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CONTENTS

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thanks to Prof. R. Kronig for many
grateful to Prof. H. B. G. Casimir

Eindhoven, January 1952

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SOME REMARKS CONCERNING CLOSE-PACKING OF EQUAL SPHERES

by A. H. BOERDIJK

513.468

Summary

For estimating the mean density in local regions of configurations of equal spheres three criteria are stated. Some configurations are described, which have in certain regions a local mean density exceeding that of close-packing. These regions may even have an infinite volume. Further it is proved that the maximal number of spheres simultaneously touching a sphere is twelve. A conjecture of Fejes concerning a fourteenth sphere added to this configuration is shown to be false.

Resumé

Définition de trois critères pour estimer la densité moyenne dans des régions locales des configurations des sphères identiques. Quelques configurations sont décrites ayant une densité locale dans certain régions dépassant celle de l'empillage plus dense des sphères identiques. Ces régions peuvent avoir une volume infinie. Ensuite on prouve que le nombre maximum des sphères tangentes à une sphère est douze. Une supposition de Fejes concernant une quatorzième sphère adjointe à cette configuration paraît incorrecte.

Zusammenfassung

Drei Kriterien werden aufgestellt, die dazu dienen die Dichte von Kugelpackungen (Kugeln gleichen Durchmessers) in endlich großen Bereichen miteinander zu vergleichen. Einige Anordnungen werden angegeben, die gemäß dieser Kriterien in gewissen Bereichen eine größere Dichte aufweisen als die bekannte dichteste gitterförmige Kugelpackung. Diese Bereiche können selbst einen unendlich großen Rauminhalt haben. Weiter wird bewiesen, daß die größte Anzahl der Kugeln, die gleichzeitig eine Kugel berühren können, zwölf ist. Es wird bewiesen, daß eine Vermutung von Fejes über eine vierzehnte Kugel, zu diesen dreizehn Kugeln hinzugefügt, falsch ist.

1. Criteria for estimating the mean density in local regions

Let us consider the configuration of equal spheres (of diameter 1) known as "close-packing" or "normal piling". If the density inside the spheres is unity, and zero outside the spheres, the mean density D of the configuration will be

$$D = (\pi/6)\ \sqrt{2} \approx 0{\cdot}7405\,. \qquad (1)$$

Is this the maximal mean density obtainable in any possible configuration of equal spheres, or may this density be surpassed, either in all space or locally? It may well be surpassed locally. For in close-packing we can extend the configuration uniformly in any direction over any distance and we shall see that there are denser local configurations that cannot be so extended.

For the investigation of this problem we shall need means for estimating the densities of configurations in local regions. We shall give three criteria. I. One measure for the local mean density is the number N of centres of spheres of the configuration located inside or on a sphere with radius R and centre M.

II. Another measure for the density is the mean density d in such a sphere.

These criteria are correlated inter alia in the following way. If we have two configurations C and C' for which

$$N \geqslant N', \quad (0 \leqslant R < R_1)$$

and

$$N > N', (R_1 \leqslant R < R_2),$$

then it follows easily that

$$d > d', (R_1 - \tfrac{1}{2} < R < R_2 - \tfrac{1}{2}). \tag{2}$$

The third criterion is based on the following argument. In close-packing the planes through the centres of adjacent spheres form a quasi-regular honeycomb. Each centre is surrounded by six regular octahedra and eight regular tetrahedra, both (in our case) having an edge-length 1. Calculating the mean densities d_4 and d_8 for the space enclosed by a tetrahedron and an octahedron we find respectively

$$d_4 = (6\varphi - \pi)\sqrt{2} \approx 0{\cdot}7797, \tag{3}$$

$$d_8 = (\tfrac{3}{4}\pi - 3\varphi)\sqrt{2} \approx 0{\cdot}7209, \tag{4}$$

where

$$\varphi = \arcsin \tfrac{1}{3}\sqrt{3} \approx 35°16', \tag{5}$$

2φ being the dihedral angle of a regular tetrahedron.

It is rather surprising to find that d_4 and d_8 differ by 8%. On the other hand it is a known fact from crystallography (see also [1])) that the diameter of the largest possible sphere inside such an octahedron (which sphere touches 6 spheres of the configuration) is almost twice that for the largest sphere inside a tetrahedron (touching 4 spheres of the configuration).

In close-packing the number of tetrahedra turns out to be twice the number of octahedra, whereas the volume of each tetrahedron is one fourth of that of each octahedron [2]). As a check we have indeed from (1,3,4)

$$D = \tfrac{1}{3}(d_4 + 2d_8). \tag{6}$$

The proportion of the numbers of tetrahedra and octahedra in a configuration is a good measure for the density. At least this should be so in the inside of the configuration, for there all this space is made up of tetrahedra and octahedra. This measure will be accurate, even if the tetrahedra and octahedra are not regular, but do not deviate "too much" from regularity, as turns out to be the case in all configurations to be considered later. For practical reasons we divide the octahedron into two four-sided pyramids. This leads to our third criterion.

III. If the numbers of three- and four-sided pyramids in a configuration

are t and f respectively, t/f can be regarded as a
density of that configuration.

For close-packing we have $t/f = 1$.

2. The density of close-packing and locally d

We will now consider six different configur
central sphere. For M we choose the centre of
configurations by addition of other spheres, i
tances from M. All n spheres of which the ce
from M form together a "shell". So we hav

$$N = \sum_0^R n.$$

The configurations 1, 2 and 3 occur in clo
conformable hexagonal close-packing, 3 is c
close-packing and 2 is a mixture of the two t
succession of planes of largest density is of t
b-c-a-b-a, in 3 of type b-c-a-b-c (M being in t
$R \leqslant \sqrt{3}$ which will be considered here, thes
possible configurations with a central sph

Configuration 4 consists of 12 spheres to
their centres on the vertices of a cuboctahe
3). Six spheres are added with their centre
square faces of the cuboctahedron and 8 sph
to the triangular faces.

Configuration 5 consists of 12 spheres, to
their centres on the vertices of a regular
do not touch each other [3]), [4]) as the edge-l

$$l_{20} = \frac{4}{\sqrt{10 + 2\sqrt{5}}} \ldots$$

Twenty spheres are added with their cent
faces of the icosahedron. For this shell R

$$a = \tfrac{2}{3}\,\frac{3\sqrt{3} + \sqrt{15} \ldots}{\sqrt{10 + 2\sqrt{5}} \ldots}$$

In configuration 6 we have one spher
its "north pole". Five spheres are added o
pole sphere, the second touching the firs
second one, and so on. Between the firs
spheres there will be a gap, as the dihedral

are t and f respectively, t/f can be regarded as a measure for the local mean density of that configuration.

For close-packing we have $t/f = 1$.

2. The density of close-packing and locally denser configurations

We will now consider six different configurations 1, 2, ..., 6, all with a central sphere. For M we choose the centre of that sphere. We build up the configurations by addition of other spheres, in the order of increasing distances from M. All n spheres of which the centres have the same distance from M form together a "shell". So we have

$$N = \sum_{0}^{R} n\,. \tag{7}$$

The configurations 1, 2 and 3 occur in close-packing. Configuration 1 is conformable hexagonal close-packing, 3 is conformable face-centred cubic close-packing and 2 is a mixture of the two types. In other words, in 1 the succession of planes of largest density is of the type *a-b-a-b-a*, in 2 of type *b-c-a-b-a*, in 3 of type *b-c-a-b-c* (M being in the middle plane). In the region $R \leqslant \sqrt{3}$ which will be considered here, these three turn out to be the only possible configurations with a central sphere occurring in close-packing.

Configuration 4 consists of 12 spheres touching the central sphere, with their centres on the vertices of a cuboctahedron (as in configurations 2 and 3). Six spheres are added with their centres on the symmetry axes of the square faces of the cuboctahedron and 8 spheres are added in the same way to the triangular faces.

Configuration 5 consists of 12 spheres, touching the central sphere, with their centres on the vertices of a regular icosahedron. These 12 spheres do not touch each other [3], [4]) as the edge-length of the icosahedron will be

$$l_{20} = \frac{4}{\sqrt{10 + 2\sqrt{5}}} \approx 1\cdot05\,. \tag{8}$$

Twenty spheres are added with their centres on the symmetry axes of the faces of the icosahedron. For this shell R will be

$$a = \tfrac{2}{3}\,\frac{3\sqrt{3} + \sqrt{15}}{\sqrt{10 + 2\sqrt{5}}} \approx 1\cdot59\,. \tag{9}$$

In configuration 6 we have one sphere touching the central sphere in its "north pole". Five spheres are added one by one, all touching the north-pole sphere, the second touching the first one, the third one touching the second one, and so on. Between the first one and the last one of these 5 spheres there will be a gap, as the dihedral angle 2φ of a regular tetrahedron

TABLE I

Shell nr	R	Num. value		Configuration 1	2	3	4	5	6
0	0	0·00	n,N	1	1	1	1	1	1
1	1	1·00	n	12	12	12	12	12	12
			N	13	13	13	13	13	13
			t	8	8	8	8	20	10
			f	6	6	6	6	0	5
			t/f	1·33	1·33	1·33	1·33	∞	2·00
2	b	1·35	n	0	0	0	0	0	1
			N	13	13	13	13	13	14
			t	8	8	8	8	20	10
			f	6	6	6	6	0	6
			t/f	1·33	1·33	1·33	1·33	∞	1·67
3	$\sqrt{2}$	1·41	n	6	6	6	6	0	4
			N	19	19	19	19	13	18
			t	8	8	8	8	20	10
			f	12	12	12	12	0	10
			t/f	0·67	0·67	0·67	0·67	∞	1·00
4	a	1·59	n	0	0	0	0	20	0
			N	19	19	19	19	33	18
			t	8	8	8	8	40	10
			f	12	12	12	12	0	10
			t/f	0·67	0·67	0·67	0·67	∞	1·00
5	$\frac{2}{3}\sqrt{6}$	1·63	n	2	1	0	8	0	10
			N	21	20	19	27	33	28
			t	10	9	8	16	40	20
			f	12	12	12	12	0	10
			t/f	0·83	0·75	0·67	1·33	∞	2·00
6	$\sqrt{3}$	1·73	n	18	21	24	0	0	0
			N	39	41	43	27	33	28
			t	28	30	32	16	40	20
			f	24	26	28	12	0	10
			t/f	1·17	1·15	1·14	1·33	∞	2·00

Table of the values of n, N, t and f as functions of R (from 0 to $\sqrt{3}$) for the configurations 1, 2, …, 6 described in the text.
R = distance of the centres of the spheres from the centre M of the central sphere
n = number of spheres with the same value of R
N = total number of spheres of which the centres are at a distance not larger than R from M
t = number of three-sided pyramids contained in the configuration
f = number of four-sided pyramids contained in the configuration a and b are defined by eq. (9) and (10) respectively.

(equation (5)) is somewhat smaller than $2\pi/$
causing the configuration to be symmetrical
plane of the central sphere. The centres of th
in the vertices of an irregular polyhedron w
triangular, 4 square and 1 rectangular. Fi
with their centres on or close to the symm
of these spheres have their centres in the
sphere. The centre of the one facing the ga
distance R from M equal to

$$b = \sqrt{3} \cos (\pi - 4\varphi)$$

For all other shells the values of R and n
will not give them separately. Table I con
N, t, f and t/f for R from 0 to $\sqrt{3}$.
We shall indicate a part of each of these
the configuration followed by the number
a sphere touched by 12 others, with their c
tahedron.
From the table it follows that the configu
have a larger local mean density than c
occurring in close-packing, according to I
tions 4-5 and 6-2 have for a certain regio
to our criterion II, as follows from (2). It
$0 \leqslant R < \sqrt{3}$ the local mean density accord
4 never falls below that of close-packing.
It turns out to be impossible to extend
6 over a larger part of space, without caus
for some shells. This effect is so strong t
tended, the density of such configuration
packing.
Comparison of configurations with ot
regular tetrahedron as a centre, leads to
too the density of close-packing can be s

3. Configurations denser than close-pack

In the configuration of face-centered
element of two adjacent tetrahedra ca
edge, but never a face. In hexagonal cl
face. However, two such tetrahedra ha
other faces in common with other tetr
build configurations of equal spheres,

TABLE I

	Configuration 1	2	3	4	5	6
n,N	1	1	1	1	1	1
n	12	12	12	12	12	12
N	13	13	13	13	13	13
t	8	8	8	8	20	10
f	6	6	6	6	0	5
t/f	1·33	1·33	1·33	1·33	∞	2·00
n	0	0	0	0	0	1
N	13	13	13	13	13	14
t	8	8	8	8	20	10
f	6	6	6	6	0	6
t/f	1·33	1·33	1·33	1·33	∞	1·67
n	6	6	6	6	0	4
N	19	19	19	19	13	18
t	8	8	8	8	20	10
f	12	12	12	12	0	10
t/f	0·67	0·67	0·67	0·67	∞	1·00
n	0	0	0	0	20	0
N	19	19	19	19	33	18
t	8	8	8	8	40	10
f	12	12	12	12	0	10
t/f	0·67	0·67	0·67	0·67	∞	1·00
n	2	1	0	8	0	10
N	21	20	19	27	33	28
t	10	9	8	16	40	20
f	12	12	12	12	0	10
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n	18	21	24	0	0	0
N	39	41	43	27	33	28
t	28	30	32	16	40	20
f	24	26	28	12	0	10
t/f	1·17	1·15	1·14	1·33	∞	2·00

nctions of R (from 0 to $\sqrt{3}$) for the configurations 1,

spheres from the centre M of the central sphere
e value of R
h the centres are at a distance not larger than R

contained in the configuration
ontained in the configuration a and b are defined

(equation (5)) is somewhat smaller than $2\pi/5$. Six more spheres are added causing the configuration to be symmetrical with respect to the equatorial plane of the central sphere. The centres of the 12 outer spheres are arranged in the vertices of an irregular polyhedron with 15 faces, of which 10 are triangular, 4 square and 1 rectangular. Fifteen more spheres are added with their centres on or close to the symmetry axes of these faces. Five of these spheres have their centres in the equatorial plane of the central sphere. The centre of the one facing the gap mentioned above will have a distance R from M equal to

$$b = \sqrt{3}\cos(\pi - 4\varphi) \approx 1{\cdot}347. \tag{10}$$

For all other shells the values of R and n can be found easily, so that we will not give them separately. Table I contains for the 6 configurations n, N, t, f and t/f for R from 0 to $\sqrt{3}$.

We shall indicate a part of each of these configurations by the number of the configuration followed by the number of the outer shell. So 3-1 will be a sphere touched by 12 others, with their centres in the vertices of a cuboctahedron.

From the table it follows that the configurations 4-5, 5-4, 5-5, 6-2 and 6-5 have a larger local mean density than configurations of the same type occurring in close-packing, according to I and III; moreover the configurations 4-5 and 6-2 have for a certain region of R a larger density according to our criterion II, as follows from (2). It is remarkable that in the region $0 \leqslant R < \sqrt{3}$ the local mean density according to all criteria of configuration 4 never falls below that of close-packing.

It turns out to be impossible to extend any of the configurations 4, 5 and 6 over a larger part of space, without causing the density to decrease locally for some shells. This effect is so strong that it seems unlikely that, if extended, the density of such configurations will be larger than that of close-packing.

Comparison of configurations with other types of centre, e.g., with a regular tetrahedron as a centre, leads to similar conclusions. In these cases too the density of close-packing can be surpassed locally.

3. Configurations denser than close-packing in regions with infinite volume

In the configuration of face-centered cubic close-packing the common element of two adjacent tetrahedra can be a vertex, or at the most an edge, but never a face. In hexagonal close-packing this element may be a face. However, two such tetrahedra having a face in common never have other faces in common with other tetrahedra. Of course it is possible to build configurations of equal spheres, of which the centres form the vertices of tetrahedra in such a way that we could visit each tetrahedron by perforating common faces of tetrahedra only. Such configurations are of interest, since in a connected part of space with a volume exceeding that of two tetrahedra they will have a local mean density exceeding that of close-packing according to criterion III.

As a typical example we take four spheres touching each other. Their centres are the vertices of a regular tetrahedron. We add four spheres with their centres on the symmetry axes of the faces. So we obtain five tetrahedra, forming a polyhedron with $4 \times 3 = 12$ faces. Six more spheres can be added with their centres on the symmetry axes of 6 of these 12 faces. Now $t = 17, f = 0$. Of course such a configuration will not occur in close-packing. Alternatively, it can be regarded as built up from "five-rings", consisting of five spheres of which the centres are arranged on a circle of radius of $\frac{1}{2}\sqrt{3} \approx 0.865$. These five-rings play here the role of the configurations of six spheres of which the centres are arranged on a circle of radius 1, in the planes of maximal density in close-packing ("six-rings"). Here our configuration contains 6 of such five-rings.

Let us compare the six-ring and the five-ring. In the former we can place one sphere, touching all 6 spheres. In the latter two spheres can be placed, each touching the plane of the centres of the 5 spheres, and also the 5 spheres themselves. Both configurations so obtained contain 7 spheres. In the six-ring there are 12 contact points, whereas this number may be 15 ("almost 16") for the five-ring. The six-ring is the base of six regular tetrahedra in close-packing, whereas the five-ring may contain four regular tetrahedra and a slightly irregular one in itself.

The configuration considered above has been built around a point (the centre of the first tetrahedron). It turns out to be possible to build up close configurations along a straight line also. We will consider three types of such "needles".

There is a needle that contains regular tetrahedra only. This is possible because a regular tetrahedron can be screwed around a certain axis in such a way that one of its faces in the original position coincides with another of its faces in the second position. In fact a regular tetrahedron has 12 of such screw-axes, 6 right-handed and 6 left-handed. The lines joining the mid-points of opposite edges in a regular tetrahedron are mutually perpendicular and all pass through the centre of the tetrahedron. Each of the screw-axes intersects one of these lines perpendicularly in a point located at a distance of $\frac{1}{10}\sqrt{2}$ from the centre of the tetrahedron and encloses an acute angle arctan $3 \approx 71°34'$ with the direction of the nearest of the two opposite edges. In fig. 1 we have drawn a regular tetrahedron projected onto a plane at right angles to one of its screw-axes Z. Along this axis identical tetrahedra can be piled so that they fit together exactly (fig. 2a).

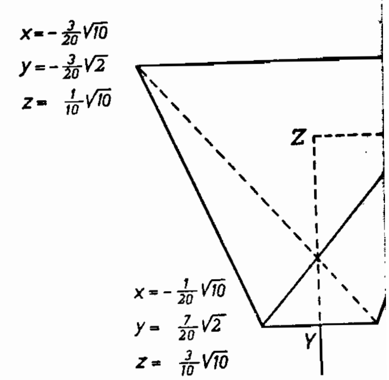

Fig. 1. Projection of a regular tetrahedron of edge
to one of its 12 screw-axes.

Each tetrahedron is rotated relative to it
$\alpha = 2$ arctan $\sqrt{5} \approx 131°49'$ around Z. T
direction of the screw-axis is $\frac{1}{10}\sqrt{10}$. It ap
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A second more complicated needle ma
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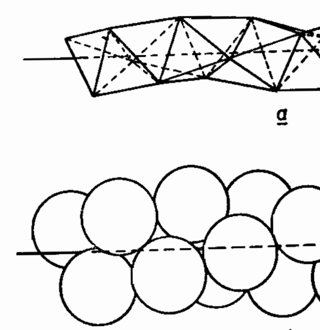

Fig. 2. a. Regular tetrahedra piled along a strai
This line is a screw-axis common to all these tetra
by equal spheres of radius 1, of which the centr
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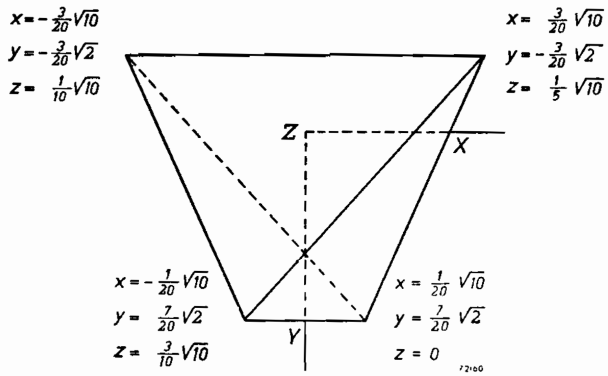

Fig. 1. Projection of a regular tetrahedron of edge-length 1 onto a plane at right angles to one of its 12 screw-axes.

Each tetrahedron is rotated relative to its neighbours through an angle $\alpha = 2 \arctan \sqrt{5} \approx 131°49'$ around Z. The relative displacement in the direction of the screw-axis is $\frac{1}{10}\sqrt{10}$. It appears that α is incommensurable with π. So the orientations of no two tetrahedra can be the same. Placing equal spheres of diameter 1 with their centres on the vertices of the tetrahedra we obtain the first needle configuration (fig. 2b). In this type of needle every sphere has 6 contact points. Since the 12 screw-axes occur in 6 pairs of members crossing at right angles two such needles can cross each other at right angles, having in common four spheres with their centres on the vertices of one tetrahedron.

A second more complicated needle may be obtained by adding to each of the two free faces of each tetrahedron in the above-mentioned needle

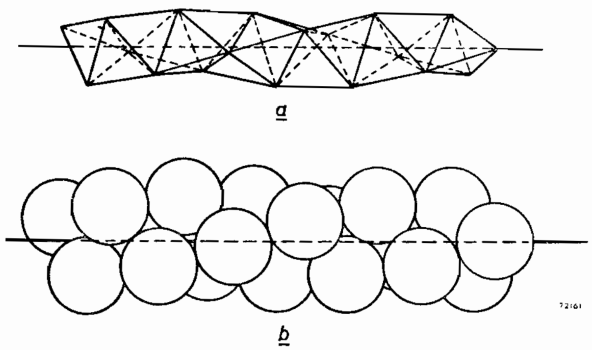

Fig. 2. a. Regular tetrahedra piled along a straight line so that they fit together exactly. This line is a screw-axis common to all these tetrahedra. b. First needle configuration formed by equal spheres of radius 1, of which the centres are on the vertices of the tetrahedra of fig. 2a.

a sphere with its centre on the symmetry axis of that face. This configuration contains interwoven five-rings.

A third type of needle can be obtained by piling identically oriented five-rings, filled up by spheres with their centres on a straight line at distances 1. Our configuration 6-1 is a part of this needle.

The latter needle is especially interesting, since it can be used as a building stone for a close plane configuration. In fig. 3 the plane of this configuration and the axes of the needles are perpendicular to the plane of the drawing.

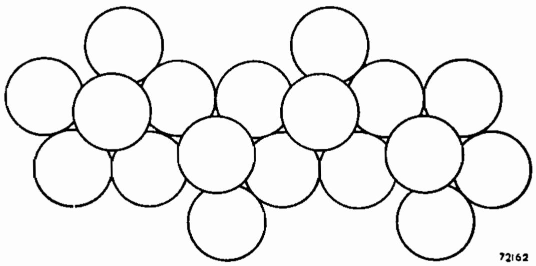

Fig. 3. Close plane configuration obtained by piling needle configurations of the third type with alternate orientations of the five-rings. The plane and the axes of the needles are perpendicular to the plane of the drawing.

To investigate the density of the above-mentioned configurations our criteria I and II can be extended in an obvious way. Instead of considering the volume of a sphere of radius R we have to consider the volumes of a cylinder of unit length and radius R, or that of a rectangular parallelopiped with edge-lengths 1, 1 and $2R$. In this way we find that in the needles and in the plane configurations the local mean density surpasses that of any configurations of the same type occurring in close-packing, according to one or more of our three criteria.

Piling of the plane configuration, or parts of it, leads to configurations filling all space. The mean density however does not exceed that of close-packing. Further investigation shows that it is likely that 3/4 of all space could be filled by parallel needles of the first type. In that 3/4 part the mean density is about 0·73. This is rather high, considering the absence of periodicity in the direction of the axes of the needles.

4. Proof that the maximal number of spheres simultaneously touching a sphere is twelve

Fejes [3]) has shown that for arbitrary configurations of equal spheres we must have for the mean density

$$D < \frac{4\pi}{30\sqrt{2(65-29\sqrt{5})}} < 0{\cdot}7547\,. \qquad (11)$$

This inequality of course only holds for the
So (3) is not in contradiction with it. His pr
that no more than 12 spheres can touch a
equal diameters). He considers this an exp
here a brief account of a proof of this assum

Using central projection, with the centre o
we can transform the problem to a spheri
follows. How many circles with a geodetical
on the surface of a sphere with radius 1?
overlap.

The area of each of the circles will be

$$A_1 = 2\pi\,(1-\tfrac{1}{2}\sqrt{3}) >$$

The number C of circles therefore cannot

$$\frac{4\pi}{A_1} = \frac{2}{1-\frac{1}{2}\sqrt{3}}$$

i.e. C will be 14 or less.

With each of the circles however a certai
As we have for the dihedral angle 2φ of a r

$$\frac{2\pi}{6} < 2\varphi <$$

the number of circles that can simultaneo
exceed 5. The closest configuration in t
a certain circle will be like fig. 4. This is a
drawn in stereographic projection, in whic
es the sphere at the centre of the central
areas A_2 is associated in the same way w

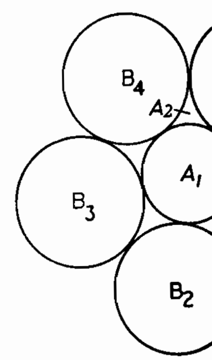

Fig. 4. Stereographic projection of a part of th
6 equal circles of radius arcsin ½ have been dra

ymmetry axis of that face.This configuration
be obtained by piling identically oriented
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is a part of this needle.
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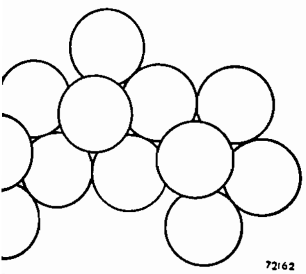
tion. In fig. 3 the plane of this configuration
perpendicular to the plane of the drawing.



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$$\frac{\ldots\pi}{\ldots-29\sqrt{5})} < 0{\cdot}7547\,. \qquad (11)$$

This inequality of course only holds for the mean density over all space. So (3) is not in contradiction with it. His proof is based on the assumption that no more than 12 spheres can touch a central sphere (all spheres of equal diameters). He considers this an experimental fact. We shall give here a brief account of a proof of this assumption.

Using central projection, with the centre of the central sphere as a centre, we can transform the problem to a spherical one. This can be stated as follows. How many circles with a geodetical radius of arcsin $\frac{1}{2}$ can be drawn on the surface of a sphere with radius 1? Of course the circles may not overlap.

The area of each of the circles will be

$$A_1 = 2\pi\,(1 - \tfrac{1}{2}\sqrt{3}) > 0{\cdot}8417\,. \qquad (12)$$

The number C of circles therefore cannot exceed

$$\frac{4\pi}{A_1} = \frac{2}{1-\frac{1}{2}\sqrt{3}} < 15\,, \qquad (13)$$

i.e. C will be 14 or less.

With each of the circles however a certain empty area will be associated. As we have for the dihedral angle 2φ of a regular tetrahedron

$$\frac{2\pi}{6} < 2\varphi < \frac{2\pi}{5}\,, \qquad (14)$$

the number of circles that can simultaneously touch a certain circle cannot exceed 5. The closest configuration in the immediate neighbourhood of a certain circle will be like fig. 4. This is a part of the surface of the sphere drawn in stereographic projection, in which the plane of the drawing touches the sphere at the centre of the central circle in the drawing. Each of the areas A_2 is associated in the same way with each of the three circles sur-

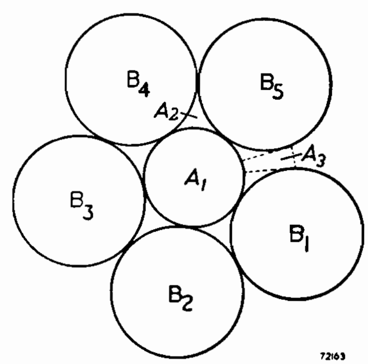

Fig. 4. Stereographic projection of a part of the surface of a sphere of radius 1, on which 6 equal circles of radius arcsin $\frac{1}{2}$ have been drawn, five of which touch a central one.

rounding it. As there are 5 of such areas, the empty area associated with the central circle will be at least

$$\tfrac{5}{3}\,A_2 = \tfrac{5}{3}\left(6\varphi - \pi - \frac{6\varphi}{2\pi}\,A_1\right) > 0{\cdot}0942\,. \tag{15}$$

It is obvious that this empty area will be associated with each of the circles. So C cannot exceed

$$\frac{4\pi}{A_1 + \tfrac{5}{3}\,A_2} < 14\,, \tag{16}$$

i.e. C will be 13 or less.

An investigation shows that with each of the circles also an empty area A_3 will be associated, which area has a radial shape. Around the centre of the central circle it will have an extension

$$\gamma = 2\pi - 10\varphi > 7°21'\,. \tag{17}$$

Of course it may be divided into several sectors by spacing the 5 circles in an arbitrary way along the circumference of the central circle. This will not influence the total angular extension of the sum of the radial areas. The radial length however will have a minimal value if this empty area is not divided between the circles. In that case it will extend over a distance of

$$\delta = \arctan \{\sqrt{3}\cos(\pi - 4\varphi)\} > 53°24' \tag{18}$$

from the centre of the central circle. So we have

$$A_3 \geqslant \gamma\,(\tfrac{1}{2}\sqrt{3} - \cos\delta) > 0{\cdot}0346\,. \tag{19}$$

It turns out that the area A_3 can never be effectively common to two or more circles, nor can any part of it. This can be seen as follows. Building the configuration further we can consider the circle B_1 as a new centre. We have already three circles (A_1, B_2 and B_5) around B_1. Of the other two one (say B_6) could touch B_1 and B_2, the other one (B_7) could touch B_1 and B_5. Then there must be a gap between B_6 and B_7. Calculation shows that this gap gives a bigger empty space than A_3. Of course we may displace the circles, but this only displaces the gaps and does not make them smaller. Therefore C cannot exceed

$$\frac{4\pi}{A_1 + \tfrac{5}{3}\,A_2 + A_3} < 13\,, \tag{20}$$

i.e. C will be 12 or less.

Returning to our original problem we find that the maximal number C of spheres that may simultaneously touch a sphere (all spheres of equal diameters) is 12. This result is in agreement with the suggestion of Schütte

and van der Waerden [5]) (i.e. that for 13
of the central sphere has to be 1·045 at
the assertion of Gregory (in an unpubli
13 equal nonoverlapping spheres can t
diameter. Since each of the contact poi
the number of contact points for any
exceed $6K$. So in close-packing it is alr

5. Proof that the conjecture of Fejes co
to a configuration of twelve spheres

Fejes states as a conjecture that, if
(all spheres of diameter 1), a 14th sphe
have a distance from the centre of th
He mentions "a crude experiment" in s
Apparently he has considered a config
5-1, to which the 14th sphere is added
axis of one of the faces of the icosahedr
sphere until the other 12 spheres are i
surface of the central sphere. This con
jecture.

From our configuration 6-2 and fro
is false, as the 14th sphere is here at
of the central sphere. We have not su
tances are impossible. With the aid of
of a lower limit of this distance. This

We express our gratitude to Dr Wi
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REFER

1) D. R. Hudson. Density and packing in a
20, 154-162, 1949.
2) H. S. M. Coxeter. Regular Polytopes, I
3) L. Fejes. Über die dichteste Kugellage
4) A. H. Boerdijk, Thesis Delft 1951, stell
5) K. Schütte and B. L. van der Waerd
9 Punkte mit Mindestabstand Eins Platz
6) H. S. M. Coxeter, reviewing 5) in Math.

and van der Waerden [5]) (i.e. that for 13 equal outer spheres the diameter of the central sphere has to be 1·045 at least), but in contradiction with the assertion of Gregory (in an unpublished note mentioned in [6])) that 13 equal nonoverlapping spheres can touch another sphere of the same diameter. Since each of the contact points is associated with two spheres, the number of contact points for any configuration of K spheres cannot exceed $6K$. So in close-packing it is already maximal [2]).

5. Proof that the conjecture of Fejes concerning a fourteenth sphere added to a configuration of twelve spheres touching a central sphere is false

Fejes states as a conjecture that, if a sphere is touched by 12 spheres (all spheres of diameter 1), a 14th sphere added to this configuration will have a distance from the centre of the central sphere in excess of 1·38. He mentions "a crude experiment" in support of his conjecture.

Apparently he has considered a configuration similar to our configuration 5-1, to which the 14th sphere is added with its centre on the symmetry axis of one of the faces of the icosahedron and pressed towards the central sphere until the other 12 spheres are in their most close position on the surface of the central sphere. This configuration indeed supports his conjecture.

From our configuration 6-2 and from (10) however it is clear that this is false, as the 14th sphere is here at a distance of 1·347 from the centre of the central sphere. We have not succeeded in proving that shorter distances are impossible. With the aid of (20) an estimation may be obtained of a lower limit of this distance. This is much lower than 1·347.

We express our gratitude to Dr Wise for informing us in detail of the inspiring contents of his paper on close-packing of unequal spheres.

Eindhoven, January 1952

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6) H. S. M. Coxeter, reviewing [5]) in Math. Rev. **13**, 61, 1952.