Multilaterals in configurations

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Abstract

We investigate the existence of g-laterals in geometric and combinatorial configurations. First we can show that within a special family of configurations any of the eight possible combinations of the existence or non-existence of g-laterals for $3 \leq g \leq 5$ may arise. Moreover, this is true for arbitrary large configurations belonging to this family. We also present geometric realizations of the two smallest trilateral-, quadrilateral- and pentalateral-free (v_3) configurations (generalized hexagons). Finally, we consider (v_4) configurations and present the smallest-known geometric trilateral-free (v_4) configuration.

1 Introduction

This paper is concerned with r-configurations, that is, incidence systems of objects we call points and lines, with the restriction that each object is incident with precisely r objects of the other kind; some other restrictions are convenient, and we shall give precise definition of this concept and the others mentioned below in Section 2. Our special interest concerns multilaterals, that is cyclic sequences of alternating points and lines, with each incident with its two neighbors in the sequence, and all distinct. The study of configurations started in the last quarter of the nineteenth century, and multilaterals were among the first topics studied – albeit under the misleading designation as "polygons" [12]. For additional information about the history of multilaterals see Sections 5.2 to 5.4 of [8].

A g-lateral is a multilateral that consists of g points and g lines; colloquially, we also speak of trilaterals, quadrilaterals, and so on. The early studies concerned mainly either trilaterals, or else "Hamiltonial multilaterals"; the latter contain all points and all lines, each exactly once. We are trying to answer some questions of the following kind: Are there configurations which do have g-laterals for a certain set of values of g, and at the same time do not have any g-laterals for another set of values of g.

As sample results we may mention that in certain families of 3-configurations there always exist 6-laterals, while in the same family there exist configurations for which any chosen subset of $\{3, 4, 5\}$ corresponds to values of g such that the configuration contains g-laterals and does not contain g-laterals for g in the complementary subset.

2 Definitions

A *(combinatorial) configuration* $\mathcal{C}$ of type (v_r), or a (v_r) *configuration*, is an incidence structure with sets $\mathcal{P}$ and $\mathcal{L}$ of objects called points and lines respectively, such that

1. $|\mathcal{P}| = |\mathcal{L}| = v$.
2. each line is incident with r points,
3. each point is incident with r lines,
4. two distinct points are incident with at most one common line.

A *geometric* (v_r) *configuration* is a set of v points and v (straight) lines in the Euclidean plane, such that precisely r of the lines pass through each of the points, and precisely r of the points lie on each of the lines. It is clear that each geometric configuration determines a combinatorial configuration, while the converse is not true, see [8].

To shorten the notation, we will frequently omit the word "combinatorial" when referring to combinatorial configurations while we retain the adjective geometric when we will speak of geometric configurations.

A configuration $\mathcal{C}$ of type (v_r) fully determines its *Levi graph* $L(\mathcal{C})$ (also called the *incidence graph*), which is an r-regular bipartite graph with the vertex set $\mathcal{P} \cup \mathcal{L}$, and the point $p \in \mathcal{P}$ is adjacent to the line $\ell \in \mathcal{L}$ whenever p and ℓ are incident in $\mathcal{C}$. The consequence of the part 4. in the definition above is that the girth of the Levi graph is always even (i.e. $6, 8, 10, \dots$). Conversely, each bipartite r-regular graph with girth at least 6 determines a pair of mutually dual (v_r) configurations. We say that a configuration is *connected* whenever its Levi graph is connected.

A *g-lateral* in a configuration is a cyclically ordered set $\{p_0, \ell_0, p_1, \ell_1, \dots, \ell_{g-2}, p_{g-1}, \ell_{g-1}\}$ of pairwise distinct points p_i and pairwise distinct lines ℓ_i such that p_i is incident with ℓ_{i-1} and ℓ_i for each $i \in \mathbb{Z}_g$. Note that g-laterals in a configuration correspond precisely to $2g$-cycles in its Levi graph. Note that for combinatorial configurations, non-existence of trilaterals is essentially the girth question of the corresponding Levi graph, see for instance [2].

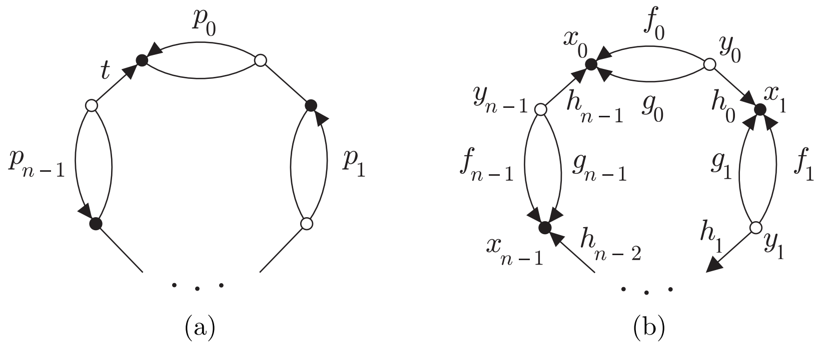


Figure 1: The reduced Levi graph of the configuration $\mathcal{C}_3(k, p, t)$ showing the non-zero voltages and the directions of the edges (a); the same graph with labeled vertices and edges (b), that will be used in the proofs.

Here, we will mainly focus on two special families of (v_3) and (v_4) configurations which were introduced in [5]. Let $k \geq 3$, and $0 \leq t < k$ be two integers and $p = (p_0, p_1, \ldots, p_{n-1})$, $n \geq 2$, a vector of integers with $0 < p_i < \frac{k}{2}$. The configuration $\mathcal{C}_3(k, p, t)$ is defined as a configuration whose Levi graph is a $\mathbb{Z}_k$-covering graph over a graph G in Figure 1(a). In this connection G is called a *reduced Levi graph*. For related concepts see also [8].

Similarly, we define a configuration $\mathcal{C}_4(k, p, q, t)$ as a configuration with its Levi graph being a $\mathbb{Z}_k$-covering graph over the graph shown in Figure 4. Here, the four parameters are: an integer $k \geq 7$, an integer $0 \leq t < k$, and integer vectors $p = (p_0, p_1, \ldots, p_{n-1})$, $q = (q_0, q_1, \ldots, q_{n-1})$, where $n \geq 2$ and $0 < p_i, q_i < \frac{k}{2}$. In works of L. Berman and B. Grünbaum geometric realizations of our $\mathcal{C}_4$ configurations are called *celestial configurations*, see for instance [1]. Note also that $\mathcal{C}_3$ configurations are $((kn)_3)$ configurations while $\mathcal{C}_4$ configurations are $((kn)_4)$ configurations.

We can exploit special structure of $\mathcal{C}_3$ and $\mathcal{C}_4$ configurations to try to obtain so-called rotational realizations in the Euclidean plane [5]. Precisely, a fact that there exists a cyclic automorphism α of order k in both $\mathcal{C}_3$ and $\mathcal{C}_4$ configurations (because their Levi graph is a $\mathbb{Z}_k$ covering graph) can be used to realize α as a rotation through $2\pi/k$ about the origin by drawing the points of the same α-orbit as vertices of a regular k-lateral. We will call such geometric realizations of $\mathcal{C}_3$ and $\mathcal{C}_4$ configurations simply *geometric $\mathcal{C}_3$ configurations* (*geometric $\mathcal{C}_4$ configurations*). In this case the values p_i and q_i indicate spans that a line makes between the points of particular orbits, see [1, 5, 8] for more details. An example of a geometric $\mathcal{C}_4$ configuration can be found in Figure 5(b).

3 Results on (v_3) configurations

Proposition 1. *Let $\mathcal{C}$ be a combinatorial $\mathcal{C}_3(k, (p_0, p_1, \ldots, p_{n-1}), t)$ configuration, where $n \geq 6$. There is a g-lateral, $g = 3, 4, 5$, in $\mathcal{C}$ precisely when one of the following expressions equals 0 modulo k:*

$$g = 3: \quad 3p_i, \tag{1}$$
$$p_i \pm p_{i+1}, \tag{2}$$

$$
\begin{aligned}
g = 4: \quad & 4p_i, && (3)\\
& 2p_i \pm p_{i+1}, && (4)\\
& p_i \pm 2p_{i+1}, && (5)\\
g = 5: \quad & 5p_i, && (6)\\
& p_i \pm 3p_{i+1}, && (7)\\
& 2p_i \pm 2p_{i+1}, && (8)\\
& 3p_i \pm p_{i+1}, && (9)\\
& p_i \pm p_{i+2}, && (10)\\
& p_i \pm p_{i+1} \pm p_{i+2}, && (11)
\end{aligned}
$$

for some i, $i = 0, 1, \ldots, n-1$ (additions in indices are performed modulo n).

Proof. A g-lateral in a configuration corresponds precisely to a $2g$-cycle in its Levi graph. If a configuration is described using a reduced Levi graph, $2g$-cycles can be obtained as $\mathbb{Z}_k$-lifts of closed walks in the reduced Levi graph which are of the following form:

$$v_0\, e_0\, v_1\, e_1 \ldots v_{2g-1}\, e_{2g-1}\,,$$

$e_i \neq e_{i+1}$, such that the vertices v_i and v_{i+1} are the end-points of the edge e_i (additions in indices are always performed modulo n) and that the voltages on the walk sum to 0 in $\mathbb{Z}_k$, i.e. $\sum_{i=0}^{2g-1} \xi'(e_i) = 0 \pmod{k}$. Here

$$\xi'(e_i) = \begin{cases} \xi(e_i) & \text{if } e_i = v_i v_{i+1} \\ -\xi(e_i) & \text{if } e_i = v_{i+1} v_i. \end{cases}$$

and $\xi(e_i)$ denotes the voltage on the edge e_i.

Let the voltages of the reduced Levi graph of $\mathcal{C}_3$ configurations be denoted as in Figure 1(a) and let the vertices and edges be labeled as in Figure 1(b). Moreover, let us assume that the length of the "main" cycle in G has length at least 12 (i.e. $n \geq 6$).

Case $g = 3$. Any closed walk of length 6 (or its inverse) has one of the following forms

$$
\begin{aligned}
W_1 &= y_i\, f_i\, x_i\, g_i\, y_i\, f_i\, x_i\, g_i\, y_i\, f_i\, x_i\, g_i,\\
W_2 &= x_i\, g_i\, y_i\, h_i\, x_{i+1}\, g_{i+1}\, y_{i+1}\, f_{i+1}\, x_{i+1}\, h_i\, y_i\, f_i,\\
W_3 &= x_i\, g_i\, y_i\, h_i\, x_{i+1}\, f_{i+1}\, y_{i+1}\, g_{i+1}\, x_{i+1}\, h_i\, y_i\, f_i
\end{aligned}
$$

with the voltage sums

$$
\begin{aligned}
\xi(W_1) &= p_i - 0 + p_i - 0 + p_i - 0 = 3p_i,\\
\xi(W_2) &= -0 + 0 - 0 + p_{i+1} - 0 + p_i = p_i + p_{i+1},\\
\xi(W_3) &= -0 + 0 - p_{i+1} + 0 - 0 + p_i = p_i - p_{i+1},
\end{aligned}
$$

which gives equations (1) and (2). Note that for $i = n-1$ the voltage corresponding to $y_i\, h_i\, x_{i+1}$ is t, but it cancels to 0 with the voltage $-t$ of $x_{i+1}\, h_i\, y_i$.

Cases $g = 4$ and $g = 5$ can be dealt similarly; we have to consider all possible ways to obtain essentially different closed walks of length 8 and 10. This was done by hand testing and verified with a simple computer program written in *Mathematica*.

Note the fact that $n \geq 6$ is only needed to reduce the number of possible different closed walks, since it prevents any closed walk of length less than 12 to "encircle" the graph along the main cycle. □

Proposition 2. *Every combinatorial $\mathcal{C}_3(k, (p_0, p_1, \ldots, p_{n-1}), t)$ configuration contains a hexalateral. In case when $n = 2$ it also contains a 5-lateral.*

Proof. We have to prove that there is always a closed walk of length 12 in the reduced Levi graph G of $\mathcal{C}_3$ configurations shown in Figure 1 which lifts to a 12 cycle, and that there is always a closed walk of length 10 in G which lifts to a 10 cycle when $n = 2$. A closed walk of length 12 with voltage 0 is (see Figure 1 for labels)

$$x_0\, g_0\, y_0\, h_0\, x_1\, g_1\, y_1\, f_1\, x_1\, h_0\, y_0\, g_0\, x_0\, f_0\, y_0\, h_0\, x_1\, f_1\, y_1\, g_1\, x_1\, h_0\, y_0\, f_0\, x_0$$

while a closed walk of length 10 and voltage 0 when $n = 2$ is

$$x_0\, f_0\, y_0\, h_0\, x_1\, g_1\, y_1\, h_1\, x_0\, g_0\, y_0\, f_0\, x_0\, h_1\, y_1\, g_1\, x_1\, h_0\, y_0\, g_0\, x_0\, .$$

□

Theorem 3. *Considering combinatorial $\mathcal{C}_3(k, (p_0, p_1, \ldots, p_{n-1}), t)$ configurations, any combination of existence or non-existence of g-lateral, $3 \leq g \leq 5$, is possible for particular values of k and p. Moreover, for each N and each of the combinations of existence of g-laterals a configuration on $\geq N$ points can be found.*

Proof. The following table gives combinatorial $\mathcal{C}_3$ configurations for all combinations of existence (+) and non-existence (−) of 3-, 4-, and 5-laterals (values of t are arbitrary):

3-lat.	4-lat.	5-lat.	$\mathcal{C}_3$ configuration
−	−	−	$\mathcal{C}_3(21, (1, 4, 6, 1, 4, 6), t)$
−	−	+	$\mathcal{C}_3(8, (1, 3, 1, 3, 1, 3), t)$
−	+	−	$\mathcal{C}_3(9, (1, 2, 4, 1, 2, 4), t)$
−	+	+	$\mathcal{C}_3(5, (1, 2, 1, 2, 1, 2), t)$
+	−	−	$\mathcal{C}_3(21, (1, 4, 7, 1, 4, 7), t)$
+	−	+	$\mathcal{C}_3(5, (1, 1, 1, 1, 1, 1), t)$
+	+	−	$\mathcal{C}_3(21, (1, 2, 7, 1, 2, 7), t)$
+	+	+	$\mathcal{C}_3(3, (1, 1, 1, 1, 1, 1), t)$

For each $\mathcal{C}_3(k, p, t)$ configuration in the table we have to check whether its parameters k, p, t satisfy or do not satisfy the equations of Proposition 1 corresponding to a particular g.

For example, we claim that the configuration $\mathcal{C}_3(9, (1, 2, 4, 1, 2, 4), t)$, line $- + -$ in the table, does not have any trilateral, has at least one quadrilateral and does not have any 5-lateral. This means that for $k = 9$, $p_0 = 1$, $p_1 = 2$, $p_2 = 4$, $p_3 = 1$, $p_4 = 2$, $p_5 = 4$

1. None of the expressions (1), (2) evaluates to 0 modulo 9 for any i;
2. At least one of the expressions (3)–(5) evaluates to 0 modulo 9 for some i — here the expression (4) for $i = 2$ evaluates to $2p_2 + p_3 = 2 \cdot 4 + 1 = 9 \equiv 0 \pmod 9$;

$3,4,5$-lat.	smallest $\mathcal{C}_3$ cfg.	comb.	geom.
$- - -$	$\mathcal{C}_3(27,(1,8,10),5)$	$\mathbf{(63_3)}$	$\mathbf{(63_3)}$
$- - +$	$\mathcal{C}_3(20,(1,9),4)$	$\mathbf{(35_3)}$	$\mathbf{(35_3)}$
$- + -$	$\mathcal{C}_3(9,(1,2,4),8)^*$, $\mathcal{C}_3(17,(1,2,8),12)$	$\mathbf{(27_3)}$	$\mathit{(51_3)}$
$- + +$	$\mathcal{C}_3(9,(2,4),3)$	$\mathbf{(15_3)}$	$\mathbf{(15_3)}$
$+ - -$	$\mathcal{C}_3(27,(1,4,10),0)$	$\mathit{(81_3)}$	$\mathit{(81_3)}$
$+ - +$	$\mathcal{C}_3(7,(2,2,2),3)$	$\mathit{(21_3)}$	$\mathit{(21_3)}$
$+ + -$	$\mathcal{C}_3(15,(1,2,5),4)$	$\mathit{(45_3)}$	$\mathit{(45_3)}$
$+ + +$	$\mathcal{C}_3(4,(1,1),3)^{**}$, $\mathcal{C}_3(3,(1,1,1),1)$	$\mathbf{(7_3)}$	$\boldsymbol{(9_3)}$

Table 1: For each of the eight possible combinations of the existence/non-existence of g-laterals, $g = 3,4,5$, we list, in order: The smallest $\mathcal{C}_3$ configuration, the smallest known combinatorial configuration, and the smallest known geometric configuration. All $\mathcal{C}_3$ configurations listed are geometric, except the ones marked by asterisks; see Figure 2 and Remark 1. Bold-faced symbols are known to denote the smallest possible configurations of the appropriate kind. Symbols in italics denote configurations $\mathcal{C}_3$ listed in the second column; the smallest configurations for these positions have not been determined.

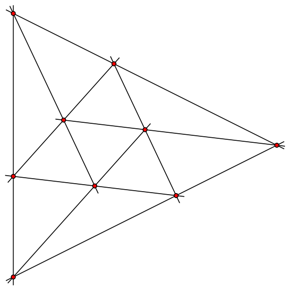
3. None of the expressions (6)–(11) evaluates to 0 modulo 9 for any i.

Analogous conditions for other rows in the table can also be easily verified. This gives the proof of the claim that combinatorial C_3 configurations exist for each combination of existence and non-existence of 3-, 4-, and 5-laterals. Note that these are not the smallest combinatorial $\mathcal{C}_3$ configuations satisfying the conditions. See Table 1 and Remark 1 for the smallest combinatorial and geometric examples.

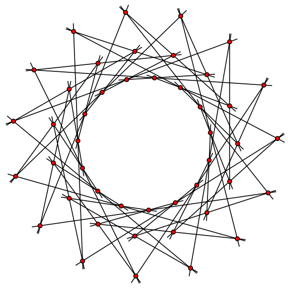
Configurations with $n = 6m$, $m > 1$, i.e. arbitrarily large combinatorial configurations corresponding to each possibility are defined by the same values of k while the sequence p is extended by repeating it $m - 1$ times. This is true since the extension of p by repetition gives exactly the same values of the expressions in Proposition 1. □

Remark 1. All $\mathcal{C}_3$ configurations in Table 1 are realizable as geometric $\mathcal{C}_3$ configurations, except of those denoted by * and **, see Figure 2. The realizations were obtained using the theory developed in [5]. The configuration denoted by * is not realizable as geometric $\mathcal{C}_3$ configuration since, if we respect the $\mathcal{C}_3$ symmetry, additional incidences occur. In fact, we can recognize it as a subconfiguration of the configuration $\mathcal{C}_4(9,(1,2,4),(3,3,3),7)$. The fact that this (27_3) configuration is the smallest such configuration was proved by G. Brinkmann (personal communication). The configuration ** is the Möbius-Kantor (8_3) configuration. The claim that $\mathcal{C}_3$ configurations in the table are the smallest of its kind has been proven by examining all admissible values for k, p, and t using a computer program written in *Mathematica*.

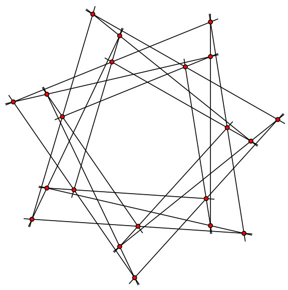
Remark 2. The configuration $\mathcal{C}_3(9,(2,4),3)$ listed in Table 1, a (18_3) configuration, is one of the small trilateral-free (v_3) configurations which where studied in [4]. The smallest (v_3) configuration without trilaterals is the Cremona-Richmond (15_3) configuration. There exist no (16_3) trilateral-free configurations while there is only one (17_3) trilateral-free configuration.



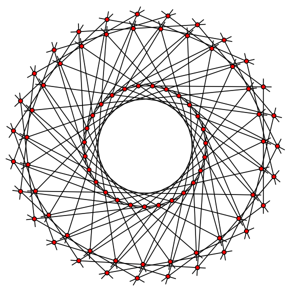
$\mathcal{C}_3(3,(1,1,1),1)$



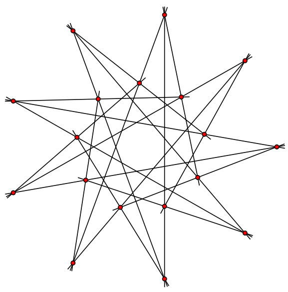
$\mathcal{C}_3(15,(1,2,5),4)$



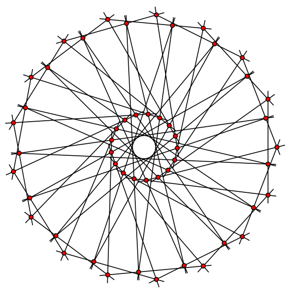
$\mathcal{C}_3(7,(2,2,2),3)$



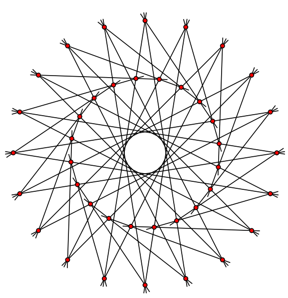
$\mathcal{C}_3(27,(1,4,10),0)$



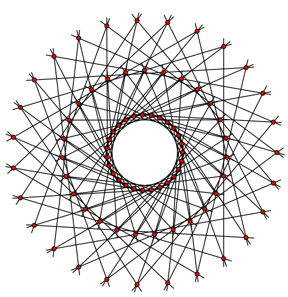
$\mathcal{C}_3(9,(2,4),3)$



$\mathcal{C}_3(17,(1,2,8),12)$



$\mathcal{C}_3(20,(1,9),4)$



$\mathcal{C}_3(27,(1,8,10),5)$

Figure 2: Geometric $\mathcal{C}_3$ configurations from Table 1.

Remark 3. By Table 1, the smallest trilateral- and quadrilateral-free $\mathcal{C}_3$ configuration is the (40_3) configuration $\mathcal{C}(20,(1,9),4)$. In general, the smallest trilateral- and quadrilateral-free (v_3) configurations result from (bipartite) 10-cages. There are 5 non-isomorphic such configurations on 35 points and all of them are geometric, see [10].

4 Generalized hexagons

According to Table 1, the smallest $\mathcal{C}_3$ configuration without 3-, 4- and 5-laterals is the $\mathcal{C}_3(27,(1,8,10),5)$ configuration on 81 points. However, the smallest cubic graph of girth 12, also called a 12-cage, has 126 vertices and is unique. It is also a bipartite graph and is therefore the Levi graph of the smallest 3-, 4- and 5-lateral-free 3-configuration. In fact, it determines a pair of dual (63_3) configurations which are also called *generalized hexagons.* In [11] it is discussed how to draw the hexagons, using their symmetries, but the presented drawings are not realizations. The question of their realization was answered simultaneously with the smallest trilateral- and quadrilateral-free configurations, but it was published only in [3]. Here we state the result again.

Theorem 4. *Both smallest* 3-, 4- *and* 5-*lateral-free* (v_3) *configurations, which are* (63_3) *configurations, are geometrically realizable.*

Proof. We can produce a $\mathcal{C}_3$ realization of these two dual configurations; both the realization of one of the configurations and the corresponding reduced Levi graph are shown in Figure 3. Numerical values of the coordinates of one point of each of the 9 orbits are:

$$
\begin{array}{lll}
P_1 = (0.1416, 0.3908) & P_2 = (-1.3574, -1.4168) & P_3 = (-2.6793, 0.9596) \\
P_4 = (-1, 3) & P_5 = (-1.6789, 2.1752) & P_6 = (-0.6435, 0.8454) \\
P_7 = (-0.1218, -0.3037) & P_8 = (-1.0093, -0.4332) & P_9 = (1, 0)
\end{array}
$$

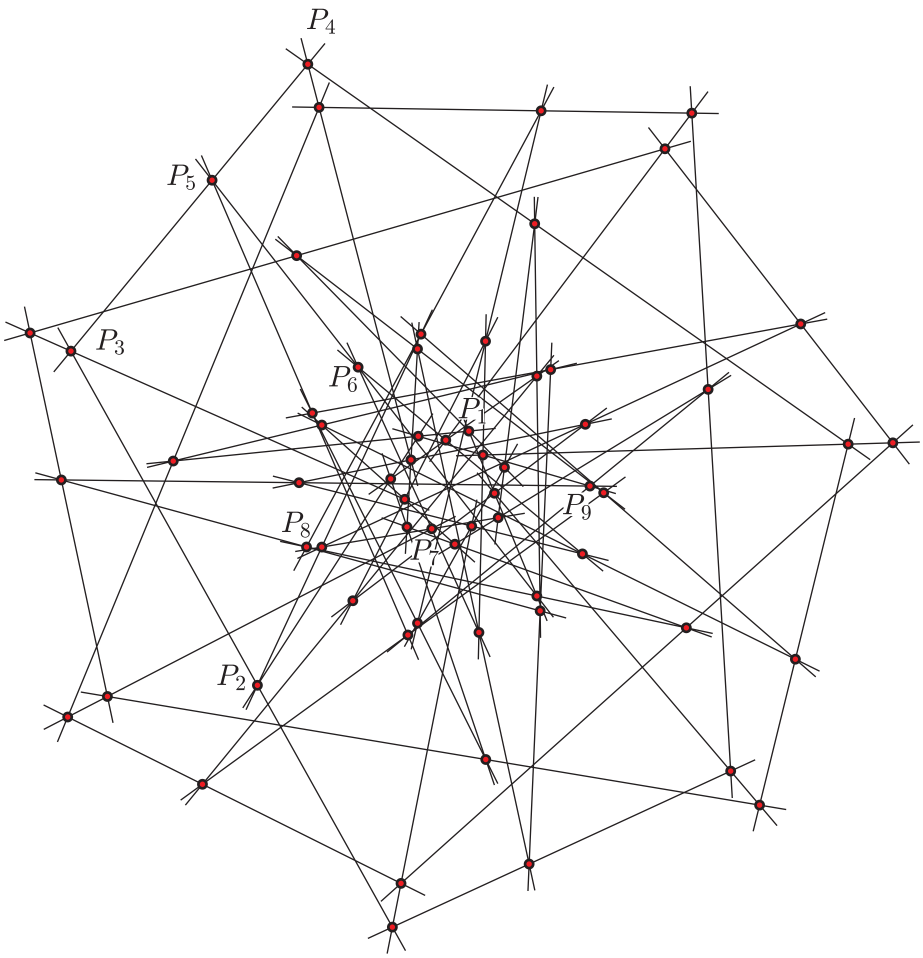
other points can are obtained as rotations for $2\pi/7$ about the origin.

The above coordinates were obtained using an adaptation of the algorithm described in [7]. Following this algorithm, a necessary condition for the existence of a geometric realization can be reduced to finding real parameters which are zeroes of a so-called *final polynomial.* The algorithm is adapted in such way that we do not need to consider all lines of a configuration, but only lines from different line orbits which reduces the number of parameters and simplifies the computation. □

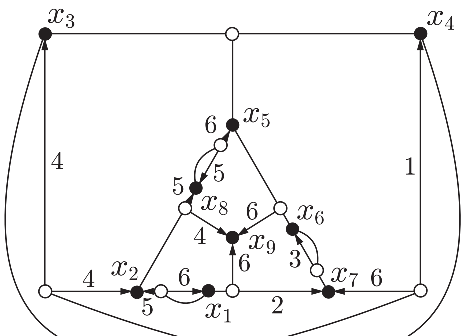
Remark 4. Note that the points $P_1, P_2, \dots, P_9$ of the configuration in Figure 3(a) correspond to the vertices $x_1, x_2, \dots, x_9$ of the voltage graph in Figure 3(b), i.e., the point P_i corresponds, say, to the vertex x_i^0 of the covering graph if we denote the vertices of the fiber over x_i by $x_i^0, x_i^1, \dots, x_i^6$.

5 Results on (v_4) configurations

In Section 2 we gave a definition of combinatorial and geometric $\mathcal{C}_4(k,p,q,t)$ configurations which can be described as configurations admitting a reduced Levi graph of the form depicted in Figure 4.



(a)



(b)

Figure 3: Geometric realization of a (63_3) configuration, the smallest 3-, 4- and 5-lateral-free (v_3) configuration or a generalized hexagon (a) and the corresponding reduced Levi graph of this realization (b). The points $P_1, P_2, \ldots, P_9$ of the configuration (a) correspond to the vertices $x_1, x_2, \ldots, x_9$ of the voltage graph (b).

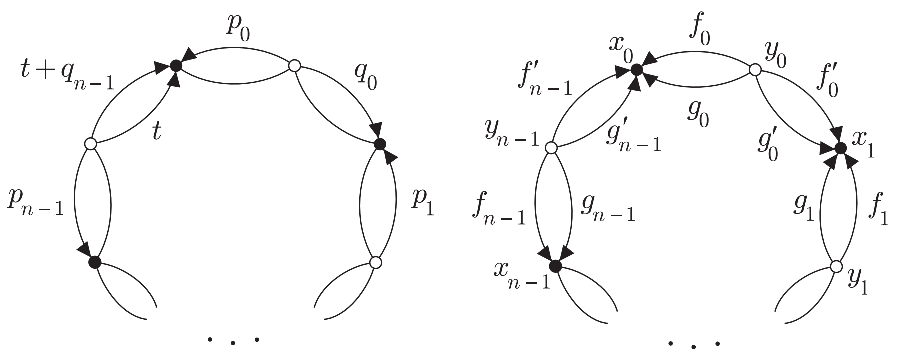


Figure 4: The reduced Levi graph of the configuration $\mathcal{C}_4(k, p, q, t)$ showing the non-zero voltages and the directions of the edges (left); the same graph with labeled vertices and edges (right).

The conditions on the parameters k, p, q, t to give a combinatorial (v_4) configuration which are given in [5] are the following.

Theorem 5 (Theorem 15, Lemma 17 in [5])**.** *For given $n \geq 2$, $k \geq 7$ the sequences $p = (p_0, p_2, \ldots, p_{n-1})$, $q = (q_0, q_2, \ldots, q_{n-1})$, $1 \leq p_i, q_i < k/2$, and the number t determine a $((nk)_4)$ configuration $\mathcal{C}_4(k, p, q, t)$ if and only if*

$$p_i \neq q_i, \quad p_i \neq q_{i-1}, \qquad i = 0, 2, \ldots, n-1 \tag{12}$$

For $n = 2$, in addition to (12), there are conditions

$$a - b + c - d \not\equiv 0 \pmod{k}$$

for any possible choice of a, b, c, d, where $a \in \{0, p_0\}$, $b \in \{0, q_0\}$, $c \in \{0, p_1\}$, $d \in \{t, t + q_1\}$.

Moreover, we also have a necessary condition for $\mathcal{C}_4$ configuration to be realizable as geometric $\mathcal{C}_4$ configuration.

Theorem 6 (Theorem 15, Lemma 22 in [5])**.** *If a geometric $\mathcal{C}_4(k, p, q, t)$ configuration exists then the equation*

$$\cos\frac{p_0\pi}{k}\,\cos\frac{p_1\pi}{k}\cdots\cos\frac{p_{n-1}\pi}{k} = \cos\frac{q_0\pi}{k}\,\cos\frac{q_1\pi}{k}\cdots\cos\frac{q_{n-1}\pi}{k}. \tag{13}$$

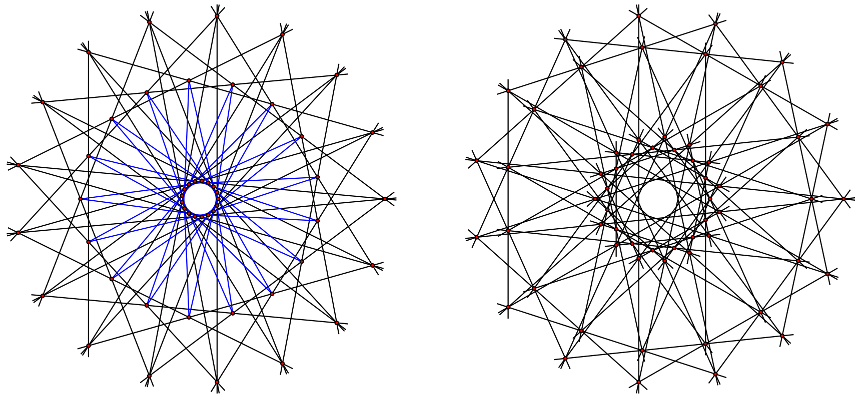
holds and

$$t = \frac{1}{2}\sum_{i=0}^{n-1}(p_i - q_i) \tag{14}$$

is an integer.

The smallest combinatorial $\mathcal{C}_4$ configuration is the configuration $\mathcal{C}_4(5, (1, 1, 1), (2, 2, 2), 0)$, a (15_4) configuration, while the smallest geometrical $\mathcal{C}_4$ configuration is $\mathcal{C}_4(7, (1, 2, 3), (3, 1, 2), 0)$, a (21_4) configuration; see [8, 9] for its realization. The smallest known geometric 4-configuration is found by J. Bokowski, see [8].

In the next theorem we show that there is always a 4-lateral in a $\mathcal{C}_4$ configuration, i.e. there is no quadrilateral-free $\mathcal{C}_4$ configuration, and present the smallest trilateral-free $\mathcal{C}_4$ configuration.



(a) (b)

Figure 5: The smallest combinatorial trilateral-free $\mathcal{C}_4$ configuration realized with pseudolines (a), and the smallest geometric trilateral-free $\mathcal{C}_4$ configuration (b).

Theorem 7. *Every combinatorial* $\mathcal{C}_4(k,(p_0,p_1,\ldots,p_{n-1}),(q_0,q_1,\ldots,q_{n-1}),t)$ *configuration contains a quadrilateral. The smallest trilateral-free combinatorial* $\mathcal{C}_4$ *configuration is a* (51_4) *configuration* $\mathcal{C}_4(17,(2,5,8),(8,2,3),1)$ *and can be realized with pseudolines. The smallest geometric* $\mathcal{C}_4$ *configuration is* $\mathcal{C}_4(15,(1,2,4,7),(6,3,6,3),11)$*, a* (60_4) *configuration.*

Remark 5. Both configurations are shown in Figure 5. The presented (60_4) configuration is currently the smallest known geometric trilateral-free (v_4) configuration. The smallest combinatorial trilateral-free (v_4) configuration arises from the $(4,8)$-cage, which has 80 vertices, and is thus a (40_4) configuration.

Proof of Theorem 7. A closed walk of length 8 with voltage 0 in the reduced Levi graph of $\mathcal{C}_4$ configurations, see Figure 4 for labels, is

$$x_0\, f_0\, y_0\, f_0'\, x_1\, g_0'\, y_0\, f_0\, x_0\, g_0\, y_0\, g_0'\, x_1\, f_0'\, y_0\, g_0\, x_0\,.$$

It gives an 8 cycle in the Levi graph and a quadrilateral in the configuration.

Cycles of length 6 in the Levi graph of a $\mathcal{C}_4$ configuration (trilaterals in the configuration) arise from closed walks of length 6 in the reduced Levi graph, see Figure 4. In the case $n \geq 4$ all different possibilities for the voltages of the closed walks of length 6 are

$3p_i$	$3q_i$
$p_i \pm p_{i+1}$	$q_i \pm q_{i+1}$
$p_i \pm 2q_i$	$q_i \pm 2p_i$
$p_{i+1} \pm 2q_i$	$q_i \pm 2p_{i+1}$
$p_i + p_{i+1} \pm q_i$	$p_i \pm p_{i+1} - q_i$
$q_i \pm q_{i+1} + p_{i+1}$	$q_i + q_{i+1} \pm p_{i+1}$

and their negative values. For example, the walks with voltage $p_i + q_i + p_{i+1}$ are

$$x_i\, g_i\, y_i\, f_i'\, x_{i+1}\, g_{i+1}\, y_{i+1}\, f_{i+1}\, x_{i+1}\, g_i'\, y_i\, f_i\, x_i\,.$$

For $n < 4$ we get more different closed walks of length 6, and thus more different voltages, since we have to consider the closed walks containing the "main" cycle. Note that those walks can contain the voltage t while in the walks considered above it cancels out.

If we check all possible values for p, q and t at some k satisfying conditions of Theorem 5 and such that none of the voltages above has value $0 \pmod k$ (i.e. there is no cycle of length 6 and hence no trilateral) we find out the following. The smallest values for k when this happens are:

$n = 2$. For $k = 30$ we get three combinatorially non-isomorphic (60_4) configurations

$$\mathcal{C}_{2,1} = \mathcal{C}_4(30,(1,11),(7,13),26), \qquad \mathcal{C}_{2,2} = \mathcal{C}_4(30,(1,2),(8,11),7),$$
$$\mathcal{C}_{2,3} = \mathcal{C}_4(30,(1,7),(11,13),22).$$

None of them satisfies the conditions of Theorem 6 and are not realizable as $\mathcal{C}_4$ configurations.

$n = 3$. For $k = 17$ we get a unique combinatorial (51_4) configuration $\mathcal{C}_4(17,(2,5,8),(8,2,3),1)$. It does not satisfy the condition 13 of Theorem 6. Thus, it is not realizable as a geometric $\mathcal{C}_4$ configuration with straight lines, although it is realizable with pseudolines, see Figure 5(a). For $k = 18$ and $k = 20$ we also get trilateral-free configurations but none of them is realizable as a $\mathcal{C}_4$ configuration by Theorem 6.

$n = 4$. For $k = 15$ we get four combinatorially different (60_4) configurations

$$\mathcal{C}_{4,1} = \mathcal{C}_4(15,(1,2,4,7),(6,3,6,3),11), \quad \mathcal{C}_{4,2} = \mathcal{C}_4(15,(1,2,4,7),(6,3,6,3),1),$$
$$\mathcal{C}_{4,3} = \mathcal{C}_4(15,(1,2,4,7),(6,3,6,3),3), \quad \mathcal{C}_{4,4} = \mathcal{C}_4(15,(1,2,4,7),(6,3,6,3),13).$$

Only $\mathcal{C}_{4,1}$ satisfies the conditions of Theorem 6. In this case, the conditions are also sufficient (we do not get accidental incidences); the realization is found in Figure 5(b). These four are not isomorphic to any of the configurations in case $n = 2$. □

References

[1] L. Berman, Movable (n_4) configurations, *Electron. J. Combin.* **13** (2006), #R104. 3

[2] A. Betten, G. Brinkmann, T. Pisanski, Counting symmetric configurations v_3, *Discrete Appl. Math.* **99** (2000), 331–338. 2

[3] M. Boben, Uporaba teorije grafov pri kombinatoričnih in geometričnih konfiguracijah, Ph.D. thesis (in Slovene), Ljubljana, 2003. 8

[4] M. Boben, B. Grünbaum, T. Pisanski, and A. Žitnik. Small triangle-free configurations of points and lines, *Discrete Comput. Geom.* **35(3)** (2006), 405–427. 6

[5] M. Boben and T. Pisanski, Polycyclic configurations, *European J. Combin.* **24** (2003), 431–457. 3, 6, 10

[6] M. Boben, T. Pisanski, and A. Žitnik, *I*-graphs and their corresponding configurations, *J. Combin. Designs* **13** (2005), 406–424.

[7] J. Bokowski, B. Sturmfels, Computational Synthetic Geometry, *Lecture Notes in Mathematics 1355*, Springer, Heidelberg, 1989. 8

[8] B. Grünbaum, Configurations of points and lines, Graduate Studies in Mathematics, 103, American Mathematical Society, Providence, RI, 2009. 2, 3, 10

[9] B. Grünbaum, J. F. Rigby, The real configuration (21_4), *J. London Math. Soc.* **41** (1990), vol 2, 336–346. 10

[10] T. Pisanski, M. Boben, D. Marušič, A. Orbanić, A. Graovac, The 10-cages and derived configurations, *Discrete Math.* **275** (2004), 265–276. 8

[11] A. Schroth, How to draw a hexagon, *Discrete Math.* **199** (1999), 167–171. 8

[12] E. Visconti, Sulle configurazioni piane atrigone, *Giornale di Matematiche di Battaglini* **54** (1916), 27–41. 2