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Hyperplane arrangements in preference modeling

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Abstract

A standard representation of a family $\mathscr{B}$ of partial orders on a given finite set X is as a set of vertices of a cube. The metric and order structures on $\mathscr{B}$ inherited from the cube are often used in applications. In this paper, following Stanley [(1996). Hyperplane arrangements, interval orders, and trees. *Proceedings of the National Academy of Sciences of the United States of America*, *93*, 2620–2625], we represent relations in $\mathscr{B}$ by regions and cells of a hyperplane arrangement arising from numerical representations of the partial orders in $\mathscr{B}$. To illustrate this approach, we establish wellgradedness of some families of generalized semiorders. Although the families of linear and weak orders are not well graded, our approach allows the recasting of such concepts as well graded families of sets.

Keywords: Media theory; Well graded families; Partial cubes

1. Introduction

Various kinds of partial orders, like linear and weak orders, and generalizations of semiorders, are widely used in social and behavioral sciences to model preference relations.

Let $\mathscr{B}$ be a family of partial orders (irreflexive and transitive binary relations) on a set X of cardinality n. Elements of the family $\mathscr{B}$ can be represented by their characteristic functions as vertices of the n^2-cube Q_{n^2}. Then $\mathscr{B}$ inherits metric and order structures from Q_{n^2}. These structures play an important role, for instance, in group choice theory (Mirkin, 1979). Originally, metric structures on families of binary relations were introduced by Barbut and Monjardet (1970) for linear orders, Kemeny and Snell (1972) for weak orders, and Bogart (1973) for some other families of partial orders.

Another revealing geometric structure on the set $\mathscr{B}$ is the order polytope associated with $\mathscr{B}$. This polytope is the convex hull of the vertices of Q_{n^2} representing elements from $\mathscr{B}$. Combinatorial and geometric properties of the order polytope play an important role, for example, in random utility theory (see, for instance, Fiorini & Fishburn (2004) and references there).

In this paper, we propose to model a family $\mathscr{B}$ of partial orders by the collection of cells of a hyperplane arrangement arising from the representational mode defining $\mathscr{B}$. "The *representational mode* defines membership in a class (of partial orders) by the existence of a map from X into real intervals that are ordered in a way that preserves (partial order)", Fishburn (1997). In a more general setting, elements of a family of partial orders can be defined by a system of real functions on X satisfying certain conditions as it is done in Section 4 of the paper.

To illustrate how hyperplane arrangements appear naturally in the representational mode, let us consider the definition of a linear order L on the set X: a binary relation L on X is a *linear order* if there is a one-to-one function $f : X \to \mathbb{R}$ such that

$$aLb \Leftrightarrow f(a) > f(b).$$

For $a \neq b$, equations in the form $f(a) = f(b)$ define hyperplanes in the vector space $\mathbb{R}^n$ of all real functions on X. The regions (see Section 2 for details) of this

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arrangement $\mathscr{A}$ of hyperplanes are open polyhedral cones in $\mathbb{R}^n$ separated by these hyperplanes. The set **LO** of linear orders on X is in one-to-one correspondence with the set of regions of $\mathscr{A}$.

In the paper, we use hyperplane arrangements and related geometric and combinatorial objects to study representations of families of partial orders by looking at well graded families of sets. A family $\mathscr{F}$ of subsets of a finite set Y is said to be *well graded* (Doignon & Falmagne, 1997) if, for any two distinct sets $A, B \in \mathscr{F}$, there is a sequence of sets in the family $\mathscr{F}$

$$Y_0 = A, Y_1, \ldots, Y_k = B,$$

such that $|Y_i \Delta Y_{i+1}| = 1$, for all $0 \leqslant i \leqslant k-1$, and $|A \Delta B| = k$, where Δ stands for the symmetric difference between two subsets of Y. Note that $|A \Delta B|$ is the standard *distance* between subsets A and B.

Some families of partial orders are well graded. These families include the set of all partial orders on X (Bogart, 1973; Kuzmin & Ovchinnikov, 1975a,b; Doignon & Falmagne, 1997) and the families of semiorders and interval orders on X (Doignon & Falmagne, 1997). On the other hand, the families of linear and weak orders on X are not well graded, but can be modeled in a rather natural way by well graded families of sets as shown in Section 7. Applications of the wellgradedness property are found, for instance, in the theory of group choice (Ovchinnikov, 1983) and in media theory (Falmagne, 1997; Falmagne & Ovchinnikov, 2002).

Some remarks about graph terminology are in order. As usual, we do not distinguish between isomorphic graphs.

The direct product of n copies of the complete graph K_2 is the *n-cube* Q_n. Vertices of Q_n can be modeled as 0/1-vectors in $\{0,1\}^n$ or, equivalently, as subsets of an n-element set. The graph distance on Q_n is the usual Hamming distance. Recall that the Hamming distance between vertices $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ is defined by

$$d_H(x, y) = \sum_{i=1}^{n} |x_i - y_i|.$$

A subgraph G' of a graph G is an *isometric subgraph* of G if $d_{G'}(u, v) = d_G(u, v)$ for all vertices u and v in G'. Here, $d_{G'}$ and d_G stand for the graph distances (shortest path distances) in G' and G, respectively. Isometric subgraphs of cubes are called *partial cubes* (Imrich & Klavžar, 2000). Clearly, a graph is a partial cube if and only if it is the Hasse diagram of a well graded family of sets ordered by inclusion. In general, a graph is a partial cube if it can be isometrically embedded into a cube.

In Section 2, we introduce arrangements of hyperplanes and prove that the region graph of an arrangement is a partial cube. This fact is instrumental in the proof of the main result in Section 4, where we establish the wellgradedness property for a wide class of families of partial orders. To illustrate our approach, we prove in Section 3 that the family of biorders from a finite set A to a finite set B is well graded (a different proof is given in Doignon & Falmagne (1997)).

In Section 4, we introduce a broad class of families of partial orders in the representational mode. The main result of this section (Theorem 4.1) asserts that families satisfying the Distinguishing Property are well graded. (Distinguishing representations for split semiorders and interval orders were introduced in Fishburn & Trotter (1999).)

The labeled interval orders introduced by Stanley (1996) are natural generalizations of semiorders; so are the interval orders themselves. In Sections 5 and 6, we use the theorem from Section 4 to establish the wellgradedness property of families of these relations. Note that for semiorders and interval orders this was shown in Doignon and Falmagne (1997).

Although the families of linear and weak orders are not well graded, they fit the general framework presented in Section 4. In particular, weak orders are represented by the set of cells of a braid arrangement and linear orders are represented by the set of regions of the same arrangement. We construct representations of families of weak and linear orders by well graded families of sets in Section 7.

2. Arrangements and their graphs

A family $\mathscr{A} = \{H_1, \ldots, H_n\}$ of distinct affine hyperplanes in a real vector space V of dimension d is a *hyperplane arrangement*. In the next two paragraphs, we introduce basic concepts of the theory of hyperplane arrangements. For details, the reader is referred to Björner, Las Vergnas, Sturmfels, White, and Ziegler (1999), Bourbaki (2002), Zaslavsky (1975), and Ziegler (1995).

The *regions* (chambers) of $\mathscr{A}$ are the connected components of the complement of the union $\cup\mathscr{A}$. Each region is a d-dimensional open polyhedron. Relatively open faces of regions are *cells* of $\mathscr{A}$. Thus each region is a maximal cell of the arrangement $\mathscr{A}$. Every cell is a convex set. The *face poset* $\mathscr{F}(\mathscr{A})$ is the set of all cells ordered by inclusion of their closures. Two regions P and Q are *adjacent* if their closures share a facet, i.e., a $(d-1)$-dimensional cell.

Let $\mathscr{R}$ be the set of regions of $\mathscr{A}$. The *region graph* G of the arrangement $\mathscr{A}$ has $\mathscr{R}$ as the set of vertices; edges of G are pairs of adjacent regions in $\mathscr{R}$.

Remark. In a more general setting, the graph G is the *tope graph* of the oriented matroid associated with the arrangement $\mathscr{A}$. It follows from Proposition 4.2.3 in Björner et al. (1999) that G is an isometric subgraph of the n-cube, where n is the number of hyperplanes in $\mathscr{A}$. We present a direct proof of this fact in this section (Proposition 2.1 below).

We define an *orientation* of $\mathscr{A}$ by selecting for every hyperplane $H_i \in \mathscr{A}$ one of the two open half spaces defined by that hyperplane. Let us assign a vector $Z^P \in \{0,1\}^n$ for every region P in $\mathscr{R}$, where $Z_i^P = 1$, if P is on the selected "side" of H_i, and $Z_i^P = 0$, otherwise. Clearly, the one-to-one function

$$\varphi : P \mapsto Z^P \tag{1}$$

is an adjacency preserving mapping, i.e., φ is a graph homomorphism from G into the n-cube Q_n. The graphs G and $\varphi(G)$ are isomorphic.

Let us show that $\varphi(G)$ is a partial cube. Let P and Q be two distinct regions. We have $d_G(P,Q) \geqslant d_H(\varphi(P), \varphi(Q))$, since $\varphi(G)$ is a subgraph of Q_n.

Let p be a point in P and let S be the union of all line segments connecting p with points in Q. The dimension of S is d. Let S' be the union of all line segments in S (as defined in the preceding sentence) that intersect cells of dimension less than $d-1$. The dimension of S' is less then d. Thus we can choose a point $q \in Q$ in such a way that different hyperplanes separating regions P and Q intersect the line segment $[p,q]$ at different points. Let us number these points together with points p and q in the direction from p to q as follows:

$$p = r_0, r_1, \ldots, r_{k+1} = q.$$

Thus, k is the number of hyperplanes in $\mathscr{A}$ separating points p and q. Each open interval (r_i, r_{i+1}) is an intersection of $[p,q]$ with a unique region R_i. Clearly, regions R_i and R_{i+1} are adjacent. Thus the sequence

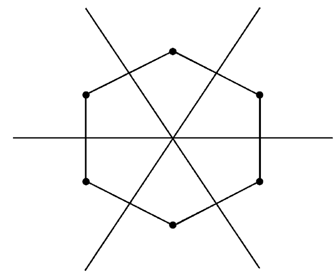
$$P = R_0, R_1, \ldots, R_k = Q$$

is a shortest path in the region graph connecting P and Q. The length of this path is the Hamming distance $d_H(\varphi(P), \varphi(Q))$. Indeed, since there are k hyperplanes separating P and Q, the vectors $\varphi(P)$ and $\varphi(Q)$ differ at exactly k coordinates. Hence, $d_G(P,Q) = d_H(\varphi(P), \varphi(Q))$ and φ is an isometry.

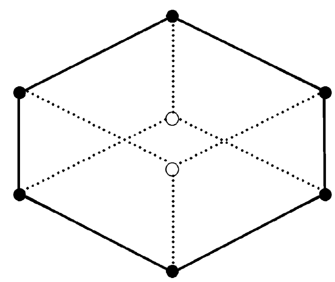
We proved the following proposition which is instrumental in our constructions.

Proposition 2.1. *The region graph G of an arrangement $\mathscr{A}$ is a partial cube. The distance $d_G(P,Q)$ is equal to the number of hyperplanes in $\mathscr{A}$ separating P and Q which in turn is equal to the Hamming distance $d_H(\varphi(P), \varphi(Q))$ between vertices $\varphi(P)$ and $\varphi(Q)$ in Q_n.*

To illustrate this result let us consider an arrangement of three lines on a plane and the corresponding region graph depicted on the figure below (see also a diagram in Section 7).



Clearly, the region graph of this arrangement is a cycle. This graph is isomorphic to a subgraph of the cube Q_3 obtained by deleting two opposite vertices as it is shown on the figure below. It follows that it is a partial cube.



In a more general setting, let W be a relatively open convex set in a nonempty affine subspace V' of V. (An affine subspace is a translation of a linear subspace of V). Let us consider those hyperplanes in $\mathscr{A}$ that have nonempty intersections with W. Distinct intersections of these hyperplanes with V' form an arrangement $\mathscr{A}'$ of hyperplanes in V'. (Note that two or more distinct hyperplanes in $\mathscr{A}$ may define a single hyperplane in V'.) We say that $\mathscr{A}'$ is an *arrangement of hyperplanes in W* and denote the region graph of $\mathscr{A}'$ by G_W. We call intersections of regions of $\mathscr{A}'$ with W *regions of $\mathscr{A}'$ in W*. Since W is a convex set, the construction used in the proof of Proposition 2.1 can be applied to prove the following proposition. (Note that the mapping φ in (1) is now defined for the arrangement $\mathscr{A}'$.)

Proposition 2.2. *The region graph G_W of the arrangement $\mathscr{A}'$ in W is a partial cube. The distance $d_{G_W}(P,Q)$ is equal to the number of hyperplanes in $\mathscr{A}'$ separating regions P and Q in W which in turn is equal to the Hamming distance $d_H(\varphi(P), \varphi(Q))$ between vertices $\varphi(P)$ and $\varphi(Q)$ in the cube $Q_{n'}$ where n' is the number of hyperplanes in $\mathscr{A}'$.*

3. An example: the family of biorders

In this section, we show how Proposition 2.1 can be used to prove that the family **BO** of biorders from a finite set A to a finite set B is well graded (Doignon & Falmagne, 1997).

In the representational mode a biorder relation is defined as follows (Ducamp & Falmagne, 1969).

Definition 3.1. Let A and B be two finite sets. A relation $R \subseteq A \times B$ is a *biorder from A to B* if there are functions $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ such that

$$aRb \Leftrightarrow f(a) > g(b), \tag{2}$$

for all $a \in A$ and $b \in B$.

For any $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$, we regard ordered pairs (f, g) as elements of the vector space $V = \mathbb{R}^A \times \mathbb{R}^B$. Two pairs are said to be (*biorder*) *equivalent* if they define the same biorder R by means of (2). This equivalence relation, which we denote by $\sim$, partitions the space V into classes which are in one-to-one correspondence with elements of **BO**.

Let $\mathscr{A}$ be an arrangement of hyperplanes $H_{(a,b)}$ in V defined by

$$H_{(a,b)} = \{(f, g) \in V : f(a) = g(b)\}, \tag{3}$$

for $(a, b) \in A \times B$. Each region of $\mathscr{A}$ is a subset of one of the classes of the relation $\sim$. The arrangement $\mathscr{A}$ is oriented by open half-spaces in the form $\{(f, g) \in V : f(a) > g(b)\}$ for $(a, b) \in A \times B$ (cf. (2)).

Lemma 3.1. *Any vector in V is equivalent to a vector in some region of* $\mathscr{A}$.

A proof of this lemma as well as similar proofs of Lemmas 5.1 and 6.1 are found in the appendix.

We consider the family **BO** as a set of vertices of the cube $\mathbf{2}^{A\times B}$. It follows from the previous lemma that the set **BO** is in one-to-one correspondence with the set of regions of $\mathscr{A}$. It is easy to see that this correspondence is the same as defined by the mapping φ in (1). By Proposition 2.1, we have the following theorem.

Theorem 3.1. *The family* **BO** *of biorders from A to B is well graded.*

4. Wellgradedness in the representational mode

An approach used in the previous section can be applied to a wide class of families of binary relations on a finite set X. We begin with a general definition.

Definition 4.1. A *representational structure* is a quintuple $\mathfrak{R} = \langle X, V, W, \alpha, \beta\rangle$ where

- X is an n-element set ($n > 2$).
- $V = \{f : X \to \mathbb{R}^m\}$ is an (mn)-dimensional ordered vector space with the order defined by

  $$f < g \Leftrightarrow f_i(x) < g_i(x), \quad \text{for all } x \in X,\ 1 \leqslant i \leqslant m,$$

  where $f = (f_1, \ldots, f_m)$, $g = (g_1, \ldots, g_m)$.
- W is a relatively open convex subset of an affine subspace V' of V.
- $\alpha = (\alpha_i)$ and $\beta = (\beta_i)$, $1 \leqslant i \leqslant m$, are m-dimensional nonzero vectors.

For a given $f \in W$ we define a binary relation R on X by

$$aRb \Leftrightarrow \sum_i \alpha_i f_i(a) > \sum_i \beta_i f_i(b) \tag{4}$$

and say that f *represents* R. We denote $\mathbf{B}(\mathfrak{R})$ the family of binary relations on X defined by (4) for $f \in W$.

Two functions $f, g \in W$ are said to be ($\mathfrak{R}$) *equivalent* if they represent the same relation R in $\mathbf{B}(\mathfrak{R})$. In the rest of the paper we denote, for a given representational structure, this equivalence relation by $\sim$.

Let $\mathscr{A} = \{H_{(a,b)}\}$ be an arrangement of hyperplanes in V that are defined by equations

$$\sum_i \alpha_i f_i(a) = \sum_i \beta_i f_i(b), \quad a \neq b, \quad a, b \in X \tag{5}$$

and let $\mathscr{A}'$ be the restriction of $\mathscr{A}$ to W. In the rest of this section we assume that $\beta \neq \pm\alpha$. Then half-spaces (in V') in the form

$$\left\{ f \in V' : \sum_i \alpha_i f_i(a) > \sum_i \beta_i f_i(b) \right\}$$

define an orientation of $\mathscr{A}'$ which we will use in the definition of the mapping φ in (1) for $\mathscr{A}'$.

Following Fishburn and Trotter (1999) we introduce the concept of a distinguishing function.

Definition 4.2. A function $f \in W$ is a *distinguishing function* if f does not belong to any of hyperplanes (5). A family $\mathbf{B}(\mathfrak{R})$ satisfies the *Distinguishing Property* if any function in W is equivalent to a distinguishing function in W.

If $\mathbf{B}(\mathfrak{R})$ satisfies the Distinguishing Property, then it is in one-to-one correspondence with the family of regions of $\mathscr{A}'$ in W. An example of a family of binary relations on a given set X which does not satisfy the Distinguishing Property is given by the family of weak orders on X (see Section 7).

Let us identify the relations in $\mathbf{B}(\mathfrak{R})$ with vertices of the cube $\mathbf{2}^{X\times X}$. Let G_W be the graph defined by the arrangement $\mathscr{A}$ and the set W (see Section 2). Clearly, (4) defines a mapping $\varphi : G_W \to \mathbf{2}^{X\times X}$, which is the same as in (1). By Proposition 2.2, the mapping φ is an isometry between G_W and $\mathbf{B}(\mathfrak{R})$. Thus we have the following theorem.

Theorem 4.1. *A family of binary relations* $\mathbf{B}(\mathfrak{R})$ *satisfying the Distinguishing Property is well graded.*

It follows that in order to establish the wellgradedness property of a family of binary relations defined by a representational structure $\mathfrak{R}$ it suffices to prove that this family satisfies the Distinguishing Property. In the next two sections we show that the families of semiorders, labeled interval orders, and interval orders on a finite set X all satisfy the Distinguishing Property, and so are well graded. Note again that for semiorders and interval orders this was established in Doignon and Falmagne (1997).

5. Labeled interval orders and semiorders

We recall the definitions of semiorders (Luce, 1956) and labeled interval orders (Stanley, 1996). In the rest of the paper X is a given set of cardinality n.

Definition 5.1. A binary relation R on X is a *semiorder* if there exists a function $f : X \to \mathbb{R}$ such that

$$aRb \Leftrightarrow f(a) > f(b) + 1, \tag{6}$$

for all $a, b \in X$.

The family of semiorders on X is denoted by **SO**.

Definition 5.2. Let ρ be a positive function on X (length function, threshold). A binary relation R on X is a *labeled interval order* if there exists a function $f : X \to \mathbb{R}$ such that

$$aRb \Leftrightarrow f(a) > f(b) + \rho(b), \tag{7}$$

for all $a, b \in X$.

Clearly, a constant length function ρ defines semiorders on X. Thus we consider the more general case of labeled interval orders. We denote by $\mathbf{IO}_\rho$ the family of labeled interval orders on X defined by the length function ρ.

The components of the representational structure $\mathfrak{R}$ for the family $\mathbf{IO}_\rho$ are defined as follows:

- $V = \{f : X \to \mathbb{R}^2\}$, so $f = (f_1, f_2)$,
- $W = \{f \in V : f_2 = f_1 + \rho\}$. Thus $W = V'$ is an affine subspace of V,
- $\alpha = (1, 0), \ \beta = (0, 1)$.

In terms of $\mathfrak{R}$, a binary relation R is a labeled interval order if there exists $f \in W$ such that

$$aRb \Leftrightarrow f_1(a) > f_2(b), \tag{8}$$

for all $a, b \in X$. Clearly, (8) is equivalent to (7).

Eqs. (5) defining the arrangement $\mathscr{A}$ have the following form in the case of the family $\mathbf{IO}_\rho$:

$$H_{(a,b)} = \{f \in V' : f_1(a) = f_2(b)\} \tag{9}$$

for $a \neq b$ in X.

Lemma 5.1. *The family* $\mathbf{IO}_\rho$ *satisfies the Distinguishing Property.*

A proof of this lemma is found in the appendix.

By applying Theorem 4.1, we obtain the following result.

Theorem 5.1. *For a given length function* ρ, *the family* $\mathbf{IO}_\rho$ *of all labeled interval orders on a finite set X is well graded. In particular, the family* **SO** *of all semiorders on X is well graded.*

6. Interval orders

In the representational mode, an interval order on X is defined as follows.

Definition 6.1. A binary relation R on X is an *interval order* if there exist functions $f < g$ on X such that

$$aRb \Leftrightarrow f(a) > g(b),$$

for all $a, b \in X$.

We denote by **IO** the family of interval orders on the set X and define the components of its representational structure $\mathfrak{R}$ as follows:

- $V = \{f : X \to \mathbb{R}^2\}$, so $f = (f_1, f_2)$,
- $W = \{f \in V : f_1 < f_2\}$. Thus W is an open cone in $V' = V$,
- $\alpha = (1, 0), \ \beta = (0, 1)$.

Clearly, a binary relation R is an interval order if there is $f \in W$ such that

$$(a, b) \in R \Leftrightarrow f_1(a) > f_2(b), \tag{10}$$

for all $a, b \in X$. Note that condition (10) is the same as condition (8) for labeled interval orders. Thus we can use the same hyperplane arrangement $\mathscr{A}'$ as defined by (9).

Lemma 6.1. *The family* **IO** *satisfies the Distinguishing Property.*

A proof of this lemma is found in the appendix.

As in the previous section, we have the following theorem.

Theorem 6.1. *The family* **IO** *of interval orders on X is well graded.*

Remark. An interval order on X is also an irreflexive biorder from X to X. Thus the family of interval orders on X is a proper subfamily of the family of biorders from X to X. It is worthwhile noting that the well-gradedness of the latter does not imply the wellgradedness of the former.

7. Weak and linear orders

A binary relation R on X is a *weak order* if there exists $f \in V$ such that

$$aRb \Leftrightarrow f(a) > f(b), \tag{11}$$

for all $a, b \in X$ (cf. (4)).

Let $\mathfrak{R} = \langle X, V, W, \alpha, \beta \rangle$ be a representational structure with the following components:

- $V = \{f : X \to \mathbb{R}\}$ is an n-dimensional ordered vector space,
- $W = V$,
- $\alpha = 1$ and $\beta = 1$ are scalars.

The quintuple $\mathfrak{R} = \langle X, V, W, \alpha, \beta \rangle$ is a representational structure for the family **WO** of weak orders on the set X.

The family **WO**, considered as a family of subsets of $X \times X$, is not well graded. For instance, let $X = \{a, b, c\}$ and let $P = \emptyset$ and $Q = \{(a, c), (b, c)\}$. The distance between P and Q is 2, but there is no path of length 2 in **WO** connecting P and Q. Nevertheless, it is possible to 'model', in a rather natural way, the family **WO** by a well graded family of sets.

Consider an arrangement of hyperplanes in the form

$$H_{(a,b)} = \{f \in V : f(a) = f(b)\} \tag{12}$$

for all $a \neq b$ in X (note that $H_{(a,b)} = H_{(b,a)}$). This is the well known *braid arrangement* (Orlik & Terano, 1992; Stanley, 1996) denoted $\mathcal{B}_n$.

It is easy to see that two functions $f, g \in V$ are equivalent (in the sense of formula (4)) if and only if f and g belong to the same cell of the arrangement $\mathcal{A}$ (this means, in particular, that **WO** does not satisfy the Distinguishing Property). Thus the family **WO** is in one-to-one correspondence with the face poset $\mathcal{F}(\mathcal{A})$ of the arrangement $\mathcal{A}$. Actually, this correspondence is an isomorphism between posets **WO** and $\mathcal{F}(\mathcal{A})$. The latter one, in turn, is isomorphic to the family $F(\Pi_{n-1})$ of nonempty faces of the permutahedron Π_{n-1} ordered by inverse inclusion. Note, that this is a well known correspondence between weak orders and faces of a permutahedron: according to Ziegler (1995), "k-faces (of Π_{n-1}) correspond to ordered partitions of (the set X) into $n-k$ nonempty parts" (see also Barbut & Monjardet, 1970, p. 54).

The Hasse diagram of $F(\Pi_{n-1})$ (and therefore that of **WO**) is a partial cube. To prove it, we will construct a representation of **WO** by a well graded family of sets explicitly in terms of weak orders (cf. Ovchinnikov, 2004a; Janowitz, 1984).

Weak orders defined by (11) can be characterized as negatively transitive antisymmetric binary relations on X. A weak order R on the set X can be represented in the form $R = (X_1, \ldots, X_k)$, where the sets X_i are indifference classes of R and xRy if and only if $x \in X_i$ and $y \in X_j$ for some $i < j$. In this case we say that R is a *weak k-order*. In particular, weak n-orders are linear orders, and the only weak 1-order is the empty weak order. The set of all weak k-orders on X is denoted by $\mathbf{WO}(k)$.

The following proposition is the statement of Problem 19 on p. 115 in Mirkin (1979). The proof is straightforward and omitted.

Proposition 7.1. *A weak order $R = (X_1, \ldots, X_k)$ contains a weak order R' if and only if*

$$R' = \left(\bigcup_{j=1}^{i_1} X_j, \bigcup_{j=i_1+1}^{i_2} X_j, \ldots, \bigcup_{j=i_m}^{k} X_j \right)$$

for some sequence of indices $1 \leqslant i_1 < i_2 \cdots < i_m \leqslant k$.

One can say (see Mirkin, 1979, Chapter 2) that $R' \subset R$ if and only if the indifference classes of R are "enlargements of the adjacent indifference classes" of R'.

Let R be a weak order. We denote by J_R the set of all weak 2-orders that are contained in R.

Proposition 7.2. *A weak order admits a unique representation as a union of weak 2-orders, i.e., for any $R \in$ **WO** there is a uniquely defined set $J \subseteq \mathbf{WO}(2)$ such that*

$$R = \bigcup_{U \in J} U. \tag{13}$$

Proof. Clearly, the empty weak order has a unique representation in the form (13) with $J = \emptyset$.

Let $R = (X_1, \ldots, X_k)$ be a weak order with more than one indifference class. By Proposition 7.1, each weak order in J_R is in the form

$$R_i = \left(\bigcup_{1}^{i} X_j, \bigcup_{i+1}^{k} X_j \right), \quad 1 \leqslant i < k.$$

Let $(x, y) \in \bigcup_{i=1}^{k-1} R_i$. Suppose that $\neg(xRy)$. Then $x \in X_p$ and $y \in X_q$ for some $p \geqslant q$. It follows that $\neg(xR_q y)$, a contradiction. This proves (13) with $J = J_R$.

Let $R = (X_1, \ldots, X_k)$ be a weak order in the form (13). Clearly, $J \subseteq J_R$. Suppose that $R_s = (\cup_1^s X_j, \cup_{s+1}^k X_j) \notin J$, for some s. Let $x \in X_s$ and $y \in X_{s+1}$. Then xRy and $\neg(xR_i y)$, for $i \neq s$, a contradiction. Hence, $J = J_R$, which proves uniqueness of representation (13). □

We denote by $\mathcal{J}$ the family of subsets of the set $\mathbf{WO}(2)$ in the form J_W. The set $\mathcal{J}$ is a poset with respect to the inclusion relation.

The following theorem is an immediate consequence of Proposition 7.2.

Theorem 7.1. *The correspondence $R \mapsto J_R$ is an isomorphism of posets* **WO** *and $\mathcal{J}$.*

Clearly, the empty weak order on X corresponds to the empty subset of $\mathbf{WO}(2)$ and the set **LO** of all linear orders on X is in one-to-one correspondence with maximal elements in $\mathcal{J}$.

Theorem 7.2. *The family $\mathcal{J}$ is a combinatorial simplicial complex, i.e., $J \in \mathcal{J}$ implies $J' \in \mathcal{J}$ for all $J' \subseteq J$.*

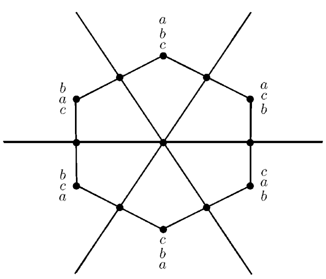
Proof. Let $J' \subseteq J = J_R$ for some $R \in$ **WO**, i.e., $R = \bigcup_{U \in J_W} U$. Consider $R' = \bigcup_{U \in J'} U$. As a union of negatively transitive relations, the relation R' itself is negatively transitive. It is antisymmetric, since $R' \subseteq R$. Thus, R' is a weak order. By Proposition 7.2, $J' = J_{R'} \in \mathscr{J}$. $\square$

It follows that $\mathscr{J}$ is a well graded family of subsets of the set **WO**(2). Thus Theorem 7.1 establishes the required representation of **WO** by a well graded family of sets.

Let now **LO** be the set of linear orders on X. Clearly, the set **LO** is in one-to-one correspondence with the regions of the braid arrangement $\mathscr{B}_n$ defined by (12). The region graph G of $\mathscr{B}_n$ is the 1-skeleton graph of the permutahedron Π_{n-1}. By Proposition 2.1, the graph G is a partial cube.

A representation of **LO** by a well graded family of sets can be obtained as follows. Let L_0 be a fixed linear order on X. It is shown in Ovchinnikov (2004b) that $\{L \cap L_0\}_{L \in \mathbf{LO}}$ is a well graded family of subsets of $X \times X$. The desired representation of **LO** is given by the correspondence $L \mapsto L \cap L_0$.

The diagram below illustrates the geometric and combinatorial features of posets **WO** and **LO** for $X = \{a, b, c\}$ (cf. Fig. 2 on p. 17 in Kemeny & Snell (1972)).



The three long lines are intersections of the planes in $\mathbb{R}^3$ defining the braid arrangement $\mathscr{B}_3$ with the plane $f(a) + f(b) + f(c) = 0$. The hexagon in this diagram is the permutahedron Π_2. To avoid cluttering the figure, only linear orders are labeled.

8. Concluding remarks

1. Melvin Janowitz has pointed out to the author that a lattice theoretical study of the join-semilattice of reflexive weak orders was presented in Janowitz (1984). In particular, it is shown there (Proposition **F1**) that the intervals above atoms of this semilattice are isomorphic to the lattice of all subsets of an $(n-1)$-element set (cf. Theorem 7.2).
2. One particular advantage of the approach presented in the paper is that it can be used to enumerate elements of a particular family of partial orders using the standard enumeration techniques of the theory of hyperplane arrangements (see Stanley (1996) and Postnikov & Stanley (2000), where some examples are given).
3. An approach to ranking patterns of the unfolding model based on hyperplane arrangements is found in Kamiya, Orlik, Takemura, and Terao (2004).

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Appendix

In this section, we present proofs of Lemmas 3.1, 5.1, and 6.1. The three proofs use essentially the same geometric idea. For a function representing a given binary relation, we construct an equivalent function which belongs to a region of the arrangement defining the corresponding family of binary relations.

We begin with a proof of Lemma 3.1.

Proof. Suppose first that (f, g) defines the empty relation by (2), i.e., that $f(x) \leqslant g(y)$ for all $x \in A$, $y \in B$. Clearly, the pair $(f - 1, g)$ defines the empty relation by (2) and this vector does not belong to any of the hyperplanes (3).

Suppose now that (f, g) defines a nonempty biorder R and let α be a number satisfying inequalities

$$0 < \alpha < \min_{(x,y) \in R} (f(x) - g(y)). \tag{14}$$

We define $f^*(x) = f(x) - \alpha$ and prove that (f^*, g) defines the same biorder R by (2) and does not belong to any of the hyperplanes (3).

Let aRb. Then $f^*(a) = f(a) - \alpha > g(b)$, by (14). On the other hand, suppose that $f^*(a) = f(a) - \alpha > g(b)$ for some $(a, b) \in A \times B$. Then, again by (14), $f(a) > g(b)$, i.e., aRb. Thus, $(f^*, g) \sim (f, g)$.

Suppose $(f^*, g) \in H_{(a,b)}$ for some (a, b). Then $f^*(a) = g(b)$, which implies $f(a) > g(b)$. Hence, aRb, implying $f^*(a) > g(b)$, a contradiction. $\square$

Now we prove Lemma 5.1.

Proof. We need to show that any function $f \in W$ is equivalent to some function $f^* \in W$ which does not belong to any of hyperplanes $H_{(a,b)}$ defined by (9).

Suppose first that $f_1(x) \leqslant f_2(y)$ for all $x, y \in X$. Then (8) defines the empty relation on X. For $\lambda = \max\{f_1(x) : x \in X\}$, we define

$$f_1^*(x) = \lambda \quad \text{and} \quad f_2^*(x) = \lambda + \rho(x),$$

for all $x \in X$. Clearly, $(f_1^*, f_2^*) \in W$ and $f_1^*(x) < f_2^*(y)$ for all $x, y \in X$. Thus, $f^* = (f_1^*, f_2^*)$ defines the empty relation by (8) and therefore $f^* \sim f$. Clearly, $f_1^*(a) \neq f_2^*(b)$ for all $a, b \in X$.

Suppose now that, for a given $f \in W$, we have $f_1(x) > f_2(y)$ for some $x, y \in X$. Let R be a labeled interval order defined by (8) and let δ be a number satisfying inequalities:

$$\max_{(x,y)\in R} \frac{\rho(y)}{f_1(x) - f_2(y) + \rho(y)} < \delta < 1. \tag{15}$$

We define $f^* \in W$ by

$$f_1^* = \delta f_1 \quad \text{and} \quad f_2^* = \delta f_1 + \rho$$

and show first that $f^* \sim f$.

Suppose that aRb. Then

$$f_1^*(a) - f_2^*(b) = \delta[f_1(a) - f_2(b) + \rho(b)] - \rho(b) > 0,$$

by the first inequality in (15). On the other hand, if $f_1^*(a) > f_2^*(b)$ for some $a, b \in X$, then

$$\begin{aligned} 0 < f_1^*(a) - f_2^*(b) &= \delta[f_1(a) - f_2(b) + \rho(b)] - \rho(b) \\ &< f_1(a) - f_2(b), \end{aligned}$$

by the second inequality in (15). Thus $(a, b) \in R$. We have proved that $f^* \sim f$.

Suppose that $f_1^*(a) = f_2^*(b)$ for some $a, b \in X$. The previous inequality shows that in this case aRb implying that $f_1^*(a) > f_2^*(b)$, a contradiction. We proved that f^* does not belong to any hyperplane in the form $H_{(a,b)}$. □

Finally, we prove Lemma 6.1.

Proof. Suppose first that $f \in W$ defines the empty interval order by (10), i.e., that $f_1(x) \leqslant f_2(y)$ for all $x, y \in X$. Clearly, for $f_1^* = f_1 - 1$, the function $f^* = (f_1^*, f_2) \in W$ defines the empty interval order and does not belong to any of the hyperplanes in the form $H_{(a,b)}$ (see (9)).

For a given $f \in W$, let $R \neq \emptyset$ be an interval order defined by (10) and let α be a number satisfying inequalities:

$$0 < \alpha < \min_{(x,y)\in R} (f_1(x) - f_2(y)). \tag{16}$$

Let us define $f_1^* = f_1 - \alpha$ and prove that $f^* = (f_1^*, f_2) \in W$ defines the same interval order R by (10).

Suppose that aRb. Then $f_1^*(a) = f_1(a) - \alpha > f_2(b)$ by (16). On the other hand, if $f_1^*(a) = f_1(a) - \alpha > f_2(b)$ for some $a, b \in X$, then $f_1(a) > f_2(b)$ by (16), i.e., aRb. Thus $(f_1^*, f_2) \sim (f_1, f_2)$.

Suppose now that $f_1^*(a) = f_1(a) - \alpha = f_2(b)$. Then $f_1(a) > f_2(b)$, i.e., aRb, which implies $f_1^*(a) > f_2(b)$, a contradiction. Thus f^* does not belong to any of hyperplanes $H_{(a,b)}$. □

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