10 ribbons of triangles.

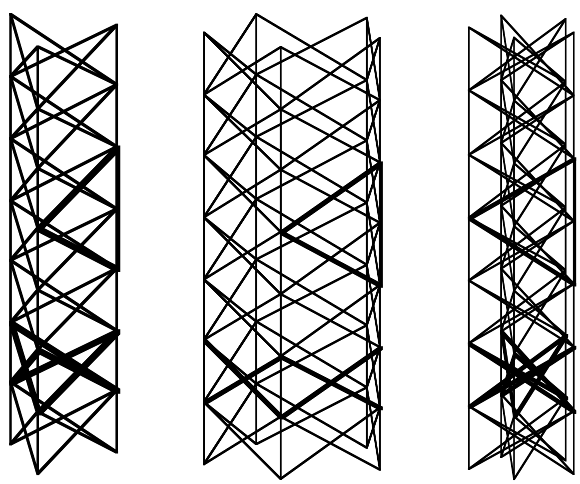

Figure 6. Unary non-acoptic columns with 6 or 10 overlapping ribbons of triangles. In each column one triangle is emphasized, as is a zigzag formed by the upper edges of a band of triangles at the same height. The zigzag is a non-planar regular polygon, called *prismatic polygon* in [G1]; it encloses the axis of the column twice. This serves to illustrate the difficulty of presenting such columns graphically, although their construction is very straightforward.

The binary ribboned columns have alternating columns of triangles and of squares. All can be constructed by folding an appropriate infinite strip of the Archimedean tiling (3.3.3.4.4). Since each column with triangles causes the two columns adjacent to it to be displaced (with respect to each other) in the vertical direction by one half of the length of an edge, the number of columns with triangles must be even. Hence there are, for each even n = 2k and each d coprime with n, binary ribboned columns with n ribbons of triangles and the same number of ribbons with squares that have as cross-section an isogonal 2n-sided polygon, with the lengths of adjacent sides in ratio $\sqrt{3}/2$. As can be seen in Figure 7, the acoptic columns of this type can easily be presented by diagrams; however, the non-acoptic one are exceedingly difficult to show. Moreover, if n is odd, n = 2k+1, there is an additional family of binary ribboned columns. It consists of 2n ribbons of triangles and 2n ribbons of squares, with a cross-section that is an isogonal 2n-gon; it accommodates the 4n ribbons since they come in overlapping pairs, displaced with respect to each other by half the edge-length. These I did not even try to show in a diagram; instead, Figure 8 shows the cross-sections.

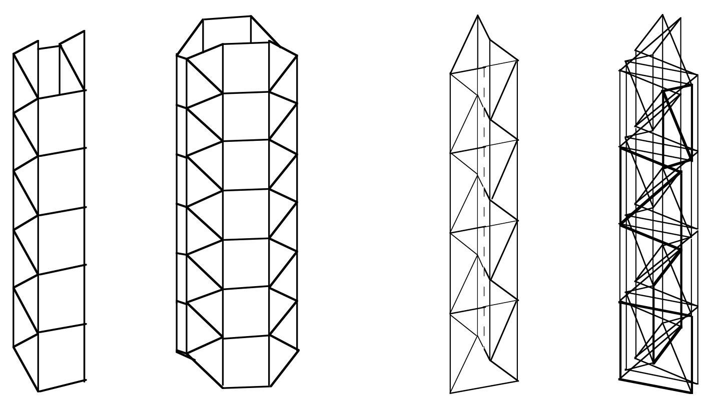

Figure 7. Binary columns with 4 or 8 ribbons, alternating between ribbons of triangles and ribbons of squares. The two on the left are acoptic, the other two have selfintersections. Note that in all cases the two ribbons with squares, that are neighboring a ribbon of triangles, are vertically displaced relative to each other by half the side of the square.

The last family consists of *helical* columns. Here the triangles form one or more ribbons that are would around a cylinder in a helix. They can be obtained by folding an appropriate oblique infinite strip of the regular tiling (3.3.3.3.3.3). In Figure 9 are shown examples of the simplest acoptic helical columns. A more detailed presentation of these and other (acoptic and non-acoptic) helical columns will appear at a later date. This will include additional bibliographic data.

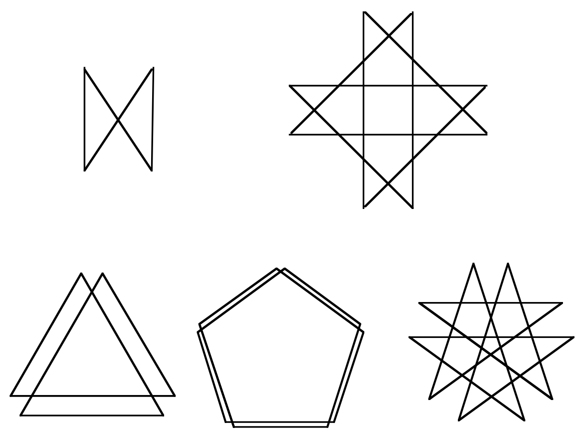

Figure 8. A cross-section of some binary non-acoptic ribboned columns. The top row corresponds to the 4- and 8-ribbon columns shown in Figure 7, while the bottom row corresponds to the 12- and 20-ribbon columns that are not shown. Throughout, the long edges represent ribbons of squares, the short ones ribbons of triangles; all edges in the bottom row are doubled-up, with displacements of the corresponding ribbons by half an edge.

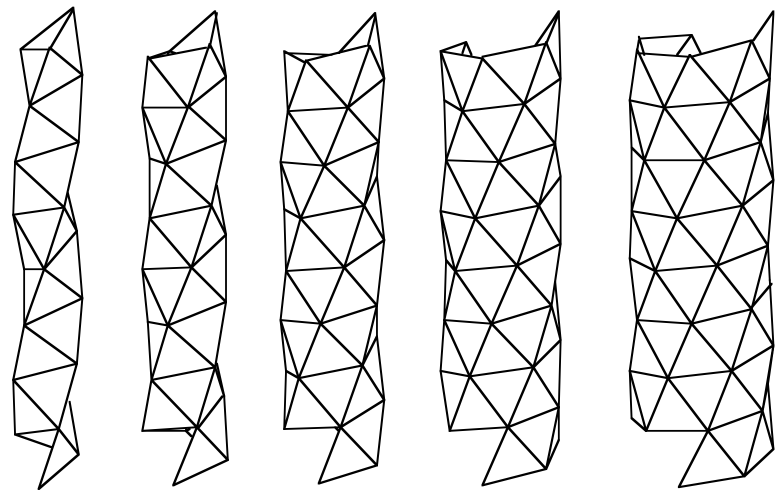

Figure 9. Example of helical acoptic columns of triangles.

References.

[1] M. Burt, *Spatial arrangement and polyhedra with curved surfaces and their architectural applications*. M. Sc. Thesis, Technion – Israel Institute of Technology, Haifa 1966.

[2] B. Grünbaum, Regular polyhedra -- old and new. *Aequationes Math*. 16(1977), 1 - 20.

[3] E. A. Lord, A. L. Mackay and S. Ranganathan, *New Geometries for New Materials*. Cambridge Univ. Press 2006.

[4] T. Tarnai, Folding of uniform plane tessellations. *Origami Science and Art. Proc. of the Second Int. Meeting of Origami Science and Scientific Origami, Otsu, Japan, 1994* (Ed.: K. Miura), Seian Univ. of Art and Design, 1997, 83–91.

[5] A. Wachman, M. Burt and M. Kleinmann, *Infinite Polyhedra*. Technion – Israel Institute of Technology, Faculty of Architecture and Town Planning. Haifa. First ed. 1974. Second (unchanged) ed. 2005.