Proposition of Pseudo-Cylindrical Concave Polyhedral Shells.

K. Miura.[1])

Summary:

The purpose of this study is to present a new category of the concave polyhedral surfaces which might be potential structural forms of the folded plate shells. These surfaces are obtained by the isometric transformation of general cylindrical surfaces. The macroscopic configuration of these surfaces are general cylindrical shape and plane shape and these are developable surfaces. These have also an intrinsic geometrical nature which provides a novel mechanism of the deployable shell. The resulting new category of folded plate shell, which is temporarily called the pseude-cylindrical concave polyhedral shell, has many useful characteristics as follows: inclusion of an arbitrary curvature distribut-ion, its midsurface being developable, intrinsically high bending rigidity, simplicity of elementary faces. The application of it to rigid shell structures, deployable shell structures, irreversible deployable shell structures, and sandwich core structures gives promise for the future. Furthermore, the discovery of the developable doubly corrugated surface evokes much theoretical interest.

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§ 1. INTRODUCTION

For a geometry-minded scholar, thinking about the definition of the folded plate shell will certainly settle down to this: the shell whose midsurface is a polyheral surface. Of course, there is no limit in numbers of the polyhedral surfaces and an architect can conceive a form for his building rather freely by intuition, experience, simulating nature, or folding a paper. An important thing is that the geometry of these surfaces almost definitely determines the primary nature of the structure which extends from strength, productivity, and economy to aesthetics.

If, then, a new geometrical form of the polyhedral surface is discovered, there is a possibility of using it as a midsurface of a folded plate shell whose characteristics might be favorable for some applications. Eventually, this will produce a new category of the folded plate shells.

The purpose of this study is to present a new category of the concave polyhedral surfaces which might be potential structural forms of the folded plate shells. The macroscopic configurations of these surfaces are general cylindrical shape and plane shape, and these have the definite common feature that these are developable everywhere, or in other words, these are isometric with a plane.

Moreover, these surfaces have an intrinsic geometrical nature which provides a novel mechanism of the deployable folded plate shell, and this suggests the future architectural as well as space engineering possibilities.

Since this category of surfaces was discovered accidentally by the present author in an effort to study the inextensional buckling deformation of general cylindrical shells, the following chapters begin with the study of buckling deformations.

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*This paper is a substantially revised version of the reference [1].

**The author wishes to thank Masamori Sakamaki and Tomoko Miura for their skilled works in preparing models.

§ 2. INTRODUCTION OF PSEUDO-CYLINDRICAL CONCAVE POLYHEDRAL SURFACES

Before getting at the kernel of the subject of this paper, we must have the recollections of the studies on the classical problem of axial buckling of thin cylindrical shells. Because the subject owes its origin to the forms of deformations encountered in the buckling process. That problem has been a target of many researchers for almost half a century and yet it has not been conclusively explored. Fortunately, such situation does by no means bother our discussion here, since the chief concern about the subject is not buckling criterion but the geometrical forms of deformations.

Now let us consider the post-buckling deformation of axially compressed circular cylindrical shells. As von Kármán and Tsien [2] assumed a buckle shape based on the observation, it consists of three trignometric terms (Fig.1):

$$w = A_{00} + A_{11}\cos(\xi\, x/r)\cos(\zeta\, s/r) + A_{20}\cos(2\xi\, x/r) + A_{02}\cos(2\zeta\, s/r) \qquad (1)$$

Here w is the radial inward displacement, r the radius of the midsurface of the shell, x and s the axial and circumferential coordinates, ξ and ζ numbers, and A's unknown coefficients. It goes without saying that the three trigonometric terms expression is not accurate enough to represent the buckle. Therefore, as the high speed digital computer became available, the computation which can includes more terms representing the shape of the buckle has been tried by, for example, Almroth [3], Hoff and his collaborators [4]; though the computation essentially follows the von Kármán-Tsien procedure. In the latter's paper, it was shown that such a procedure results in displacement patterns that approach a kind of concave polyhedral surface more and more closely as the number of terms representing the shape of the buckle is increased. Since that polyhedral surface had been predicted by Yoshimura [5] in 1951, it is called Yoshimura-pattern*(Fig.2). Indeed a typical experimental result

*The pattern was afterwards described independently by L. Kirste in Austria.

as shown in Fig.3 certainly resembles the Yoshimura-pattern. Moreover, through a certain modification of Yoshimura-pattern, the so-called local buckling pattern appeared in Fig.3 can be explained by Yoshimura-like inextensional deformation as shown in Fig.4. It appears, therefore, very likely that an infinite number of terms would yield the exact Yoshimura-pattern if a means could be found to include an infinite number of terms in the calculation.

It should be noted that this pattern can be obtained from the original circular cylindrical shape through an inextensional process. In other words, any line element of midsurface of the circular cylindrical shell does not change its length after deformation. In exact geometrical terms, these two configurations are isometric. The significance of this fact with regard to the elastic stability can easily be understood through energy consideration, but their consideration is outside the scope of the present paper. In short, the purely geometrical feature of the polyhedral surface representing the Yoshimura-pattern is the exactly isometric, axially and uniformly shortened surface that exists indefinitely close to an arbitrary circular cylindrical surface.

Now, as a natural extension of the above discussion, the existence of such isometric surfaces for general cylindrical surfaces is asked. This question was answered by the present author in his recent paper [6]. On purpose of revealing the important features of such isometric surfaces, the essential part of the paper is presented in the following.

Let us prove the presence of a surface which has the following characteristics: it is isometric with a given arbitrary cylindrical surface S; it has a uniform axial shortening; and it is indefinitely close to S. S can be either the closed or open cylindrical surface.

As shown in Fig.5. the parallel lines l,m,l,m,... with a constant interval λ_x and the parallel zones L,M,L,M,... constructed by the former are considered on a plane. The arbitrary points on the lines l and m are designated by $s_j (j = \ldots,k-2,k,k+2,\ldots)$ and s_{j+1}, respectively, with the condition $s_j < s_{j+1}$. The orthogonal projections of these points on the other group of the lines are distinguished by the superscript of dash. Connecting those points as shown, the rows of many triangles with an identical height are formed. Let us fold along the oblique sides of those triangles, for example on the L zone, so that every triangular plane may be inclined to the x axis by an angle of θ. In order to add clarity to the matter, the solid and dashed lines refer to the folding lines which are convex and concave to the reader's side, respectively. A part of the L zone formed by such a process is seen in Fig.5 as $s_k s'_k s_{k+1} s'_{k+1}$. If the line segment $\overline{s_k s'_{k+1}}$ is assumed to be on the plane l* normal to the x axis, the line segment $\overline{s'_k s_{k+1}}$ must be on the plane m* parallel to l* and as much as $\lambda_x \cos\theta$ distant from l*. This relation holds everywhere between corresponding line segments on l and m. Thus the broken lines l and m are formed on the parallel planes l* and m*, respectively; and these planes, apart as much as $\lambda_x \cos\theta$, are evidently normal to the x axis. By the same process throughout the M's and L's zones, each zone constructs a polyhedral surface that is symmetrical with the adjacencies about the plane l* and m*.

In order to study the characteristics of this cylinder-like polyhedral surface, the quantity which in a broad sense may be called "curvature" is the best conceivable means at present. In a general sense, the mean curvature is definined as the ratio of the angle between the tangents at the edges of a curve to the length of it. By the same token, it may be possible to define the "quasi-curvature" $\langle\kappa_1\rangle$ of the circumference... $s'_{k-1} s_k s'_{k+1}$... as

$$\langle\kappa_1\rangle_k = (\pi - \angle s'_{k-1} s_k s'_{k+1}) / (\overline{s'_{k-1} s_k} + \overline{s_k s'_{k+1}}) \tag{2}$$

It is, however, more convenient for the following analysis to use the angular variation between the line segments $\overline{s_{k-1} s'_k s_{k+1}}$ and $\overline{s_k s'_{k+1} s_{k+2}}$, though these are in different parallel planes. Thus

$$\langle\kappa_2\rangle_k = \beta_k / s'_k s_{k+1} = \beta_k / \lambda_{sk} \tag{3}$$

Evidently, when the broken lines converge to a curve through an infinitesimal division, these quasi-curvatures represent exactly the curvature in a general sense. By the help of Fig.5, the following relation can easily be obtained:

$$\beta_k = 2 \tan^{-1}[(\lambda_x \sin\theta) / \lambda_{sk}] \tag{4}$$

Thus, the quasi-curvature $\langle \kappa_2 \rangle$ is written as

$$\langle \kappa_2 \rangle_k = (2/\lambda_{sk})\tan^{-1}[\,(\lambda_x \sin\theta)\,/\,\lambda_{sk}] \tag{5}$$

Also, the unit shortening in the x axis direction ϵ, and the amplitude of the wave α are given as follows:

$$\epsilon = 1 - \cos\theta \tag{6}$$

$$\alpha = \lambda_x \sin\theta \tag{7}$$

Next, let us consider the limiting case where the line segments of l and m are infinitesimally small while s_j and s_{j+1} are following a given arbitrary smooth continuous function, and at the same time the amplitude of the wave is infinitesimally small such as $\alpha / \lambda_{sk} \ll 1$. In this case, the quasi-curvature $\langle \kappa_2 \rangle$, and therefore $\langle \kappa_1 \rangle$, almost coincide with the curvature in a general sense and it is given by

$$\kappa(s) = 2\lambda_x(2\epsilon)^{1/2} / [\lambda_s(s)]^2 = 2\alpha / [\lambda_s(s)]^2 \tag{8}$$

Simultaneously, the broken lines l's and m's converge to a curve; as an inevitable consequence, the zones M's and L's and then the whole surface converge indefinitely close to a cylindrical surface S, whose curvature is given by $\kappa(s)$. The limiting case of this polyhedral surface is now denoted as S_p.

One distinction between the polyhedral surface S_p and the corresponding cylindrical surface S with an identical curvature distribution is that the former has a uniform axial shortening ϵ everywhere. Also it is evident that the surface S_p is developable as it is constructed by folding a plane. Therefore, it can be said that the surface S and S_p are isometric, or S is developable on S_p.

In conclusion, the presence of a developable surface S_p is verified in relation to a smooth arbitrary cylindrical surface S with the following characteristics: S and S_p are isometric, S_p has a uniform axial shortening ϵ, and S_p is indefinitely close to S. The surface S_p is the concave polyhedral surface and the basic geometrical parameters are governed by Eq.(8). Fig.6 shows an exaggerated view (since the real surface is constructed by the infinite number of infinitesimal waves) of such a surface corresponding to an elliptical cylindrical surface.

It appears also well established that if the transfer from a surface S to the corresponding surface S_p is taken place, it should be done in the inextensional process. If the surface S is referred to the midplane of a thin arbitrary general cylindrical shell, then it is found that the surface S_p satisfies probable conditions for a surface being an inextensional buckling deformation.

In common parlance, this transfer means "the folding of cylindrical surface in the axial direction". In adition, it is indeed infinite in number of the combination of axial and circumferential wave numbers and thus resulting axial shortenings. It is interesting to note that any cylindrical surface can be shortened by an arbitrary amount through such a folding.

There is certainly something in common between this fact and the geometrical paradox that the surface area of a circular cylinder can not be obtained as the simple upper bound of the surface area of the concave polyhedron inscribing the cylinder. Let us consider a two-dimensional Euclidean complex K(p,q) inscribing a circular cylinder and denote F(p,q) as the sum of area of two-dimensional Euclidean simplexes (Fig.7). Converging p and q to zero, we will have K(p,q) indefinitely close to the cylindrical surface. On the other hand, however, the limiting value of F(p,q) depends on a process of convergence of p and q to zero and the value can take from the finite to the infinite.

But to return to the point to our subject, we assumed in the preceding analysis the uniformity of the axial shortening ϵ with regard to both circumferential and axial coordinates. It is, however, possible to relax the latter assumption and give a wide definition of polyhedral surface S_p. Then Eq.(8) is valid for x-dependent λ_x and ϵ, provided that the amplitude of the wave $\alpha = \lambda_x(2\epsilon)^{1/2}$ is kept constant; thus

$$\left.\begin{aligned} &\kappa(s) = 2\lambda_{xi}(2\epsilon_i)^{1/2} / [\lambda_s(s)]^2 \\ &[\alpha = \lambda_{xi}(2\epsilon_i)^{1/2} : \text{constant}] \end{aligned}\right] \qquad (9)$$

where the subscript i indicates the i-th zone in the axial direction. This enables us to make a pattern that resembles the local buckling pattern of the axially compressed circular cylinder shown in Fig.4.

Furthermore, the triangular division appeared in the preceding analysis is not mandatory, and instead, the trapezoidal division yields the similar conclusion. Denoting the bisections of the upper and the lower bases of a trapezoid as λ^s and λ_s, respectively (Fig.8a), the correspondents to Eqs.(5) and (8) can be written as follows:

$$\langle \kappa_2 \rangle = [2/(\lambda_s + \lambda^s)] \tan^{-1}[\lambda_x \sin\theta/(\lambda_s - \lambda^s)] \qquad (10)$$

$$\left.\begin{aligned} &\kappa(s) = 2\lambda_{xi}(2\epsilon_i)^{1/2}/[\lambda_s(s)^2 - \lambda^s(s)^2] \\ &[\ \alpha = \lambda_{xi}(2\epsilon_i)^{1/2} : \text{constant}\ ] \end{aligned}\right] \qquad (11)$$

The resulting "hexagonal-pattern" can be seen in a typical example (Fig.8b). Indeed one will sometimes observe the hexagonal-pattern instead of the Yoshimura-pattern in the collapsing shapes of circular cylindrical shells. A. V. Pogorelov has mentioned about a similar but somewhat different hexagonal pattern configuration in external pressure case. In Figs.9 and 10, the general case of diamond and hexagonal type polyhedral surfaces S_p corresponding to Eqs.(9) and (11), respectively, are shown. It is also possible to furnish a sign change in curvature of polyhedral surface S_p as shown in Fig.11.

The above argument raises a new question whether a surface with another kind of pattern might exist that also belongs to the category of the polyhedral surface S_p. In this respect, the author will not attempt to prove in an exact manner the existence or nonexistence of such a surface, but he simply points out in the following the analogous characteristics between this problem and the classical problem of regular tessellation.

The concave polyhedral surface S_p with either the diamond or the hexagonal pattern can be developed into the plane, as it is the fundamental premise. If the edges of these patterns were "printed" on the surface, while a edge insides of each pattern is excluded, the planes printed with either the diamond or the hexagonal pattern will be obtained after the deployment (Figs.12a and 12b). Then, it is always possible to find the appropriate coordinate transfer as to x and s, by which the pattern is "regularized", that is, each pattern is transferred to the same-sized regular polygon. Such coordinate transfer can be expressed formally as follows:

$$x' = f(x), \qquad s' = g(s) \tag{12}$$

The resulting regular patterns are shown in Figs.12c and 12d.

These patterns remind us the classical problem of regular tessellation studied first by Kepler [7]. For a formal definition, we may say that a tessellation is regular if it has regular faces and a regular vertex figure at each vertex. It has been proved that the triangular, diamond, and hexagonal regular tessellations are possible, and these are the only regular tessellations. Since the diamond and hexagonal tessellations have their counterparts in the patterns of the polyhedral surfaces S_p, the possibility of triangular pattern of S_p is asked now. At present, however, we are unable to find a mechanism by which a cylindrical surface with triangular pattern can be transferred to a polyhedral surface S_p. Therefore, it is most likely that the group of polyhedral surface S_p is characterized by either the diamond or the hexagonal pattern. The mixture of these patterns is possible, but it is not the fundamental pattern.

Since the group of these surfaces has the distinct characteristics, it will be convenient to give them an appropriate designation. The author temporarily uses the word "pseudo-cylindrical concave polyhedral surface", and PCCP surface as an abbreviation, because this surface is not only cylindrical in a macroscopic sense but also it has a cylindrical surface as its limit. It is also assumed that this designation of PCCP surface includes the case where fundamental parameters, λ_x, λ_s, λ^s, and α, are finite. We can say, there is two kinds of PCCP surfaces, one is the PCCP surface with diamond pattern and the other is the PCCP surface with hexagonal pattern. The geometrical characteristics of PCCP surfaces are summarized in the following.

Geometry of PCCP surface (diamond pattern)

(1) The developable concave polyhedral surface composed of triangular faces.

(2) The relations between an arbitrary triangular face 1-2-3 and the three adjacent triangular faces (Fig.13)

a) The one particular sides of every triangles are on the mutually parallel planes. Let us call them the bases of triangles. A line which is vertical to these planes is denoted as x.

b) The two triangular faces, which own a side jointly, make the identical angle to x.

c) The two triangles, which own a base jointly, have the identical orthogonal projection to a plane vertical to x. This orthogonal projection is also a triangle with the same base. The height of it is denoted as the amplitude α. The amplitude α characterizes the depth of the wavy concave polyhedral surface and is constant throughout the whole surface.

d) The x-coordinate of the base, jointly owned by two triangles, is in-between the x-coordinates of vertexes facing to the base.

(3) The macroscopic configuration composed by this surface is cylindrical, thus the quasi-curvature of the surface is approximately given by

$$\kappa\ (s) = 2\alpha / \lambda_s(s)^2$$

Geometry of PCCP surface (hexagonal pattern)

(1) The developable concave polyhedral surface composed of non-rectangular trapezoidal faces.

(2) The relation between an arbitrary trapezoidal face 1-2-3-4 and the four adjacent trapezoidal faces (Fig.14):

a) The bases of every trapezoidal faces are on the mutually parallel planes. A line which is vertical to these planes is denoted by x.

b) The two trapezoidal faces, which own a side jointly, make the identical angle to x.

c) The two trapezoidal faces, which own a base jointly, have the identical orthogonal projection to a plane vertical to x. This orthogonal projection is also a trapezoid and the height of it is denoted as the amplitude α. The amplitude α characterizes the depth of the wavy concave polyhedral surface and is constant throughout the whole surface.

d) The x-coordinate of the base, jointly owned by two trapezoids, is in-between the x-coordinates of the other two bases.

(3) The macroscopic configuration composed by this surface is cylindrical, thus the quasi-curvature of the surface is approximately given by

$$\kappa(s) = 2\alpha / [\lambda_s(s)^2 - \lambda^s(s)^2]$$

§ 3. PROPOSITION OF PSEUDO-CYLINDRICAL CONCAVE POLYHEDRAL SHELLS

A new idea is not necessarily conceived as a logical conclusion by a certain rational mathematical analysis, but rather it is frequently obtained by chance by a process transcending a logical way. This is exactly the process by which the new form of shell structure is introduced from the PCCP surface discussed in the preceding chapter.

For the purpose of illustration of PCCP surface, the author has made a model using 0.25 mm thick Kent paper. This is the one shown in Fig.6, and is representing an elliptic PCCP surface with diamond pattern. Playing unconsciously with the model, the author has noticed the considerable supporting capability of it in the axial direction. This phenomenon may be explained in an approximate manner as follows. The form of this model represents indeed a stable post-buckling equilibrium of a very thin cylindrical shell subjected to an axial load, that is, in other words, a failure configuration under such loading. But the typical load-shortening curves as shown in Fig.15 indicate the positive gradient in the post-buckling region for a fixed combination of axial and circumferential wave numbers. So it is obvious that the model could be capable of supporting substantial loads though it is not as stiff as the original cylindrical form. Therefore, if a thin structure is designed from the first in a form of the PCCP surface, the elastically stable region will be assured until the next form of failure, possibly either the material yield or the local instability, will occur. The author believes that there is indeed a great potentiality of such structures in practice.

The conversion of thought from the failed form of a structure to the potential new form of a structure is really a drastic turn. But an even more drastic turn has been done by the turn, in its literal meaning, as much as ninety degrees of the loading direction. The author has noticed immediately that the model exhibits quite a large rigidity against the load normal to the surface, in macroscopic sense, of it. In spite of the minute thickness of the paper, the model can actually support a substantial load without indicating a sign of collapsing. On the contrary, an elliptic cylindrical model made of the same paper collapses by its own weight. In short, the former has much larger circumferential bending rigidity than the latter.

This odd experience and the consideration of the intrinsic geometrical characteristics of this developable surface, both has convinced the author of that the thin structure in the form of the PCCP surface should have great possibilities for engineering applications [8]. According to precedent, such a category of structures is temporarily denominated as the pseudo-cylindrical concave polyhedral shell and the PCCP shell or CP shell for its abbreviation is used in the text. It is to be hoped that the formal designation of it is established by some authorized organization.

In the past, a folded plate shell which certainly belongs to the family of PCCP shells has been appeard. It is a shell in the configuration of Yoshimura-pattern, that is, a circular PCCP shell with diamond pattern. For instance, Salvadori discribes such a structure in his unique book entitled "Structure in Architecture"[9].

In general, the domain of form of PCCP shell is very wide. As it has already mentioned that the shell can be designed for an arbitrary curvature distribution with either the diamond or the hexagonal pattern. Also, a rather wide selection of independent parameters, which define the form in macroscopic as well as microscopic sense, is possible. Furthermore, as an extremely special case of the shell, the concave polyhedral shell with a plane configuration is possible; this will be shown in the later part of this paper. In the following chapter the characteristics of the PCCP shell is discussed in details.

§ 4. PRINCIPAL CHARACTERISTICS OF PCCP SHELLS

In this chapter we discuss in details the characteristics inherent in PCCP shells. These can be conveniently divided into the following six principal items, though these are closely connected with each other.

4-1. Versatility in forms

It goes without saying that the cylindrical shell form is the most versatile and thereby the widely used shell form. Since the PCCP shell is also in the cylindrical form at least in macroscopic sense, the merit of the cylindrical form over other types of the shell is retained in this case.

The independent form parameters which define the form of PCCP surface are as follows;

$$\alpha \;,\quad \lambda_s(s),\quad \lambda_x(x) \qquad \text{(diamond pattern)}$$

$$\alpha \;,\quad \lambda_s(s),\quad \lambda^s(s),\quad \lambda_x(x) \qquad \text{(hexagonal pattern)}$$

where the amplitude α of the wave is constant throughout the surface, while the wave-length parameters can vary stepwise with the coordinates indicated in parentheses.

The sole quantity which determines the macroscopic form of the shell is the curvature. The quasi-curvature of the PCCP surface is given approximately as follows;

$$\kappa\ (s) = 2\alpha / \lambda_s(s)^2 \qquad \text{(diamond pattern)}$$

$$\kappa\ (s) = 2\alpha / [\lambda_s(s)^2 - \lambda^s(s)^2] \qquad \text{(hexagonal pattern)}$$

As the amplitude α is constant throughout the surface, the arbitrary curvature distribution can be gained by λ_s and λ^s variations along the circumferential coordinate s.

If both α and $\lambda_s(s)^2$, or $\lambda_s(s)^2 - \lambda^s(s)^2$, are infinitesimal numbers of the same order, that is, by Landau's symbol

$$\alpha = O[\lambda_s(s)^2] \qquad (13)$$

$$\alpha = O[\lambda_s(s)^2 - \lambda^s(s)^2] \qquad (14)$$

a PCCP surface converges to a cylindrical surface. At the same time, the quasi-curvature represents exactly the curvature in general sense. In regard to the convergence of surface area, however, the above condition is not sufficient; the reason has been mentioned before in chapter 2. Then, the following condition must be satisfied simultaneously.

$$\alpha / \lambda_x \rightarrow 0 \tag{15}$$

Generally speaking, when the quantity α / λ_x is small, the PCCP shell has almost an identical surface area with the corresponding cylindrical shell; this fact might have technical as well as economic meanings.

Excluding an aesthetic problem of design, considerable differences in technical meaning will be encountered by selecting either of two patterns of PCCP shells. For instance, let us consider the case of designing a PCCP shell when the curvature distribution and the quantity α, on which the circumferential bending rigidity primarily depends, are given in advance, the rest of free parameters are one in case of the diamondpattern, and two in case of the hexagonal pattern; thus the latter provides more freedom in designing the shell.

4-2. Circumferential bending rigidity

Now the comparison is made between a circumferential strip of a circular PCCP shell with the diamond pattern and that of a circular cylindrical shell as shown in Fig.16. The width of both strips is as much as 2 λ_x, that is, a single pattern width. It is also assumed that both strips have the identical uniform thickness. The moment of inertia of a cylindrical shell strip is

$$I = (1/6)\, \lambda_x t^3 \tag{16}$$

The moment of inertia of a PCCP shell strip, I_p, is variable along the circumferential coordinate s. It takes the maximum value at nodal position and the minimum at the middle of nodal position. That is

$$(1/24)\, \lambda_x{}^3 t \sin^2\theta \leq I_p \leq (1/6)\, \lambda_x{}^3 t \sin^2\theta \tag{17}$$

where $\alpha \gg t$ is assumed.

The similar relation holds also in the case of the hexagonal pattern. Therefore, the relative magnitude of the moment of inertia can be written as follows;

$$I_p / I \geq (\alpha / 2t)^2 \tag{18}$$

This comparison on the moment of inertia can not directly be transferred to that of the circumferential bending rigidity of two shells, because the section of the PCCP shell changes periodically along the circumferential direction. The period of its variation is apparently λ_s or $\lambda_s + \lambda^s$ depending on patterns. Also an important fact we should know is that there is a completely different type of the large deformation not familiar with us. If the face angles are changeable, the large deformation is possible even without the elastic deformation of elementary faces. This somewhat peculiar characteristic is in fact essential for a new deployable structure, which will be discussed in the later part of this chapter. In the context of discussion developed here, the face angles are assumed to be constant. Under these circumstances, and as far as the rough qualitative comparison is concerned, the comparison of bending rigidities of two shells by using Eqs.(16), (17), and (18) may be justified. The analytical as well as experimental study of this subject is currently being done in the author's laboratory.

Eq.(18) clearly shows that the potential of having greater circumferential bending rigidity is almost essential in case of PCCP shells. For example, if the amplitude is taken as much as several times of the thickness of the shell, the bending rigidity of a PCCP shell could be approximately 10 times of a regular cylindrical shell. Due to the definition of the shell in general, the amplitude several times of the thickness is still a small amount comparing with other dimensions. Therefore, the PCCP shell, being cylindrical in macroscopic sense, can be designed for a structure with greater circumferential bending rigidity by a relatively small amount of the amplitude.

4-3. Being developable surface

One of the most important characteristics of PCCP shells is that the midsurface is developable. This feature is inherent in the shells with zero Gaussian curvature. Whether the shell is developable or not has a vital influence on the production process of it. Because the manufacturing of an undevelopable shell from the sheet material involves the process entails the extensional deformation of the material, while a developable shell does not.

4-4. Elements are in simple form

The whole structure of PCCP shell can be constructed by simple triangular or trapezoidal faces. This feature as well as the possibility of including a large number of same-sized elements will contribute to rationalize the production process.

4-5. Characteristic as deployable structures

The geometrical fundamental of the deployable surface structure is the inextensional transfer between two surfaces, the two occupying the different expanses in space. Among other things, the deployable surface structure with rigid plane element has important applications; these strctures such as accordion type or fan type structures are examples. It is quite interesting to note that a new mechanism of deployment is possible by using a characteristic of the PCCP surface and assuming every edges of the triangular or trapezoidal elements are revolving hinges. This mechanism is most clearly exhibited by using a paper model as shown in Fig.17. Fig.17a shows a flat plane where the subsequent edge lines of diamond pattern are marked. In other words, the quantities λ_x and λ_s are fixed and α is left unfixed. Give each triangular element a small angle θ to x axis; this is equivalent with giving a small amount of the amplitude α . This process results in the surface shown in Fig.17b. Increasing θ , or α , we have in sequence the surfaces shown in Figs.17c, 17d, and 17e.

From a theoretical point of view, this mechanism is quite different from those of accordion type or fan type deployment, because this is the deployment in two-dimensional while the latter two are one-dimensional deployments. In a sense, the deployment can be defined as the increase in the order of dimensions. Through two-dimensional deployment mechanism, therefore, it is theoretically possible to contract a plane to a point. Then, by patterns with infinitesimal wave lengths, the following conditions are perfecly realizable, that is

$$\epsilon' \longrightarrow 1 \tag{19}$$

$$\kappa \longrightarrow \infty \tag{20}$$

Viewing the models in Fig.17, we could imagine the ultimate result of these conditions as an infinitesimal coil with the axial length of zero.

4-6. Characteristic as bellows

Another distinct characteristic of PCCP shell will be found, if we view the classical problem of axial buckling of cylindrical shells from a different standpoint. Now let us observe Fig.15 again. In this figure the reduced compressed stress $\sigma r/Et$ against unit end shortening $\epsilon r/t$ for $\mu = 1.00$ and different number of waves in circumferential direction is plotted. Where $\eta = n^2t/r$ represents the non-dimensional circumferential wave number. What in particular catches our interest is not the domains of buckling and post-buckling as usual, but the region following the post-buckling. In this region, the gradient of the curves are much smaller than in the pre-buckling region. In other words, the elastic spring constant of the shell in the axial direction is comparatively small. This tendency becomes more pronounced as the circumferential wave number decreases. Based on this fact, we may deduce that a shell designed at first in the form of post-buckling configuration should have a small spring constant in the axial direction. Indeed it can be likened to a bellows. Since the PCCP shell is in a sense the idealization of post-buckling configuration of cylindrical shells, it seems probable that this characteristic as bellows is also inheritted to the PCCP shell. Especially, the PCCP shell with a large angle of inclination exhibits clearly a spring-like behavior against the axial force (Fig.18). A noticeable feature of this bellows-like structure is that its midsurface is developbale. It is also interesting to note that the bellows in usual form represent something like the axisymmetric buckling deformation of circular cylindrical shells.

§ 5. PROPOSITION OF A CONCAVE POLYHEDRAL SHELL WHOSE MACROSCOPIC CONFIGURATION IS A PLANE (DEVELOPABLE DOUBLY CORRUGATED PLATE)

As shown in section 4-6 of the preceding chapter, the PCCP shell with revolving hinges has a remarkably interesting characteristic of deployment mechanism. It suggests a great potential for the futre conception of the structure.

It can be said in that case, that the two-dimensional contraction, which inextensionaly transfers a plane into a point, is realized by a folding in axial direction and a winding in the plane vertical to the axis (Fig.19a). This opens up a new and interesting problem that whether there exists an inextensional transfer by which a plane is contracted into a point by folding in two orthogonal directions in the plane (Fig.19b).

As a first trial to search for such a transfer, let us consider a PCCP surface with zero quasi-curvature. In case of the hexagonal pattern, for example, if we put the condition $\langle \kappa_2 \rangle \rightarrow 0$ into Eq.(10), we have $\lambda_s \longrightarrow \infty$. As this corresponds to a usual corrugation type configuration, this turned out a decided failure.

After due consideration it has been found that a probable key for the solution of the problem is afforded by the arrow-feather like pattern which facilitates the sign change in curvature as shown in Fig.11. By repeating alternatively the diamond pattern and the arrow-feather like pattern as shown in Fig.19c, the original plane can indeed be folded in two orthogonal directions. Incomplete as it is, it is certainly nearing a right solution.

The right solution is obtained by this way. From the configuration of Fig.19c, let's neglect every diamond patterns and draw a new figure composed by the repetition of a "feather pattern" which is resolved into four rhomboids(Figs.19d and 20). If these rhomboidal elements are identical everywhere, though it is not necessarily so, the relationships of geometrical parameters defining the configuration can be obtained by the help of Fig.20. The axial and lateral shortenings λ_x, and λ_y, respectively, are given by

$$\epsilon_x = 1 - \cos\ \theta \qquad (21)$$

$$\epsilon_y = 1 - \cos[\tan^{-1}(\sin\theta\tan\gamma)] \qquad (22)$$

where γ is the smaller interior angle of the rhomboid. The bulk thickness defined in Fig.20 is given by

$$\delta = \lambda_y \sin[\tan^{-1}(\sin\theta\tan\gamma)] \qquad (23)$$

From these equations, the conditions by which the original plane can be transferred to a point is rigorously given by the following formula:

$$\lim\ [\ \theta \to \pi/2,\quad \gamma \to \pi/2,\quad \lambda_y \to 0,\quad \lambda_x \to 0]$$
$$\longrightarrow\ [\ \epsilon_x = 1,\quad \epsilon_y = 1,\quad \delta = 0,\quad \lambda_x = 0] \qquad (24)$$

This is apparently the infinitesimal orthogonal division of a plane. The sequential photographs of a model in Fig.21 vividly explane the remarkable mechanism of this deployment.

One of the most attractive features found in this mechanism is that the intermediately deployed configuration, as shown in Fig.22, has the undulation in two orthogonal directions. In other words, this is a developable doubly corrugated surface. To the author's knowledge, this kind of surface has never been known. It should be noted that this feature of double corrugation, when applied to a rigid type shell configuration, will give it the stiffening effect in two orthogonal directions.

In addition, from a theoretical standpoint, the limiting case of this concave polyhedral surface when λ_x and λ_y are infinitesimally small recieves added interest because of its similarity as well as difference to S_p stated before. The limiting surface and the plane, let's call S_p' and S', respectively, have the following relations; S_p' and S' are isometric, S_p' is indefinitely close to S', and S_p' has uniform shortenings in two orthogonal directions in plane. Thus, S_p' may be an inextensional buckling configuration of plates subject to the bi-axial uniform loadings.

§ 6. APPLICATIVE POSSIBILITY OF PCCP SHELLS

It would be premature to discuss at this stage the applicative possibility of the PCCP shell to structures, because in this study the principal effort is devoted to disclose the geometrical characteristics of the shell and thus our knowledge about the other aspects of it is in a state of infancy. It is, however, useful to do so, since predicting the future possibility based on the present state of knowledge will rose helpful discussions concerning the problem among professionals.

6-1. Rigid shells

It is assumed here that the rigid shell means the PCCP shell with rigid ridges of the elements and is used as the antonym of the developable PCCP shell whose ridges are revolutionable. These characteristics, such as possibility of inclusion of an arbitrary curvature distribution, intrinsically high circumferential bending rigidity, simplicity of elements, being a developable surface, and possibility of including large number of same-sized elements, are favourable for the application of rigid PCCP shells.

As stated before, several vault structures in the form of Yoshimura-pattern, that is, circular PCCP shells of diamond pattern, have been tried before. The newly discovered configurations, that is, the diamond pattern configurations of variable curvature and the hexagonal pattern configurations of both constant and variable curvatures, open up fresh posibilities for this category of folded shells. The inclusion of variable curvature makes it possible to select the best possible form, in macroscopic sense, for strength as well as utility purpose. The use of hexagonal pattern possesses the two advantages over the diamond pattern. First, four ridges concentrate in every vertexes in hexagonal case while six in diamond case; thus the singularities of strength as well as fabrication are less influential in the former case. Secondly, the hexagonal case includes more numbers of geometric parameters, while the design of diamond case is severely restricted.

A few models shown in Figs. 23, 24, and 25 illustrate the applications to large span shell structures. The catenary curvature distribution is used for models of Figs. 23 and 24. The elliptic curvature distribution is used for the one in Fig.25. In addition, Fig.8b can be considered as a model of reservoir.

It should be noted, however, that in these large-sized structures the feature that the midsurface of the shell on the whole is developable is of no importance to fabrication. Moreover, the use of reinforced concrete is open to question. Professor Salvadori of Columbia University, in his private communication to the author, wrote,"if they are prefabricated they require a very large number of joints which must be grouted and require welding of the reinforcing bars...". Indeed, the great advantage of the reinforced concrete is its possibility of forming any non-developable surface and, except in case of the mould fabrication, being a developable surface is non-essential. Therefore, it may be wise to use the materials which are usually obtainable as the form of panels, such as steel, alminum, and plastic materials.

Meanwhile, the geometry of developable doubly corrugated surface disclosed in the previous chapter will without any doubt possess ample potentialities of applications to structures in general. A doubly corrugated plate can be considered as an orthogonally stiffened plate. Its bending stiffnesses in two directions can be varied in wide range by changing its geometric parameters. As for materials, almost any material, such as the reinforced concrete, metals, plastics, and even papers can be applicable. Also, it is considered possible to extend the concept of double corrugation to the non-developable but similar configuration.

In addition, it is almost self-evident that the characteristics of these rigid shells are very suitable for the core structure of sandwitch structures.

6-2. Deployable structures and irreversible deployable structures

As stated before, the PCCP shell having revolving hinges is a deployable structure. A structure in the form of Yoshimura-pattern, that is, the circular PCCP shell with diamond pattern, may be appeared before. As in the case of rigid shells, the deployable structures in the forms of the diamond pattern configurations of variable curvature, the hexagonal pattern configurations of both constant and variable curvatures, and the feather pattern configuration of double corrugation seems novel.

There are two principal directions for use of these types of deployable structures. The first group simply uses the transfer from the small expanse to the large expanse, or its reverse action. No strength consideration is necessary. Such example is the panel structure for solar cell mounting which is deployed in the outer space. Since there is no gravity force, the strength of the structure is unnecessary. On the contrary, the second group makes good use of the strength characteristics of the PCCP shell at some intermediate stage of deployment. As a typical example of this kind of structure, a case of deployable vault structure is shown in Fig.26.

There are some cases where the repeating cycles of deployment and folding is not necessary. The deployable structure, whose hinges are temporary and after the deployment they are cemented to make a monocoque PCCP shell structure, is indeed practicable.

§ 7. CONCLUSION

A new category of concave polyhedral surfaces, which might be potential structural forms of the folded plate shells, is presented. The macroscopic configurations of these surfaces are general cylindrical shape and plane shape, and these are developable surfaces. These have an intrinsic geometrical nature which provides a novel mechanism of deployment. It has been shown that the resulting new category of folded plate shell has many useful characteristics as follows; inclusion of an arbitrary curvature distribution, its midsurface being developable, intrinsically high bending rigidity, simplicity of elementary faces. The application of it to rigid shell structures, deployable structures, irreversible deployable structures, and sandwitch core structures gives promise for the future.

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SYMBOLS

l, m	=	parallel lines
l*, m*	=	parallel planes
n	=	circumferential wave number
p, q	=	base and height of two-dimensional Euclidean simplex, respectively
r	=	radius of circular cylindrical shell
s	=	circumferential coordinate
s' = g(s)	=	coordinate transfer
t	=	thickness of shell
w	=	radial inward displacement
x	=	axial coordinate
x' = f(x)	=	coordinate transfer
A_{00}, A_{11}, A_{20}, A_{02}	=	coefficients
E	=	Young's modulus
F(p, q)	=	sum of area of two-dimensional Euclidea simplexes
I	=	moment of inertia of cylindrical shell in circumferential direction
I_p	=	moment of inertia of PCCP shell in circumferential direction
K(p, q)	=	two-dimensional Euclidean complex
L, M	=	parallel zones
S	=	general cylindrical surface
S_p	=	concave polyhedral surface
α	=	radial amplitude of concave polyhedral surface S_p
β	=	angle
γ	=	angle
δ	=	bulk thickness of corrugated surface
ϵ	=	unit shortening
ξ , ζ	=	numbers
η	=	circumferential wave number parameter
θ	=	angle of inclination of elementary face to x axis
κ	=	curvature
$\langle \kappa_1 \rangle$	=	quasi-curvature of circumference, Eq.(2)
$\langle \kappa_2 \rangle$	=	quasi-curvature of circumference, Eq.(3)

λ_x = half wave-length of buckle in axial direction

λ_s = half wave-length of buckle in circumferential direction

λ_s, λ^s = bisections of the lower and upper bases of trapezoids composing a hexagonal pattern, respectively

λ_y = y-directional length of rhomboid composing a feather-pattern

μ = aspect ratio of wave

ρ = radius

σ = average compressive stress in axial direction

s_j, s_{j+1} ($j = \ldots, k-2, k, k+2, \ldots$) = circumferential coordinates of nodes of concave polyhedral surface

1, 2, 3,...= nodes and semi-nodes of PCCP surface