SIAM J. DISCRETE MATH.
Vol. 12, No. 1, pp. 111-118

MEDIAN GRAPHS AND TRIANGLE-FREE GRAPHS*

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Abstract. Let $M(m,n)$ be the complexity of checking whether a graph G with m edges and n vertices is a median graph. We show that the complexity of checking whether G is triangle-free is at most $O(M(m,m))$. Conversely, we prove that the complexity of checking whether a given graph is a median graph is at most $O(m\log n + T(m\log n, n))$, where $T(m,n)$ is the complexity of finding all triangles of the graph. We also demonstrate that, intuitively speaking, there are as many median graphs as there are triangle-free graphs. Finally, these results enable us to prove that the complexity of recognizing planar median graphs is linear.

Key words. median graph, triangle-free graph, algorithm, complexity

AMS subject classifications. 68Q25, 05C12

PII. S0895480197323494

1. Introduction. All graphs considered in this paper are finite undirected graphs without loops and multiple edges. Unless stated otherwise, for a given graph G, n and m stand for the number of its vertices and edges, respectively.

The *interval* $I(u,v)$ between vertices u and v consists of all vertices on shortest paths between u and v. A *median* of a set of three vertices u, v, and w is a vertex that lies in $I(u,v)\cap I(u,w)\cap I(v,w)$. A connected graph G is a *median graph* if every triple of its vertices has a unique median. Trees, hypercubes, and grid graphs are prime examples of median graphs. It is easy to see that median graphs are bipartite.

By now a rich theory has been developed for median graphs. For instance, they are shown to be the graphs of windex 2 by Chung, Graham, and Saks [8]. They constitute the class of retracts of hypercubes (see Bandelt [5]). They have applications in location theory and consensus theory (see, e.g., McMorris, Mulder, and Roberts [15]). They are the underlying graphs of discrete structures from various areas, involving, e.g., ternary algebras, hypergraphs, convexities, semilattices, join geometries, and conflict models. For a survey of all these aspects of median graphs, the reader is referred to Klavžar and Mulder [14].

It is clear that median graphs can be recognized in polynomial time and a direct approach would yield an $O(n^4)$ algorithm. Jha and Slutzki [13] followed the convex expansion theorem of Mulder [16, 17] to obtain an $O(mn) = O(n^2\log n)$ algorithm. A simple algorithm of the same complexity was recently proposed by Imrich and Klavžar [11]. Currently, the fastest known algorithm for recognizing median graphs is by Hagauer, Imrich, and Klavžar [9] and runs in $O(m\sqrt{n}) = O(n^{3/2}\log n)$ time. The

*Received by the editors June 25, 1997; accepted for publication (in revised form) July 22, 1998; published electronically January 29, 1999.
http://www.siam.org/journals/sidma/12-1/32349.html

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last equality holds because a median graph with n vertices has at most $n \log n$ edges. For more information on these and related algorithms, see [10].

Cartesian product graphs can be recognized in $O(m \log n)$ time by the algorithm of Aurenhammer, Hagauer, and Imrich [3]. The simplest Cartesian product graphs are obtained by multiplying complete graphs on two vertices and are usually called hypercubes or n-cubes. As mentioned above, Bandelt [5] proved that median graphs are very special subgraphs of n-cubes; namely, they are precisely the retracts of the n-cubes. Hence, the natural question arises: Can the complexity $O(m\sqrt{n})$ for recognizing median graphs be improved to, say, $O(m \log^k n)$ for some $k \geq 1$? The main message of this paper claims that this is very unlikely.

Several algorithms are known which recognize triangle-free graphs or, more generally, find all triangles of a given graph. Clearly, a straightforward implementation yields an algorithm of complexity $O(mn)$. It is worthwhile to add that this simple algorithm finds a triangle in $O(n^{5/3})$ on the average (see [12]). In [12] Itai and Rodeh show that Strassen's algorithm for (Boolean) matrix multiplication can be used to solve the problem in $O(n^{\log 7})$ time. In addition, they also give an algorithm using rooted spanning trees of complexity $O(m^{3/2})$. The algorithm finds all triangles of a given graph and becomes linear in the case of planar graphs. Another algorithm which lists all the triangles of a given graph is due to Chiba and Nishizeki [7]. For a graph G its time complexity is $O(a(G)m)$, where $a(G)$ denotes the arboricity of G. They also show that $a(G) \leq O(m^{1/2})$. Thus, the algorithm of Chiba and Nishizeki is in the worst case still of complexity $O(m^{3/2})$. Recently, Alon, Yuster, and Zwick [2] proved that deciding whether a directed or undirected graph contains a triangle, and finding one if it does, can be done in $O(m^{1.41})$ time. For related results we refer to [1, 18].

We continue this paper as follows. We first recall several notions needed in the rest of the paper. Then, in the next section, we introduce the main construction of this paper which for a given triangle-free graph produces a median graph. We study this construction and show that it can be used to deduce that, intuitively speaking, there are as many median graphs as there are triangle-free graphs. In section 3 we use the construction to show that recognizing triangle-free graphs is at most as difficult as recognizing median graphs. For the converse we prove that the complexity of checking whether a graph with m edges and n vertices is a median graph is at most $O(m \log n + T(m \log n, n))$, where $T(m, n)$ is the complexity of finding all triangles of a given graph with m edges and n vertices. A consequence of this relationship is a linear algorithm for the recognition of planar median graphs. It exploits the fact that the triangles of a planar graph can be found in linear time.

The *eccentricity* $e(x)$ of a vertex x in a connected graph G is the maximum distance of x to any other vertex in G. The *radius* $r(G)$ of G is the minimum eccentricity in G, and a vertex x is a *central vertex* of G if $e(x) = r(G)$. The *periphery* of G consists of all vertices in G at distance $r(G)$ to some central vertex in G.

For an edge $e = uv$ in a graph G, the *subdivision* of e is obtained by replacing the edge e by a new vertex adjacent to both u and v. For convenience, we denote the new vertex by e and the new edges by ue and ev.

Let G be a graph. The *simplex graph* $S(G)$ of G is the covering graph of the partially ordered set of the family of simplices (i.e., complete subgraphs) in G ordered by inclusion. In other words, the vertices of $S(G)$ are the complete subgraphs of G (including the empty one), two vertices being adjacent provided they differ in at most one vertex. Simplex graphs were introduced by Bandelt and van de Vel [6].

Obviously, a simplex graph is a median graph: The median of the simplices A, B, C is the simplex $(A \cap B) \cup (A \cap C) \cup (B \cap C)$.

By Q_3^- we denote the graph obtained from the 3-cube Q_3 by deleting one vertex. The antipodal of the deleted vertex is called the *base* of the Q_3^-. In other words, the base is the only vertex of Q_3^- which is incident to three vertices of degree 3. Note that the three vertices of degree 2 in Q_3^- do not have a median.

Finally, the *Cartesian product* $G \square H$ of graphs G and H is the graph with vertex set $V(G) \times V(H)$ and $(a,x)(b,y) \in E(G \square H)$ whenever $ab \in E(G)$ and $x = y$, or $a = b$ and $xy \in E(H)$.

2. Constructing median graphs from triangle-free graphs. Let $G = (V, E)$ be a graph with $|V| = n$ and $|E| = m$. The graph $\widetilde{G} = (\widetilde{V}, \widetilde{E})$ is obtained from G by subdividing all edges of G and adding a new vertex z joined to all the original vertices of G. So we have $\widetilde{V} = V \cup E \cup \{z\}$ and

$$\widetilde{E} = \{zv \mid v \in V\} \cup \{ue \mid e \in E, u \in V \text{ and } u \text{ is incident with } e \text{ in } G\}.$$

Note that $|\widetilde{V}| = n + m + 1$, and $|\widetilde{E}| = n + 2m$. Observe also that $\widetilde{G}$ is connected, even if G is not. An example for this construction is given in Fig. 2.1.

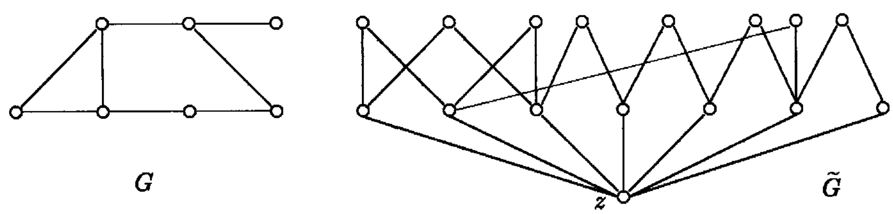

FIG. 2.1. *Illustration of the construction.*

Let d denote the degree function of G and $\widetilde{d}$ that of $\widetilde{G}$. Then we have $\widetilde{d}(z) = n$, and $\widetilde{d}(v) = d(v) + 1$, for $v \in V$, and $\widetilde{d}(e) = 2$, for $e \in E$. Note that z has maximum degree in $\widetilde{G}$, and that $\widetilde{d}(v) = n$ if and only if v is a *dominating vertex* in G, i.e., a vertex adjacent to all other vertices in G.

Since all vertices in $\widetilde{G}$ are at a distance of at most 2 from z, we have $r(\widetilde{G}) \leq 2$. Clearly, we have $r(\widetilde{G}) = 2$ if and only if $m \geq 1$. In any case, z is a central vertex of $\widetilde{G}$. Note that G is disconnected if and only if z is a cut-vertex in $\widetilde{G}$, so that $\widetilde{G}$ is 2-connected if and only if G is nontrivial and connected.

Assume that there is an edge $e = uv$ and a vertex w in G such that w is not incident with e. Then, in $\widetilde{G}$, the vertices w and e have distance 3, so neither vertex is central in $\widetilde{G}$. This implies the following facts: Either

(i) $G = K_2$, and $\widetilde{G} = C_4 = K_2 \square K_2$, and all four vertices of $\widetilde{G}$ are central, or

(ii) $G = K_{1,n-1}$ with $n \neq 2$, and $\widetilde{G} = K_2 \square K_{1,n-1}$, and the two vertices of degree n in $\widetilde{G}$ are the two central vertices of $\widetilde{G}$ (where it is understood that $K_{1,0} = K_1$), or

(iii) G is not a star and z is the unique central vertex of $\widetilde{G}$.

Thus, to reconstruct G from $\widetilde{G}$, we need only to search for a central vertex; take this to be z, take the neighbors of z to be the vertices of G, and replace the remaining vertices, which are all of degree 2, by edges.

An automorphism of a graph necessarily maps central vertices to central vertices. So, if $G = (V, E)$ is not a star, then each automorphism α of $\widetilde{G}$ fixes z. Furthermore, we have $\alpha(V) = V$ and $\alpha(E) = E$ in $\widetilde{G}$. So, essentially, $\alpha|_{V\cup E}$ is an automorphism of G, which gives us the following result.

PROPOSITION 2.1. *Let G be a graph. If $G = K_2$, then $Aut(\widetilde{G}) = Aut(C_4)$. If G is a star different from K_2, then $Aut(\widetilde{G}) \cong \mathbb{Z}_2 \times Aut(G)$. If G is not a star, then $Aut(\widetilde{G}) \cong Aut(G)$.*

Note that G contains a triangle if and only if $\widetilde{G}$ contains a Q_3^-, the base of which necessarily is z.

In case that G is triangle-free, the graph $\widetilde{G}$ is just the simplex graph of G and hence is a median graph. Since Q_3^- is a forbidden convex subgraph in a median graph, we have as an immediate consequence the following result.

THEOREM 2.2. *A graph G is triangle-free if and only if its associated graph $\widetilde{G}$ is a median graph.*

We next have a closer look at the median graphs arising in Theorem 2.2.

Let G be a triangle-free graph. Then G contains a dominating vertex if and only if G is a star. Hence, if G is not a star, then z is not only the unique central vertex but also the unique vertex of maximum degree in $\widetilde{G}$. The only vertices of degree 1 in $\widetilde{G}$ arise from components of G consisting of a single vertex. Let us ignore such components. Then the minimum degree in $\widetilde{G}$ is 2.

Conversely, let H be a median graph of minimum degree 2, with radius $r(H) = 2$, with a unique central vertex z, which is also the unique vertex of maximum degree n. Let p be any vertex in the periphery of H, whence at distance 2 from z. Since H is bipartite, all neighbors of p must be adjacent to z. Since H is median, and therefore without $K_{2,3}$, it follows that p has exactly two neighbors. Let m be the number of vertices in the periphery of H. Then H has $1 + n + m$ vertices and $n + 2m$ edges. Now we construct the graph G on the set of neighbors of z in H. We join two vertices of G by an edge if and only if in H they have a common neighbor in the periphery. Then G has n vertices and m edges and, clearly, we have $H = \widetilde{G}$. Since H is Q_3^--free, G is triangle-free.

Let $\mathcal{G}_{n,m}$ be the class of triangle-free graphs with n vertices and m edges and without singleton components. Let $\mathcal{H}_{n,m}$ be the class of median graphs with minimum degree 2 and radius 2 and a single vertex of maximum degree n, which is also the unique central vertex, and m vertices in the periphery. Thus, we have just proved the following theorem.

THEOREM 2.3. *For each n and m the mapping $G \mapsto \widetilde{G}$ is a bijection between the graph classes $\mathcal{G}_{n,m}$ and $\mathcal{H}_{n,m}$.*

Thus we have an injection of the class $\mathcal{T}$ of triangle-free graphs into the class $\mathcal{M}_2$ of median graphs of radius 2. Let $\mathcal{M}^*$ be the class of all Q_3-free median graphs and let $\mathcal{M}$ be the class of all median graphs. Then we have the following situation:

$$\mathcal{M} \subset \mathcal{T} \hookrightarrow \mathcal{M}_2 \subset \mathcal{M}^* \subset \mathcal{M}\,.$$

Intuitively speaking, we have shown that there are as many median graphs as there are triangle-free graphs. Thus median graphs are much less exotic than one would expect from the definition of median graphs and the rich structure theory by now developed for median graphs.

We conclude this section with the following observation. Let G be a triangle-free graph. Then $\widetilde{G}$ is a median graph and can be isometrically embedded into a hypercube Q_r. Let $i(\widetilde{G})$ be such an embedding. Let uv be an edge of G. Then the corresponding vertices $\widetilde{u}$ and $\widetilde{v}$ lie on a 4-cycle of $\widetilde{G}$. Since i is an isometry, it maps a 4-cycle of $\widetilde{G}$ onto a 4-cycle of Q_r. This in particular implies that $d(i(\widetilde{u}), i(\widetilde{u})) = 2$. Hence we have the following proposition.

PROPOSITION 2.4. *Let G be a triangle-free graph. Then there is an r and a mapping $j : V(G) \to V(Q_r)$, such that if uv is an edge of G, then $d(j(u), j(v)) = 2$.*

3. On the complexity of recognizing median graphs and triangle-free graphs. In this section we will show that the complexity of recognizing median graphs is closely related to the complexity of recognizing triangle-free graphs and to the complexity of finding all triangles of a graph. We first have the following corollary to Theorem 2.2.

COROLLARY 3.1. *Let $M(m, n)$ be the complexity of checking whether a graph G with m edges and n vertices is median. Then the complexity of checking whether G is triangle-free is at most $O(M(m, m))$.*

Proof. By Theorem 2.2 a graph G is triangle-free if and only if $\widetilde{G}$ is a median graph. Since $|E(\widetilde{G})| = 2m + n$ and $|V(\widetilde{G})| = n + m + 1$, $\widetilde{G}$ can be checked if it is a median graph with complexity $O(M(2m + n, n + m + 1)) = O(M(m, m))$. ◻

We can now explain why it seems unlikely that the complexity $O(m\sqrt{n})$ for recognizing median graphs can be improved to $O(m \log^k n)$ for some $k \geq 1$. For, if this were the case, then Corollary 3.1 would imply the existence of an algorithm for recognizing triangle-free graphs of time complexity in $O(m \log^k m) = O(m \log^k n)$, thus significantly improving known algorithms for recognizing triangle-free graphs. Note also that, by Corollary 3.1, the fastest known algorithm for recognizing median graphs, which is of complexity $O(m\sqrt{n})$, yields an $O(m^{3/2})$ algorithm for recognizing triangle-free graphs.

We next consider whether algorithms for recognizing triangle-free graphs can help us in recognizing median graphs, in particular by improving the performance of the algorithm of Hagauer, Imrich, and Klavžar [9] of complexity $O(m\sqrt{n})$. As this algorithm is rather involved, we shall not recall it here in detail but will state whatever is needed for our construction. We refer to this algorithm as Algorithm A.

First some notation. Let $G = (V, E)$ be a connected bipartite graph. For $u \in V(G)$, let $N(u)$ be the set of all vertices adjacent to u. For $X \subseteq V(G)$, let $\langle X \rangle$ denote the subgraph induced by X. A subgraph H of a graph G is an *isometric* subgraph, if the distance in G between any pair of vertices u and v of H is equal to the distance between u and v in H. For any edge ab of G, we write

$$\begin{aligned}
W_{ab} &= \{w \in V \mid d(w, a) < d(w, b)\}, \\
W_{ba} &= \{w \in V \mid d(w, b) < d(w, a)\}, \\
U_{ab} &= \{u \in W_{ab} \mid u \text{ is adjacent to a vertex in } W_{ba}\}, \\
U_{ba} &= \{u \in W_{ba} \mid u \text{ is adjacent to a vertex in } W_{ab}\}, \\
F_{ab} &= \{uv \mid u \in U_{ab}, v \in U_{ba}\}.
\end{aligned}$$

We refer to the set $F = F_{ab}$ as a color. In fact, if G is a median graph, then the sets of type F are a proper edge-coloring of G. Also, G is a median graph if and only if, for any edge ab, the sets U_{ab} and U_{ba} are convex. This characterization was proved by Bandelt [4] but also follows immediately from results in [16, 17]. The bottleneck in testing whether G is a median graph is testing whether the sets U_{ab} are convex. This convexity testing can be reduced to testing condition (iii) listed below. In fact, with one exception, all steps of Algorithm A require at most $O(m \log n)$ time, the exception being Step 3.4, which tests condition (iii) for U_{ab}.

THEOREM 3.2. *Let $G = (V, E)$ be a connected bipartite graph, and let $ab \in E$. Suppose the following properties hold:*

(i) *F_{ab} is a matching that defines an isomorphism between $\langle U_{ab}\rangle$ and $\langle U_{ba}\rangle$;*

(ii) *for any $u \in U_{ab}$ and $v \in U_{ba}$, $I(u, a) \subseteq U_{ab}$ and $I(v, b) \subseteq U_{ba}$, respectively;*

(iii) *for any $u \in W_{ab} \backslash U_{ab}$ and $v \in W_{ba} \backslash U_{ba}$, $|N(u) \cap U_{ab}| \leq 1$ and $|N(v) \cap U_{ba}| \leq 1$.*

Then G is a median graph if and only if $\langle W_{ab}\rangle$ and $\langle W_{ba}\rangle$ are median graphs.

As we mentioned, Algorithm A without Step 3.4 checks all conditions of Theorem 3.2 except (iii) for U_{ab}. We first describe how Algorithm A checks this condition. It first constructs a breadth first search tree, say T, with root a. Suppose that a vertex x from $W_{ab} \setminus U_{ab}$ has two neighbors in U_{ab}, say u and v. As U_{ab} is isometric, there is a vertex $w \in U_{ab}$ which is adjacent to both u and v. Moreover, because condition (i) was also tested before, there are vertices u', v', and w' in U_{ba} which are adjacent to u, v, and w, respectively, such that these six vertices together with x induce a Q_3^-.

Let $L_0, L_1, \ldots$ be the distance levels of the tree T and assume that $x \in L_{i+1}$. Then we know that u and v both belong to L_i as condition (ii) of Theorem 3.2 has already been tested at this stage. Suppose that $w \in L_{i+1}$. Then by the down-closure there is a vertex $r \in L_{i-1}$ adjacent to u and v. But then the vertices x, w, u, v, r induce a $K_{2,3}$, which has been tested before. Hence $w \in L_{i-1}$. We thus have the situation depicted in Fig. 3.1.

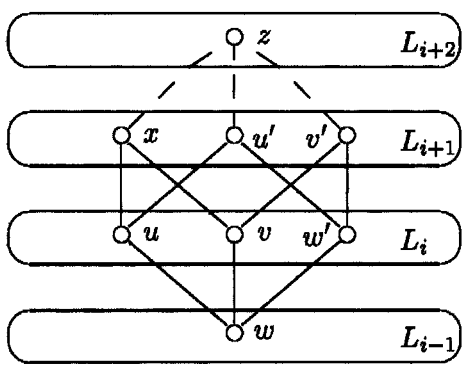

FIG. 3.1. *Testing condition* (iii).

What we need to check now is if there exists a vertex $z \in L_{i+2}$ adjacent to x, u', and v'. If this is not the case, then the test of (iii) fails and G is not a median graph. If no such situation occurs, then all the checks of Theorem 3.2 have been done and G is recognized as a median graph. We next describe how we can do these tests using an algorithm for listing all triangles of a given graph.

Let H_i be the graph on the vertex set L_i and two vertices of H_i are adjacent, if they have a common neighbor in L_{i+1}. By Corollary 4.2 of [9], H_i has at most $|L_{i+1}| \log^2 n$ edges. Thus, all the graphs H_i together have at most $n \log^2 n$ edges. Moreover, there are at most $n \log^3$ triangles in them. We now use an algorithm which finds all triangles in the graphs H_i. For each such triangle of H_i we have only to check whether the corresponding three vertices in L_{i+1} have a common neighbor z in L_{i+2}. This is easy, because G has already been embedded into a hypercube by the previous steps of Algorithm A. In other words, we know precisely the colors of the possible edges between z and the three vertices of L_{i+1}.

Suppose that we have an algorithm of complexity $T(m, n)$ which finds all triangles in a given graph with n vertices and m edges. As we wish to test whether a given graph G is a median graph, we have $m = O(n \log n)$. As already mentioned, all steps of Algorithm A, except Step 3.4, require $O(m \log n)$ time. For the test of (iii) we then follow the above approach, which takes $O(n \log^2 n, n)$ time. Thus, we have proved the following theorem.

THEOREM 3.3. *Let $T(m, n)$ be the complexity of finding all triangles of a given graph with m edges and n vertices. Then the complexity of checking whether a graph G on n vertices and m edges is a median graph is at most $O(m \log n + T(m \log n, n))$.*

As we mentioned in the introduction, the best general algorithm known for listing all triangles of a given graph is of complexity $O(m^{3/2})$. Thus, applying Theorem 3.3, we conclude that median graphs can be recognized in $O(m \log n + (m \log n)^{3/2})$ time. Since $m = O(n \log n)$ this reduces to $O(n^{3/2} \log^3 n)$ which differs only by factor $\log^2 n$ from the complexity of Algorithm A. In special cases this complexity can be further reduced. As an example we show that planar median graphs can be recognized in linear time.

The arguments leading to this result rely on the observation that the factor $\log n$ in Algorithm A without Step 3.4 is a bound on the down-degree of vertices in an isometric subgraph of the hypercube with respect to a distance tree. To be more precise, let x be vertex of in level L_{i+1} with respect to a distance tree of a graph G. Then the number of neighbors of x in L_i is at most $\log n$ if G is a subgraph of a hypercube. This number is called the down-degree of x. See [9].

In [9] it is also shown that every vertex x of down-degree k in a median graph G is contained in a hypercube Q_k. Since Q_k is nonplanar for $k > 3$ this implies that the down-degrees of planar median graphs are bounded by 3 and that Algorithm A without Step 3.4 can be executed in $O(m)$ steps.

COROLLARY 3.4. *Planar median graphs can be recognized in linear time.*

Proof. Let G be a graph on n vertices with m edges. We wish to show that the complexity of checking whether G is a planar median graph is $O(m+n)$. As it is well known that connectedness, bipartiteness, and planarity can be checked in linear time, we can assume that G is a connected, planar, bipartite graph given by its adjacency list and that we wish to check whether it is a median graph. We further observe that a distance-tree can be found in linear time and that down-degrees can be found and checked in linear time, too.

We now consider an embedding of G in the plane and the subgraph X_i spanned by L_{i+1} and L_i. In L_{i+1} there may be vertices of degree 1, 2, or 3 in X_i. Let w be a vertex in L_{i+1} of degree 3 and a, b, c be its neighbors in L_i. We split w into three vertices x, y, z and replace the edges aw, bw, cw with ax, ay, by, bz, cz, cx. We do this for every vertex of degree 3. Clearly the new graph X_i' obtained this way is still planar. Moreover, every vertex of X_i not in L_i has degree 1 or 2. We now delete the vertices of degree 1 and replace every path $x_1x_2x_3$, where $x_1, x_3 \in L_i$ and $x_2 \in L_{i+1}$, by a single edge x_1x_3. This way we obtain the graph H_i from the construction in the proof of Theorem 3.3.

Proceeding as in the proof of Theorem 3.3 we have to find the triangles in the H_i and perform certain checks, the complexity of these operations being determined by the complexity of finding all triangles. Now, the triangles in planar graphs can be found in linear time; cf. [7]. Now the proof is completed by the observation that the total number of edges in the H_i is at most $3n$, where n is the number of vertices of G. ∎

4. Concluding remark. A variant of Theorem 3.5 from [2] can be used to further improve the recognition complexity of median graphs from $O(n^{1.5} \log n)$ to $O(n^{1.41} \log^{2.82} n)$. This will be subject of a subsequent paper.

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