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A catalogue of simplicial arrangements in the real projective plane.

An *arrangement* is the complex generated in the *real projective plane* by a family of straight lines that do not form a pencil. The *vertices* of an arrangement are the intersection points of two or more lines, the *edges* are the segments into which the lines are partitioned by the vertices, and the *faces* are the connected components of the complement of the set of lines generating the arrangement. Figure 1 shows an arrangement generated by five lines; it has five vertices, twelve edges, and eight faces. A *face vector* of an arrangement is a triplet $f = (f_0, f_1, f_2)$, where f_0 is the number of vertices, f_1 the number of edges, and f_2 the number of faces of the arrangement. For the arrangement in Figure 1 we have $f = (5, 12, 8)$. As is well known, Euler's theorem implies that $f_0 - f_1 + f_2 = 1$ for every arrangement.

An arrangement is *simplicial* if all faces are triangles. Clearly, for every simplicial arrangement $2f_1 = 3f_2$. Simplicial arrangements were first introduced by Melchior [8]; an extensive account appears in [5].

Three infinite families of (isomorphism classes of) simplicial arrangements are known; we denote them by $\mathcal{R}(0)$, $\mathcal{R}(1)$ and $\mathcal{R}(2)$.

Family $\mathcal{R}(0)$ consists of all *near-pencils*. A near-pencil denoted $\mathcal{A}(n,0)$ consists of $n-1$ lines that have a point in common, the last line not being incident with that point. An example, with $n = 5$, is shown in Figure 1. Near-pencil simplicial arrangements exist for all $n \geq 3$.

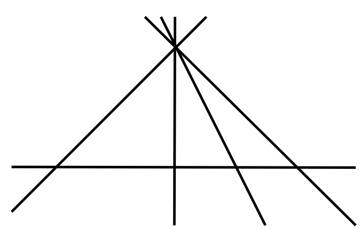

Figure 1. A view of the near-pencil $\mathcal{A}(5,0)$ from the family $\mathcal{R}(0)$.

Family $\mathcal{R}(1)$ consists of simplicial arrangements most easily described as follows. The arrangement denoted $\mathcal{A}(n,1)$ exists for all even $n = 2m$, with $m \geq 3$. Starting with a regular convex m–gon in the Euclidean plane, $\mathcal{A}(n,1)$ is obtained by taking the m lines determined by the sides of the m-gon together with the m lines of mirror symmetry of that m-gon. The arrangements $\mathcal{A}(8,1)$ and $\mathcal{A}(10,1)$ are illustrated in Figure 2.

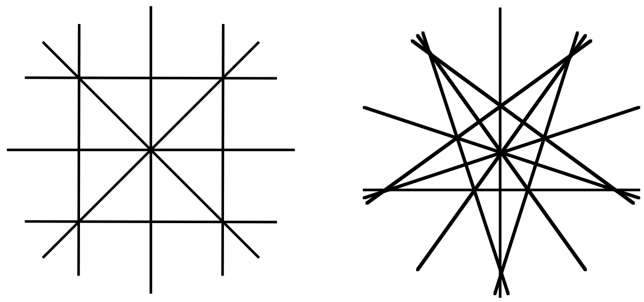

Figure 2. A view of the arrangements $\mathcal{A}(8,1)$ and $\mathcal{A}(10,1)$ from the family $\mathcal{R}(1)$.

Family $\mathcal{R}(2)$ consists of simplicial arrangements denoted $\mathcal{A}(n,1)$, for $n = 4m+1$ with $m \geq 2$. Arrangement $\mathcal{A}(4m+1,1)$ is obtained from $\mathcal{A}(4m,1)$ in the $\mathcal{R}(1)$ family by adjoining the "line-at-infinity" in the extended Euclidean plane model of the projective plane. An example with $m = 3$ is shown in Figure 3. In this illustration, and throughout the catalogue, the inclusion of the line-at-infinity as one of the lines of the arrangement is indicated by the infinity symbol ∞.

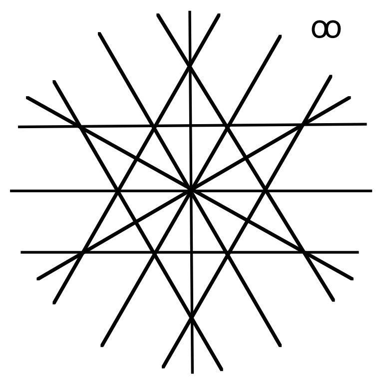

Figure 3. A view of the arrangement $\mathcal{A}(13,1)$ from the family $\mathcal{R}(2)$.

The following table lists all known (isomorphism classes of) simplicial arrangements with at most 37 lines. This is the largest number of lines in any known sporadic simplicial arrangement.

Sporadic arrangements are rather mysterious. There is no known explanation *why* the ones that exist do exist, or why others do not. In particular, there is no known explanation for the observation that no sporadic simplicial arrangement has more than 37 lines. The restriction cannot be related to the topology of the projective plane, since there are several additional infinite families of simplicial arrangements of pseudolines, besides many additional sporadic ones.

The present compilation of data about simplicial arrangements arose from several aims. First, there has been no detailed published account of the known simplicial arrangements beyond the more than a third of a century old paper [5]. The collection in which the paper appeared is not widely available, and the presentation there is not user friendly. Moreover, although the number of sporadic arrangements is still the same as quoted there, two changes have occurred. One pair of arrangements – denoted $\mathcal{A}_2(17)$ and $\mathcal{A}_7(17)$ in the paper – have been found to be isomorphic, as reported in [7] [1, p. 64]. On the other hand, an additional arrangement $\mathcal{A}(16,7)$ was found, as indicated in [6, pages 7 and 9]. The complete list of presently known (isomorphism classes of) simplicial arrangements is given in the table below, and the sporadic members are illustrated by one or more diagrams at the end of the catalog. For each diagram we also list in the table several data, as well any comments that may seem appropriate. I conjecture that the present list is a complete enumeration of isomorphism classes of sporadic arrangements; the language throughout the paper will be simplified by assuming that this is a fact, even though it is the most attractive open problem concerning simplicial arrangements.

The second reason for this collection is the possibility that a better presentation than the one in [5] may generate more interest in the challenging topic. One way of improving the presentation is by noticing that the simplicial arrangements form a partially ordered set, with respect to the relation in which lines of one are a subset of lines of another. The interesting aspect is that there is only a small number of arrangements that are maximal. This is visible from the Hasse diagram shown in Figure 4. Indeed, from their construction it is clear that none of the arrangements in the families $\mathcal{R}(0)$, $\mathcal{R}(1)$ and $\mathcal{R}(2)$ is maximal, while the diagram shows there are only ten sporadic maximal ones. In the diagram, the maximal arrangements are indicated by bold-framed numerals. The numerals with shaded backgrounds indicate "pseudo-minimal" sporadic simplicial arrangements, that is, arrangements that do not contain as subarrangements any sporadic arrangements.

Finally, the simplicial arrangements, or arrangements that are nearly simplicial, appear as examples or counterexamples in many contexts of combinatorial geometry and its applications. This is illustrated, among others, in [6], [7], [1], [2, p. 825], [3, p. 86], [4, Section 5.4].

The list of simplicial arrangement that follows includes all the arrangements with at most 37 lines. The first column, n, denotes the number of lines, the second column the notation for the isomorphism class of the arrangement. The third column, shows the f-vector $f = (f_0, f_1, f_2)$, the number of vertices, edges, and faces of the arrangement. The fourth column, $t = (t_2, t_3, t_4, \ldots)$, list the values of t_j, that is the number of vertices incident with precisely j of the lines. The fifth column, $r = (r_2, r_3, r_4, \ldots)$, similarly gives the value of r_j, the number of lines each of which is incident with precisely j of the vertices. The last column serves to indicate which of the arrangements are non-sporadic, as well as which are maximal (signaled by $\mathcal{M}$) or pseudominimal (indicated by m).

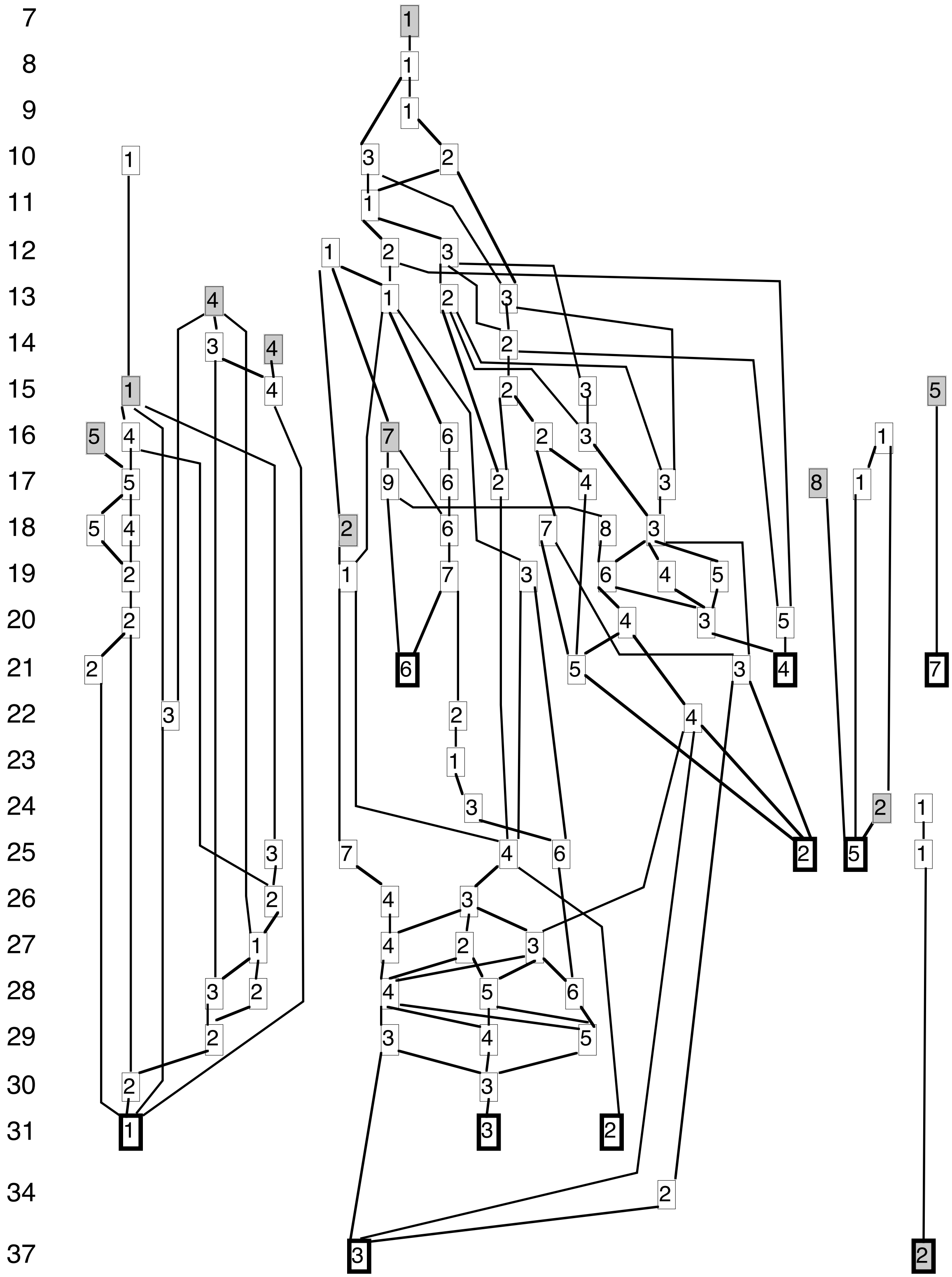

Figure 4. A Hasse diagram of the partially ordered set of sporadic simplicial arrangements, together with a few arrangements from the families $\mathcal{R}(1)$ and $\mathcal{R}(2)$. The arrangement $\mathcal{A}(n,k)$ is indicated by the entry k in row n.

References

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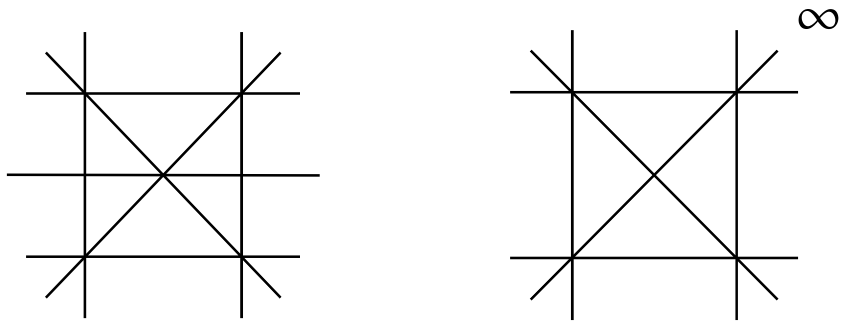
n	$\mathcal{A}$(n,0)	f = (f_0, f_1, f_2)	t = (t_2, t_3, t_4, ...)	r = (r_2, r_3, r_4, ...)	Notes
3	$\mathcal{A}$(3,0)	(3, 6, 4)	(3)	(3)	$\mathcal{R}$(0)
4	$\mathcal{A}$(4,0)	(4, 9, 6)	(3, 1)	(3, 1)	$\mathcal{R}$(0)
5	$\mathcal{A}$(5,0)	(5, 12, 8)	(4, 0, 1)	(4, 0, 1)	$\mathcal{R}$(0)
6	$\mathcal{A}$(6,0)	(6, 15, 10)	(5, 0, 0, 1)	(5, 0, 0, 1)	$\mathcal{R}$(0)
	$\mathcal{A}$(6,1)	(7, 18, 12)	(3, 4)	(0, 6)	$\mathcal{R}$(1)
7	$\mathcal{A}$(7,0)	(7, 18, 12)	(6, 0, 0, 0, 1)	(6, 0, 0, 0, 1)	$\mathcal{R}$(0)
	$\mathcal{A}$(7,1)	(9, 24, 16)	(3, 6)	(0, 4, 3)	m
8	$\mathcal{A}$(8,0)	(8, 21, 14)	(7, 0, 0, 0, 0, 1)	(7, 0, 0, 0, 0, 1)	$\mathcal{R}$(0)
	$\mathcal{A}$(8,1)	(11, 30, 20)	(4, 6, 1)	(0, 2, 6)	$\mathcal{R}$(1)
9	$\mathcal{A}$(9,0)	(9. 24, 16)	(8. 0, ... , 0, 1)	(8, 0, ... , 0, 1)	$\mathcal{R}$(0)
	$\mathcal{A}$(9,1)	(13, 36, 24)	(6, 4, 3)	(0, 0, 9)	$\mathcal{R}$(2)
10	$\mathcal{A}$(10,0)	(10, 27, 18)	(9, 0, ... , 0, 1)	(9, 0, ... , 0, 1)	$\mathcal{R}$(0)
	$\mathcal{A}$(10,1)	(16, 45, 30)	(5, 10, 0, 1)	(0, 0, 5, 5)	$\mathcal{R}$(1)
	$\mathcal{A}$(10,2)	(16, 45, 30)	(6, 7, 3)	(0, 0, 6, 3, 1)	
	$\mathcal{A}$(10,3)	(16, 35, 30)	(6, 7, 3)	(0, 1, 3, 6)	
11	$\mathcal{A}$(11,0)	(11, 30, 20)	(10, 0, ... , 0, 1)	(10, 0, ... , 0, 1)	$\mathcal{R}$(0)
	$\mathcal{A}$(11,1)	(19, 54, 36)	(7, 8, 4)	(0, 0, 4, 4, 3)	
12	$\mathcal{A}$(12,0)	(11, 33, 22)	(11, 0, ... , 0, 1)	(11, 0, ... , 0, 1)	$\mathcal{R}$(0)
	$\mathcal{A}$(12,1)	(22, 63, 42)	(6, 15, 0, 0, 1)	(0, 0, 3, 3, 6)	$\mathcal{R}$(1)
	$\mathcal{A}$(12,2)	(22, 63, 42)	(8, 10, 3, 1)	(0, 0, 3, 3, 6)	
	$\mathcal{A}$(12,3)	(22, 63, 42)	(9, 7, 6)	(0, 0, 3, 3, 6)	
13	$\mathcal{A}$(13,0)	(12, 36, 24)	(12, 0, ... , 0, 1)	(12, 0, ... , 0, 1)	$\mathcal{R}$(0)
	$\mathcal{A}$(13,1)	(25, 72, 48)	(9, 12, 3, 0, 1)	(0, 0, 3, 0, 10)	$\mathcal{R}$(2)
	$\mathcal{A}$(13,2)	(25, 72, 48)	(12, 4, 9)	(0, 0, 3, 0, 10)	
	$\mathcal{A}$(13,3)	(25, 72, 48)	(10, 10, 3, 2)	(0, 0, 1, 4, 8)	
	$\mathcal{A}$(13,4)	(27, 78, 52)	(6, 18, 3)	(0, 0, 0, 0, 13)	m
14	$\mathcal{A}$(14,0)	(14, 39, 26)	(13, 0, ... , 0, 1)	(13, 0, ... , 0, 1)	$\mathcal{R}$(0)
	$\mathcal{A}$(14,1)	(29, 84, 56)	(7, 21, 0, 0, 0, 1)	(0, 0, 0, 7, 0, 7)	$\mathcal{R}$(1)
	$\mathcal{A}$(14,2)	(29, 84, 56)	(11, 12, 4, 2)	(0, 0, 1, 4, 4, 4, 1)	
	$\mathcal{A}$(14,3)	(30, 87, 58)	(9, 16, 4, 1)	(0, 0, 0, 0, 11, 3)	

	$\mathcal{A}(14,4)$	(29, 84, 56)	(10, 14, 4, 0, 1)	(0, 0, 0, 4, 6, 4)	m
15	$\mathcal{A}(15,0)$	15, 42, 28)	(14, 0, ... , 0, 1)	(14, 0, ..., 0, 1)	$\mathcal{R}(0)$
	$\mathcal{A}(15,1)$	(31, 90, 60)	(15, 10, 0, 6)	(0, 0, 0, 0, 15)	m
	$\mathcal{A}(15,2)$	(33, 96, 64)	(13, 12, 6, 2)	(0, 0, 1, 4, 2, 4, 4)	
	$\mathcal{A}(15,3)$	(34, 99, 66)	(12, 13, 9)	(0, 0, 0, 0, 9, 3, 3)	
	$\mathcal{A}(15,4)$	(33, 96, 64)	(12, 14, 6, 0, 1)	(0, 0, 0, 0, 10, 4, 1)	
	$\mathcal{A}(15,5)$	(34, 99, 66)	(9, 22, 0, 3)	(0, 0, 0, 0, 9, 3, 3)	m
16	$\mathcal{A}(16,0)$	(16, 45, 30)	(15, 0, ... , 0, 1)	(15, 0, ... , 0, 1)	$\mathcal{R}(0)$
	$\mathcal{A}(16,1)$	(37, 108, 72)	(8, 28, 0, 0, 0, 0, 1)	(0, 0, 0, 4, 4, 0, 8)	$\mathcal{R}(1)$
	$\mathcal{A}(16,2)$	(37, 108, 72)	(14, 15, 6, 1, 1)	(0, 0, 1, 2, 4, 2, 7)	
	$\mathcal{A}(16,3)$	(37, 108, 72)	(15, 13, 6, 3)	(0, 0, 0, 0, 10, 0, 6)	
	$\mathcal{A}(16,4)$	(36, 105, 70)	(15, 15, 0, 6)	(0,0,0,0,10,5,0,0,1)	
	$\mathcal{A}(16,5)$	(37, 108, 72)	(14, 16, 3, 4)	(0, 0, 0, 2, 4, 8, 0, 2)	m
	$\mathcal{A}(16,6)$	(37, 108, 72)	(15, 12, 9, 0, 1)	(0, 0, 0, 0, 7, 6, 3)	
	$\mathcal{A}(16,7)$	(38, 111, 74)	(12, 19, 6, 0, 1)	(0, 0, 0, 3, 3, 2, 8)	m
17	$\mathcal{A}(17, 0)$	(17, 48, 32)	(16, 0, ... , 0, 1)	(16, 0, ... , 0, 1)	$\mathcal{R}(0)$
	$\mathcal{A}(17,1)$	(41, 120, 80)	(12,24,4,0,0,0,1)	(0, 0, 0, 0, 8, 0, 9)	$\mathcal{R}(2)$
	$\mathcal{A}(17,2)$	(41, 120, 80)	(16, 16, 7, 0, 2)	(0, 0, 1, 0, 6, 0, 10)	
	$\mathcal{A}(17,3)$	(41, 120, 80)	(18, 12, 7, 4)	(0, 0, 0, 0, 8, 0, 9)	
	$\mathcal{A}(17,4)$	(41, 120, 80)	(16, 16, 7, 0, 2)	(0, 0, 1, 0, 6, 0, 10)	
	$\mathcal{A}(17,5)$	(41, 120, 80)	(16, 18, 1, 6)	(0,0,0,0,6,8,1,0,2)	
	$\mathcal{A}(17,6)$	(42, 123, 82)	(16, 15, 10, 0, 1)	(0,0,0,0,6,3,7,0,1)	
	$\mathcal{A}(17,7)$	(43, 126, 84)	(13, 22, 7, 0, 1)	(0,0,0,0,6,0,10,0,1)	was $\mathcal{A}_9(17)$
	$\mathcal{A}(17,8)$	(43, 126, 84)	(14, 20, 7, 2)	(0, 0, 0, 0, 1, 8, 8)	m
18	$\mathcal{A}(18,0)$	(18, 51, 34)	(17, 0, ... , 0, 1)	(17, 0, ... , 0, 1)	$\mathcal{R}(0)$
	$\mathcal{A}(18,1)$	(46, 135, 90)	(9,36,0,0,0,0,0,1)	(0, 0, 0, 0, 9, 0, 0, 9)	$\mathcal{R}(1)$
	$\mathcal{A}(18,2)$	(46, 135, 90)	(18, 18, 6, 3, 1)	(0, 0, 0, 0, 3, 3, 12)	m
	$\mathcal{A}(18,3)$	(46, 135, 90)	(19, 16, 6, 5)	(0,0,0,0,6,2,6,3,1)	
	$\mathcal{A}(18,4)$	(46, 135, 90)	(18, 19, 3, 6)	(0,0,0,0,3,9,3,0,3)	
	$\mathcal{A}(18,5)$	(46, 135, 90)	(18, 19, 3, 6)	(0,0,0,0,3,9,3,0,3)	
	$\mathcal{A}(18,6)$	(47, 138, 92)	(18, 16, 12, 0, 1)	(0,0,0,0,5,2,7,2,2)	
	$\mathcal{A}(18,7)$	(46, 135, 90)	(18, 18, 6, 3, 1)	(0,0,0,0,3,3,0,6,6)	
	$\mathcal{A}(18,8)$	(47, 138, 92)	(16, 22, 6, 2, 1)	(0,0,0,0,6,0,7,4,1,)	
19	$\mathcal{A}(19,0)$	(19, 54, 36)	(18, 0, ... , 0, 1)	(18, 0, ... , 0, 1)	$\mathcal{R}(0)$
	$\mathcal{A}(19,1)$	(49, 144, 96)	(21, 18, 6, 0, 4)	(0, 0, 0, 0, 4, 0, 15)	
	$\mathcal{A}(19,2)$	(51, 150, 100)	(21, 18, 6, 6)	(0,0,0,0,1,8,6,0,4)	
	$\mathcal{A}(19,3)$	(49, 144, 96)	(24, 12, 6, 6, 1)	(0, 0, 0, 0, 4, 0, 15)	

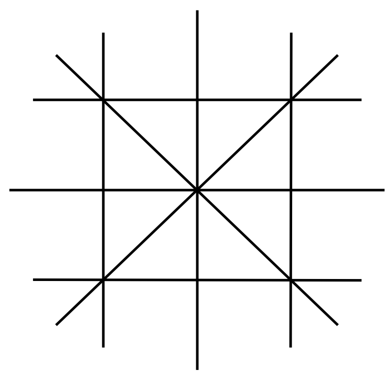
	$\mathcal{A}$(19,4)	(51, 150, 100)	(20, 20, 6, 4, 1)	(0,0,0,0,4,4,4,4,3)	
	$\mathcal{A}$(19,5)	(51, 150, 100)	(20, 20, 6, 4, 1)	(0,0,0,0,4,4,4,4,3)	
	$\mathcal{A}$(19,6)	(51, 150, 100)	(20, 20, 6, 4, 1)	(0,0,0,0,6,0,6,4,3)	
	$\mathcal{A}$(19,7)	(52, 153, 102)	(21, 15, 15, 0, 1)	(0,0,0,0,4,3,3,6,3)	
20	$\mathcal{A}$(20,0)	(20, 57, 38)	(19, 0, ... , 0, 1)	(19, 0, ... , 0, 1)	$\mathcal{R}$(0)
	$\mathcal{A}$(20,1)	(56, 165, 110)	(10,45,0,0,0,0,0,0,1)	(0,0,0,0,5,5,0,0,10)	$\mathcal{R}$(1)
	$\mathcal{A}$(20,2)	(56, 165, 110)	(25, 15, 10, 6)	(0,0,0,0,0,5,10,0,5)	
	$\mathcal{A}$(20,3)	(56, 165, 110)	(21, 24, 6, 4, 0, 1)	(0,0,0,0,4,2,4,6,3,1)	
	$\mathcal{A}$(20,4)	(56, 165, 110)	(23, 20, 7, 5, 1)	(0,0,0,0,5,1,4,4,6)	
	$\mathcal{A}$(20,5)	(55, 162, 108)	(20, 26, 4, 4, 0, 0, 1)	(0,0,0,2,2,0,4,12)	
21	$\mathcal{A}$(21,0)	(21, 60, 40)	(20, 0, ... , 0, 1)	(20, 0, ... , 0, 1)	$\mathcal{R}$(0)
	$\mathcal{A}$(21,1)	(61, 180, 120)	(15, 40, 5, 0, ..., 0, 1)	(0,0,0,0,5,0,5,0,11)	$\mathcal{R}$(2)
	$\mathcal{A}$(21,2)	(61, 180, 120)	(30, 10, 15, 6)	(0,0,0,0,0,0,15,0,6)	
	$\mathcal{A}$(21,3)	(61, 180, 120)	(24, 24, 9, 0, 4)	(0,0,0,0,6,0,3,0,12)	
	$\mathcal{A}$(21,4)	(61, 180, 120)	(22, 28, 6, 4, 0, 0, 1)	(0,0,0,0,4,0,4,8,4,0,1)	$\mathcal{M}$
	$\mathcal{A}$(21,5)	(61, 180, 120)	(26, 20, 9, 4, 2)	(0,0,0,0,5,0,3,4,9)	
	$\mathcal{A}$(21,6)	(63, 186, 124)	(25, 20, 15, 2, 1)	(0,0,0,0,1,0,11,0,8,0,1)	$\mathcal{M}$
	$\mathcal{A}$(21,7)	(64, 189, 126)	(24, 22, 15, 3)	(0,0,0,0,0,0,12,0,6,3)	$\mathcal{M}$
22	$\mathcal{A}$(22,0)	(22, 63, 42)	(21, 0, ... , 0, 1)	(21, 0, ... , 0, 1)	$\mathcal{R}$(0)
	$\mathcal{A}$(22,1)	(67, 198, 132)	(11, 55, 0, ... , 0, 1)	(0,0,0,0,0,11,0,0,0,11)	$\mathcal{R}$(1)
	$\mathcal{A}$(22,2)	(70, 207, 138)	(24, 30, 12, 3, 1)	(0,0,0,0,1,0,6,3,9,0,3)	
	$\mathcal{A}$(22,3)	(67, 198, 132)	(27, 28, 0, 12)	(0,0,0,0,0,0,12,0,9,0,1)	
	$\mathcal{A}$(22,4)	(67, 198, 132)	(27, 25, 9, 3, 3)	(0,0,0,0,4,0,6,0,6,6)	
23	$\mathcal{A}$(23,0)	(23, 66, 44)	(22, 0, ... , 0, 1)	(22, 0, ... , 0, 1)	$\mathcal{R}$(0)
	$\mathcal{A}$(23(1)	(75, 222, 148)	(27, 32, 10, 4, 2)	(0,0,0,0,1,0,6,2,7,4,3)	
24	$\mathcal{A}$(24,0)	(24, 69, 46)	(23, 0, ... , 0, 1)	(23, 0, ... , 0, 1)	$\mathcal{R}$(0)
	$\mathcal{A}$(24,1)	(79, 234, 156)	(12, 66, 0, ... , 0, 1)	(0,0,0,0,0,6,6,0,0,0,12)	$\mathcal{R}$(1)
	$\mathcal{A}$(24,2)	(77, 228, 152)	(32, 32, 0, 12, 0, 0, 1)	(0,0,0,0,0,4,0,0,20)	m
	$\mathcal{A}$(24,3)	(80, 237, 158)	(31, 32, 9, 5, 3)	(0,0,0,0,1,0,6,1,6,6,4)	
25	$\mathcal{A}$(25,0)	(25, 72, 48)	(24, 0, ... , 0, 1)	(24, 0, ... , 0, 1)	$\mathcal{R}$(0)
	$\mathcal{A}$(25,1)	(85, 252, 168)	(18, 60, 6, 0,..., 0, 1)	(0,0,0,0,0,0,12,0,0,0,13)	$\mathcal{R}$(2)
	$\mathcal{A}$(25,2)	(85, 252, 168)	(36, 28, 15, 0, 6)	(0,0,0,0,4,0,3,0,6,0,12)	$\mathcal{M}$
	$\mathcal{A}$(25,3)	(91, 270, 180)	(30, 40, 15,6)	(0,0,0,0,0,0,0,0,15,0,10)	
	$\mathcal{A}$(25,4)	(85, 252, 168)	(36, 30, 9, 6, 4)	(0,0,0,0,1,0,9,0,3,0,12)	
	$\mathcal{A}$(25,5)	(81, 240, 160)	(36, 32, 0, 8, 4, 0, 1)	(0,0,0,0,0,0,5,0,20)	$\mathcal{M}$
	$\mathcal{A}$(25,6)	(85, 252, 168)	(36, 30, 9, 6, 4)	(0,0,0,0,1,0,6,0,6,6,6)	

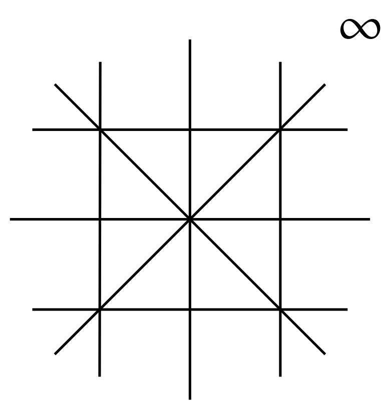
	$\mathcal{A}(25,7)$	(85, 252, 168)	(33, 34, 12, 2, 3, 0,1)	(0,0,0,0,2,0,4,4,4,0,1)	
26	$\mathcal{A}(26,0)$	(26, 75, 50)	(25, 0, ... , 0, 1)	(25, 0, ... , 0, 1)	$\mathcal{R}(0)$
	$\mathcal{A}(26,1)$	(92, 273, 182)	(13, 78, 0, ... , 0, 1)	(0,...,0, 13, 0,...,0, 13)	$\mathcal{R}(1)$
	$\mathcal{A}(26,2)$	(96, 285, 190)	(35, 40, 10, 11)	(0,0,0,0,0,0,0,0,11,5,10)	
	$\mathcal{A}(26,3)$	(92, 273, 182)	(37, 36, 9, 6, 3, 1)	(0,0,0,0,1,0,7,2,2,1,8,4,1)	
	$\mathcal{A}(26,4)$	(92, 273, 182)	(35, 39, 10, 4, 3, 0,1)	(0,0,0,0,1,1,4,4,2,2,7,4,1)	
27	$\mathcal{A}(27,0)$	(27, 78, 52)	(26, 0, ... , 0, 1)	(26, 0, ... , 0, 1)	$\mathcal{R}(0)$
	$\mathcal{A}(27,1)$	(101, 300, 200)	(40, 40, 6, 14, 1)	(0,0,0,0,0,0,0,0,8,8,11)	
	$\mathcal{A}(27,2)$	(99, 294, 196)	(39, 40, 10, 6, 2, 2)	(0,0,0,0,1,0,5,4,1,2,4,8,2)	
	$\mathcal{A}(27,3)$	(99, 294, 196)	(39, 40, 10, 6, 2, 2)	(0,0,0,0,1,0,6,2,2,2,5,6,3)	
	$\mathcal{A}(27,4)$	(99, 294, 196)	(38, 42, 9, 6, 3, 0,1)	(0,0,0,0,1,0,5,4,2,0,7,4,4)	
28	$\mathcal{A}(28,0)$	(28, 81, 54)	(27, 0. ... , 0, 1)	(27, 0, ... , 0, 1)	$\mathcal{R}(0)$
	$\mathcal{A}(28,1)$	(106, 315, 210)	(14, 91, 0, ... , 0, 1)	(14, 91, 0, ... , 0, 1)	$\mathcal{R}(1)$
	$\mathcal{A}(28,2)$	(106, 315, 210)	(45, 40, 3, 15, 3)	(0,0,0,0,0,0,0,0,6,9,13)	
	$\mathcal{A}(28,3)$	(106, 315, 210)	((45, 40, 3, 15, 3)	(0,0,0,0,0,0,0,0,6,9,13)	
	$\mathcal{A}(28,4)$	(106, 315, 210)	(41, 44, 11, 6, 2,1,1)	(0,0,0,0,1,0,4,4,2,1,4,6,6)	
	$\mathcal{A}(28,5)$	(106, 315, 210)	(42, 42, 12, 6, 1, 3)	(0,0,0,0,1,0,4,4,1,3,1,10,4)	
	$\mathcal{A}(28,6)$	(106, 315, 210)	(42, 42, 12, 6, 1, 3)	(0,0,0,0,1,0,6,0,3,3,3,6,6)	
29	$\mathcal{A}(29,0)$	(29, 84, 56)	(28, 0, ... , 0, 1)	(28, 0, ... , 0, 1)	$\mathcal{R}(0)$
	$\mathcal{A}(29,1)$	(113, 336, 224)	(21, 84, 7, 0,..., 0, 1)	(0,0,0,0,0,0,7,0,7,0,0,0,15)	$\mathcal{R}(2)$
	$\mathcal{A}(29,2)$	(113, 336, 224)	(50, 40, 1, 14, 6)	(0,0,0,0,0,0,0,0,5,8,16)	
	$\mathcal{A}(29,3)$	(113, 336, 224)	(44, 46, 13, 6, 2,0,2)	(0,0,0,0,1,0,3,4,3,0,4,4,10)	
	$\mathcal{A}(29,4)$	(113, 336, 224)	(45, 44, 14, 6,1,2,1)	(0,0,0,0,1,0,3,4,2,2,1,8,8)	
	$\mathcal{A}(29,5)$	(113, 336, 224)	(45, 44, 14, 6,1,2,1)	(0,0,0,0,1,0,4,2,3,2,2,6,9)	
30	$\mathcal{A}(30,0)$	(30, 87, 58)	(29, 0, ... , 0, 1)	(29, 0, ... , 0, 1)	$\mathcal{R}(0)$
	$\mathcal{A}(30,1)$	(121, 360, 240)	(15, 105, 0, ... , 0, 1)	(0,0,0,0,0,0,0,15,0,0,0,0,0,15)	$\mathcal{R}(1)$
	$\mathcal{A}(30,2)$	(116, 345, 230)	(55, 40, 0, 11, 10)	(0,0,0,0,0,0,0,0,5,5,20)	
	$\mathcal{A}(30,3)$	(120, 357, 238)	(49, 44, 17, 6, 1,1,2)	(0,0,0,0,1,0,3,2,4,1,2,4,13)	
31	$\mathcal{A}(31,0)$	(31, 90, 60)	(30, 0, ... , 0, 1)	(30, 0, ... , 0, 1)	$\mathcal{R}(0)$
	$\mathcal{A}(31,1)$	(121, 360, 240)	(60, 40, 0, 6, 15)	(0,0,0,0,0,0,0,0,6,0,25)	$\mathcal{M}$
	$\mathcal{A}(31,2)$	(127, 378, 252)	(54, 42, 21, 6,1,0,3)	(0,0,0,0,1,0,0,0,9,0,6,0,15)	$\mathcal{M}$
	$\mathcal{A}(31,3)$	(127, 378, 252)	(54, 42, 21, 6,1,0,3)	(0,0,0,0,1,0,3,0,6,0,3,0,18)	$\mathcal{M}$
32	$\mathcal{A}(32,0)$	(32, 93, 62)	(31, 0, ... , 0, 1)	(31, 0, ... , 0, 1)	$\mathcal{R}(0)$
	$\mathcal{A}(32,1)$	(137, 408, 272)	(16, 120, 0, ... , 0, 1)	(0,0,0,0,0,0,0,8,8,0,0,0,0,0,16)	$\mathcal{R}(1)$
33	$\mathcal{A}(33,0)$	(33, 96, 63)	(32, 0, ... , 0, 1)	(32, 0, ..., 0, 1)	$\mathcal{R}(0)$

	$\mathcal{A}(33,1)$	(145, 432, 288)	(24, 112, 8,0, ...,0,1)	(0,0,0,0,0,0,0,0,16,0,0,0,0,0,17)	$\mathcal{R}(2)$
34	$\mathcal{A}(34,0)$	(34, 99, 66)	(33, 0, ... , 0, 1)	(33, 0, ... , 0, 1)	$\mathcal{R}(0)$
	$\mathcal{A}(34,1)$	(154, 459, 306)	(17, 136, 0, ... , 0, 1)	(0,0,0,0,0,0,0,0,17,0,0,0,0,0,0,17)	$\mathcal{R}(1)$
35	$\mathcal{A}(35,0)$	(35, 102, 68)	(34, 0, ... , 0, 1)	(34, 0, ... , 0, 1)	$\mathcal{R}(0)$
36	$\mathcal{A}(36,0)$	(36, 105, 70)	(35, 0, ... , 0, 1)	(35, 0, ... , 0, 1)	$\mathcal{R}(0)$
	$\mathcal{A}(36,1)$	(172, 513, 342)	(18, 153, 0, ... , 0, 1)	(0,0,0,0,0,0,0,0,9,9,0,0,0,0,0,0,18)	$\mathcal{R}(1)$
37	$\mathcal{A}(37,0)$	(37, 108, 72)	(36, 0, ... , 0, 1)	(36, 0, ... , 0, 1)	$\mathcal{R}(0)$
	$\mathcal{A}(37,1)$	(181, 540, 360)	(27, 144, 9, 0,...,0,1)	(0,0,0,0,0,0,0,0,9,0,9,0,0,0,0,0,19)	$\mathcal{R}(2)$
	$\mathcal{A}(37,2)$	(181, 540, 360)	(72,72,12,24,0,0,0,0,0,0,1)	(0,0,0,0,0,0,0,0,0,0,13,0,0,0,24)	m, $\mathcal{M}$
	$\mathcal{A}(37,3)$	(181, 540, 360)	(72,72,24,0,10,0,3)	(0,0,0,0,0,0,3,0,6,0,4,0,0,0,12,0,12)	$\mathcal{M}$

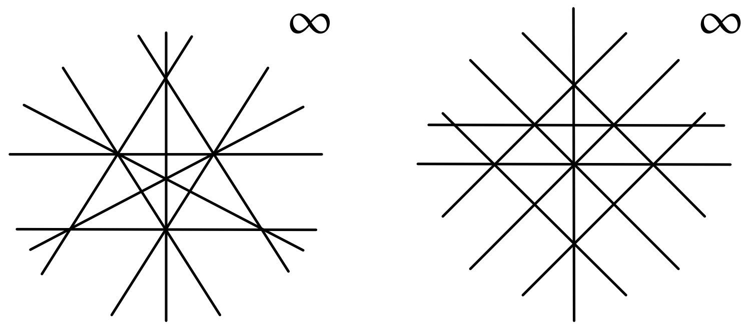


$\mathcal{A}(7, 1)$

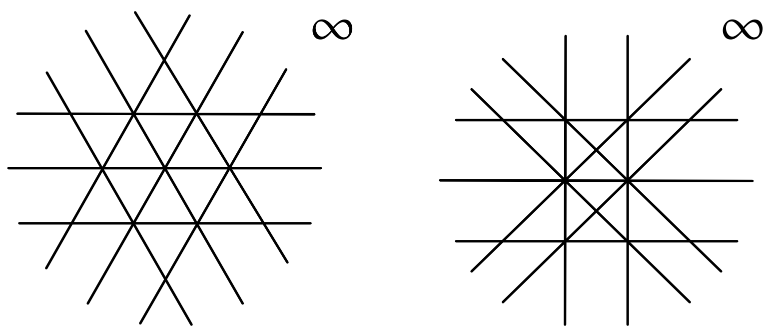

$\mathcal{A}(8, 1)$



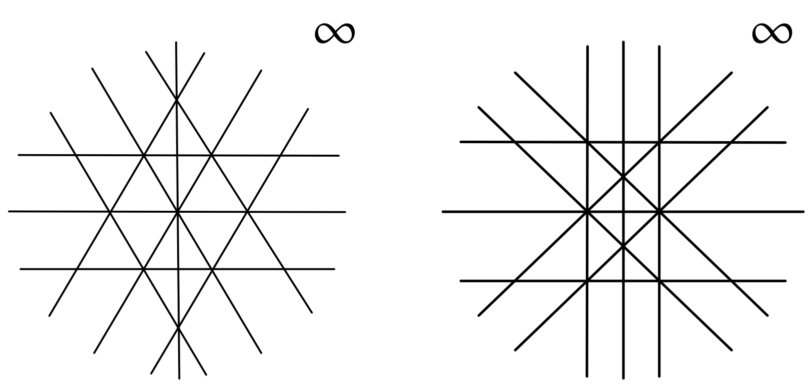
$\mathcal{A}(9, 1)$



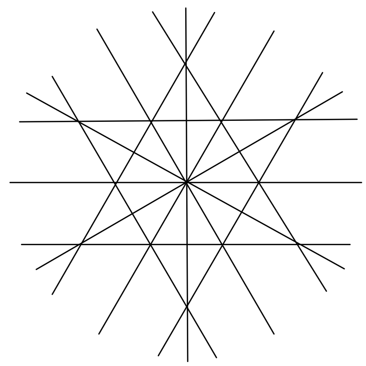
$\mathcal{A}(10, 2)$



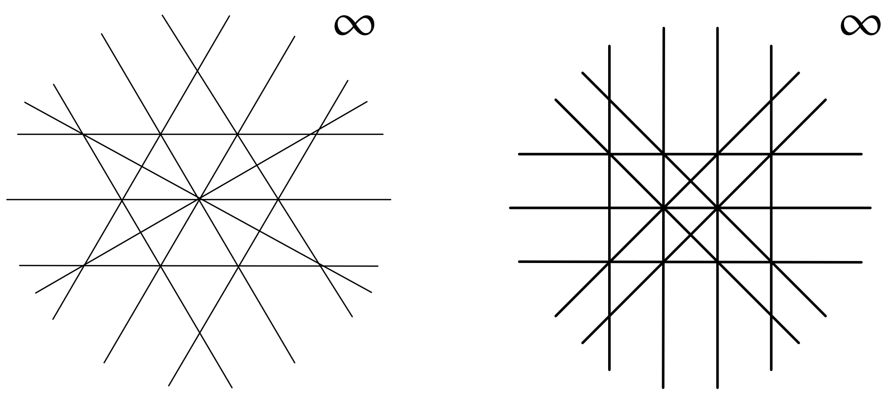
$\mathcal{A}(10, 3)$



$\mathcal{A}(11, 1)$



$\mathcal{A}(12, 1)$



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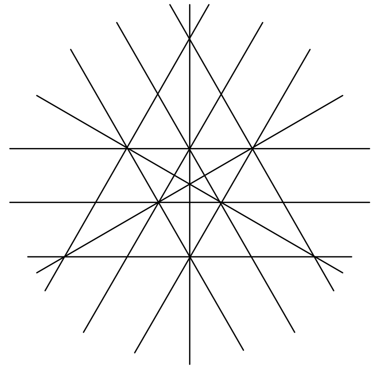

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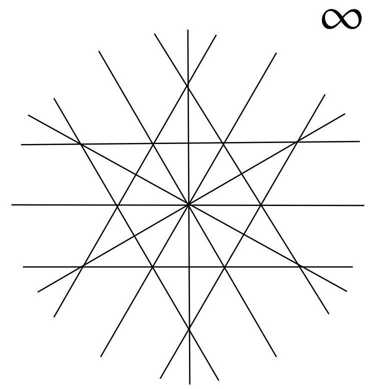
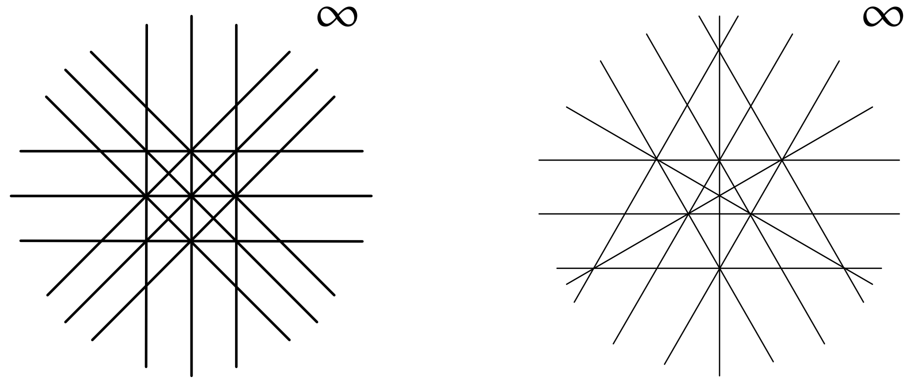


$\mathcal{A}(13, 1)$



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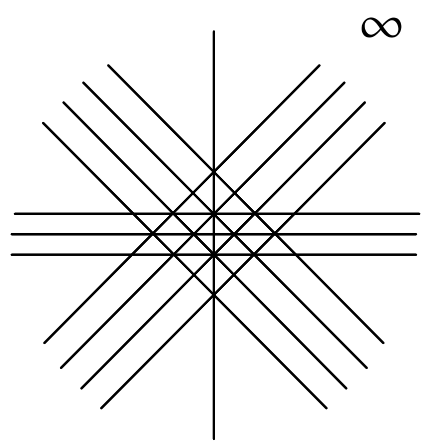
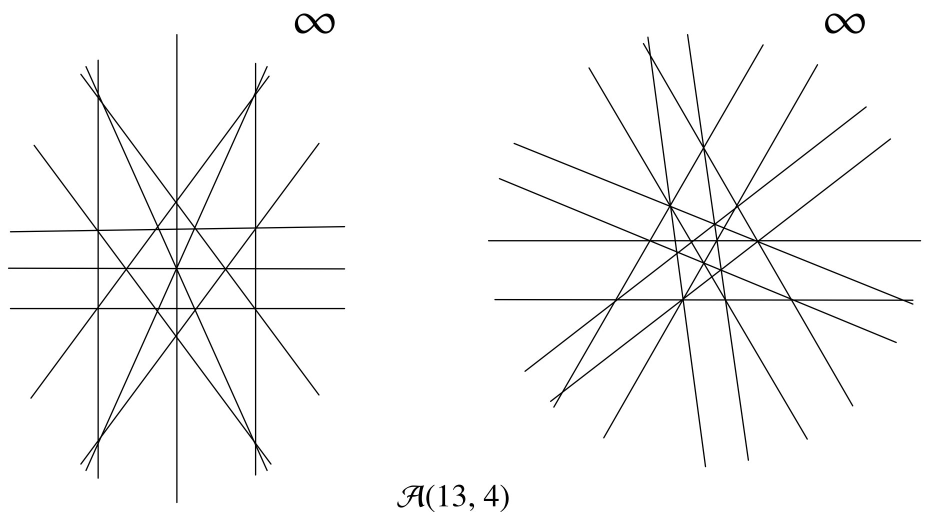
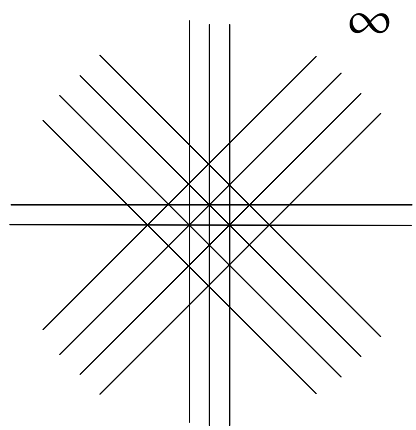


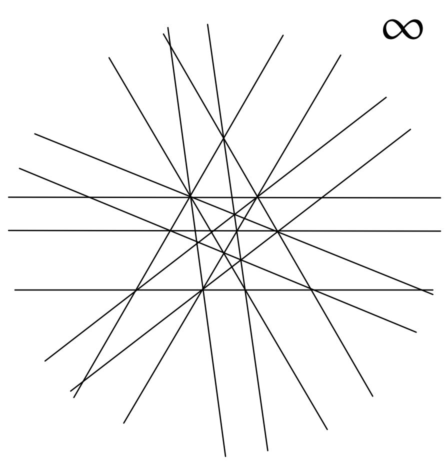
$\mathcal{A}(13, 3)$



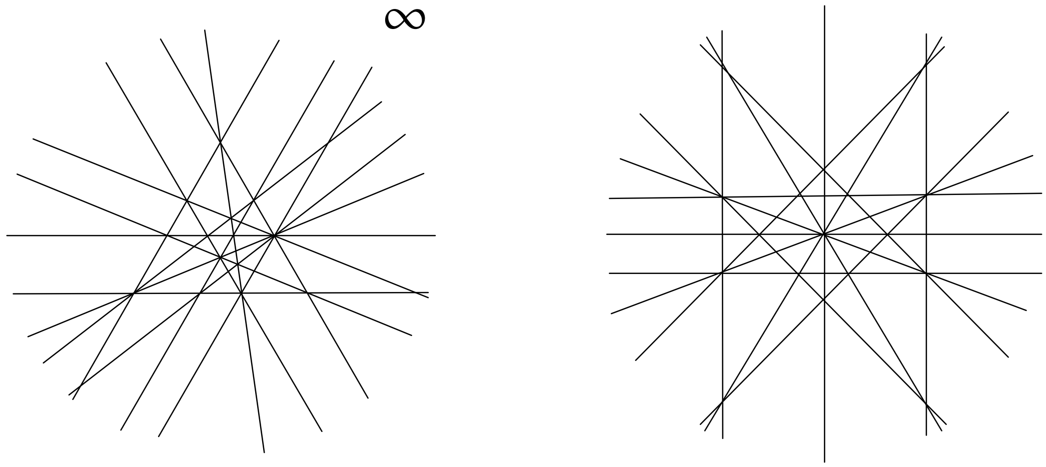
$\mathcal{A}(13, 4)$



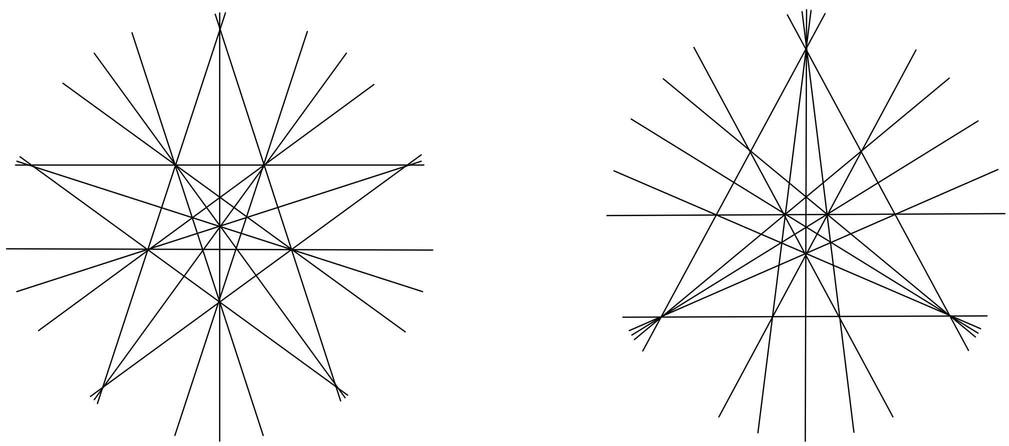
$\mathcal{A}(14, 2)$



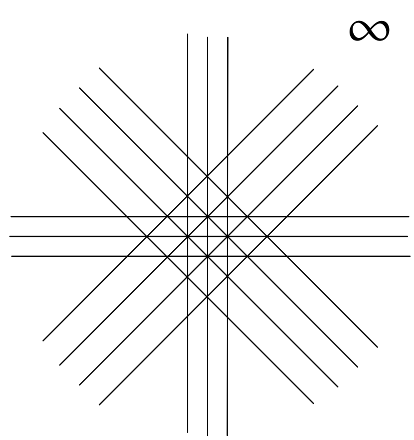
$\mathcal{A}(14, 3)$



$\mathcal{A}(14, 4)$



$\mathcal{A}(15,1)$



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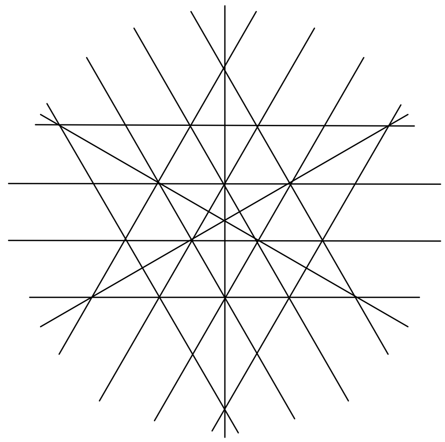
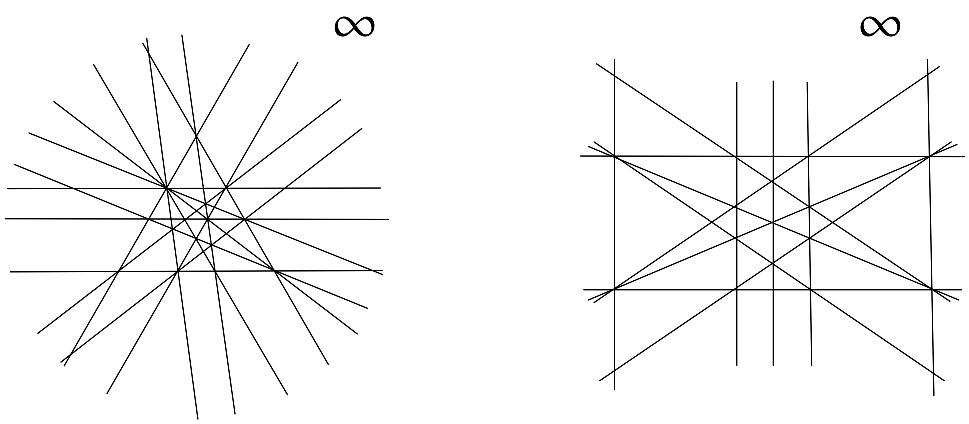
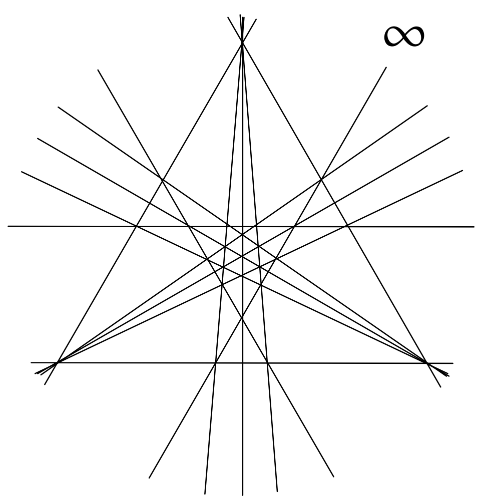


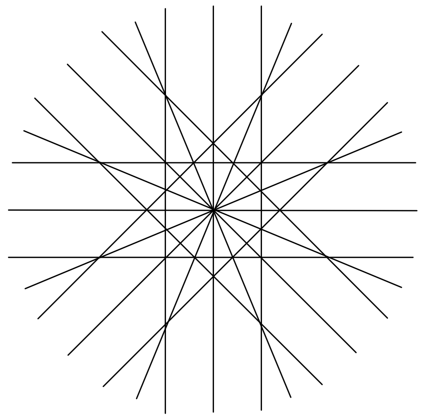
$\mathcal{A}(15,3)$



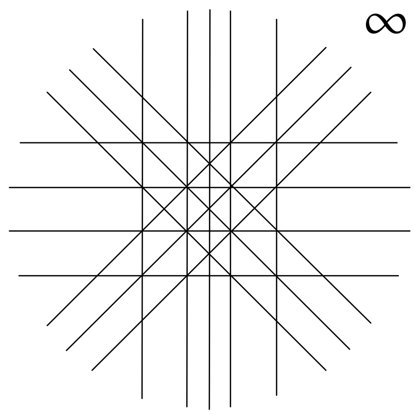
$\mathcal{A}(15,4)$



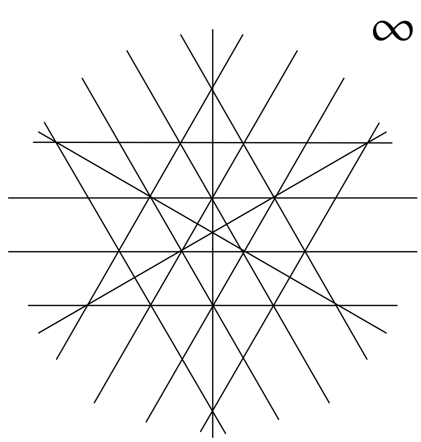
$\mathcal{A}(15,5)$



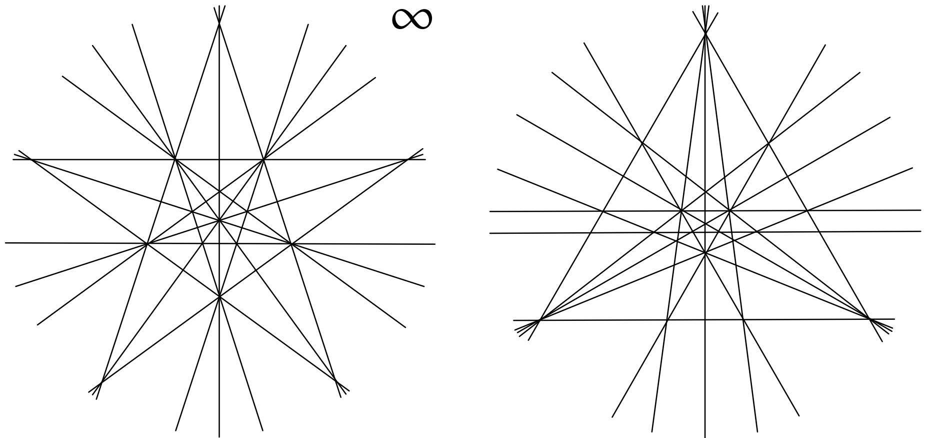
$\mathcal{A}(16, 1)$



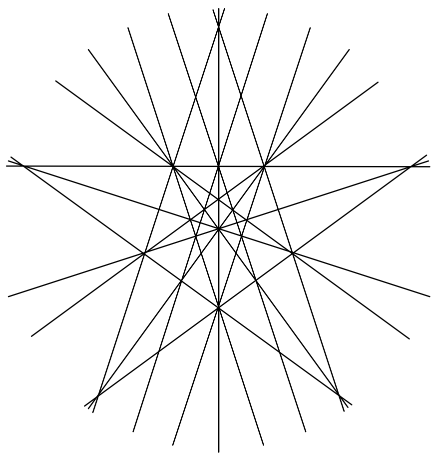
$\mathcal{A}(16, 2)$



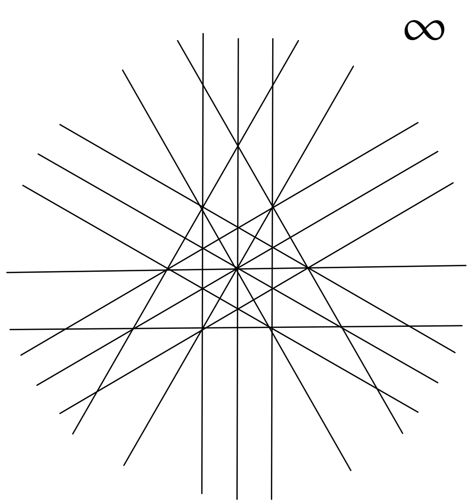
$\mathcal{A}(16, 3)$



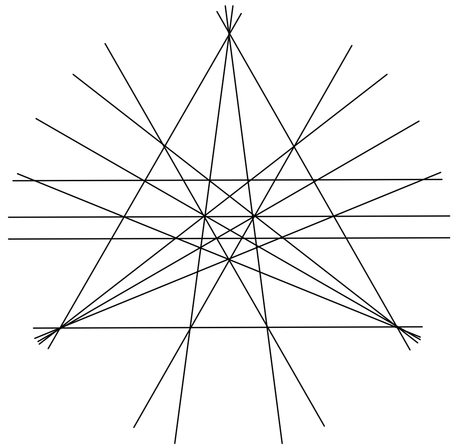
$\mathcal{A}(16, 4)$



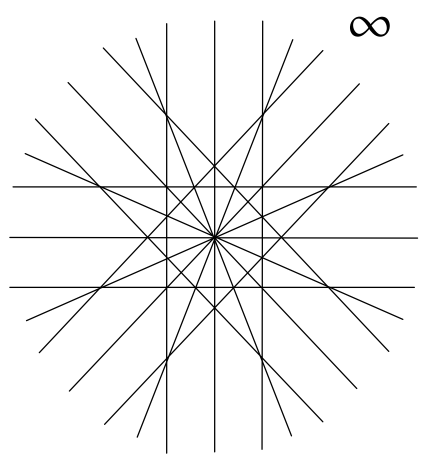
$\mathcal{A}(16, 5)$



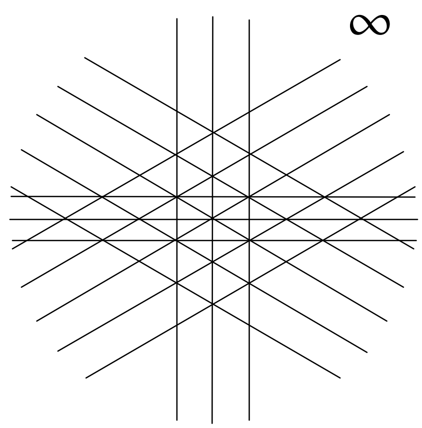
$\mathcal{A}(16, 6)$



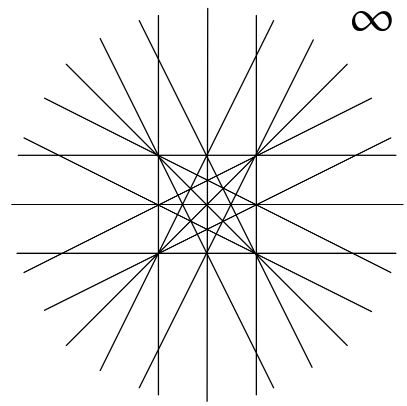
$\mathcal{A}(16, 7)$



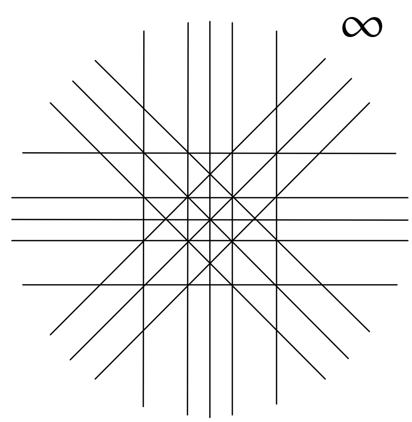
$\mathcal{A}(17, 1)$



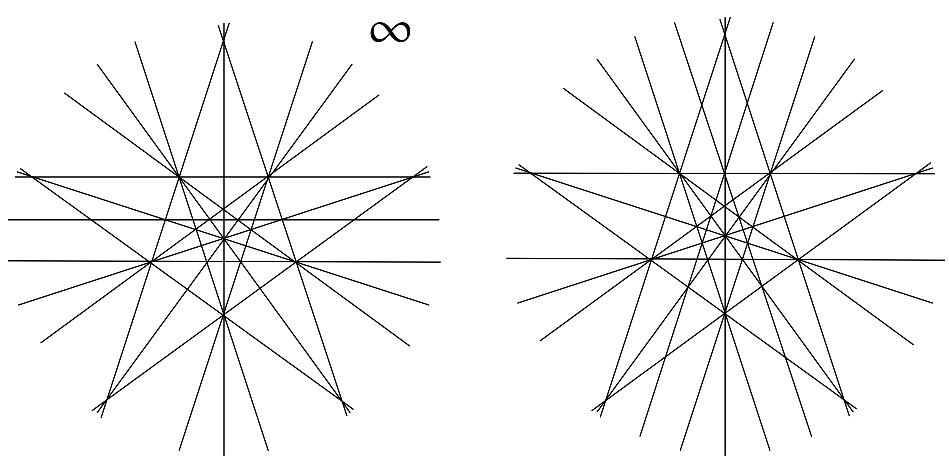
$\mathcal{A}(17, 2)$



$\mathcal{A}(17, 3)$



$\mathcal{A}(17, 4)$



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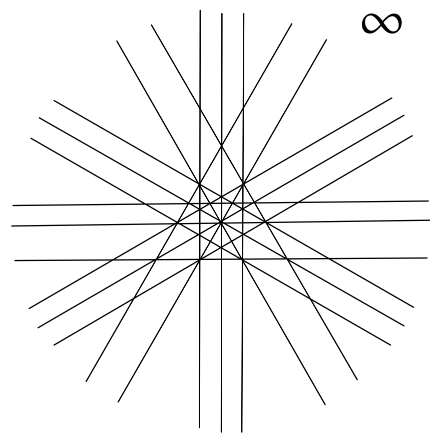
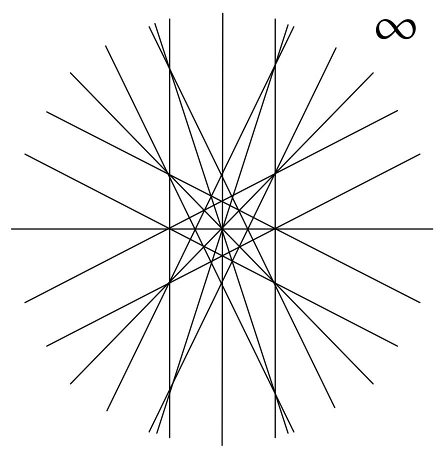
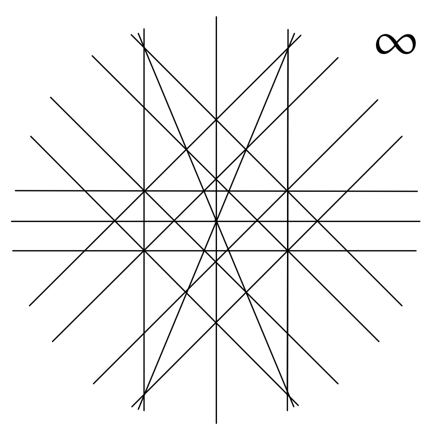


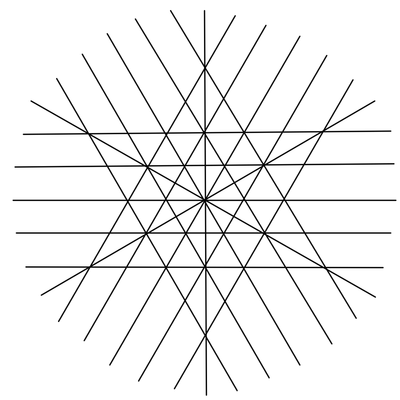
$\mathcal{A}(17, 6)$



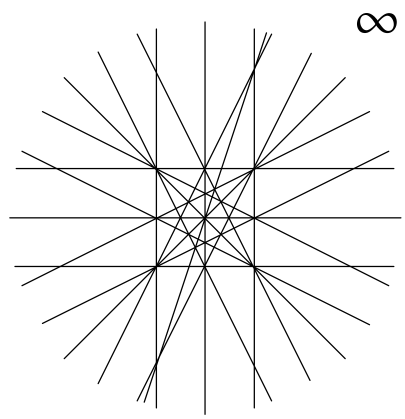
$\mathcal{A}(17, 7)$



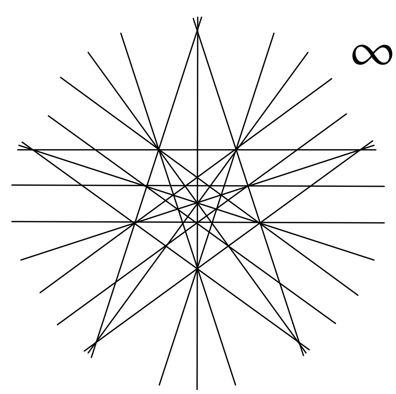
$\mathcal{A}(17, 8)$



$\mathcal{A}(18, 2)$



$\mathcal{A}(18, 3)$



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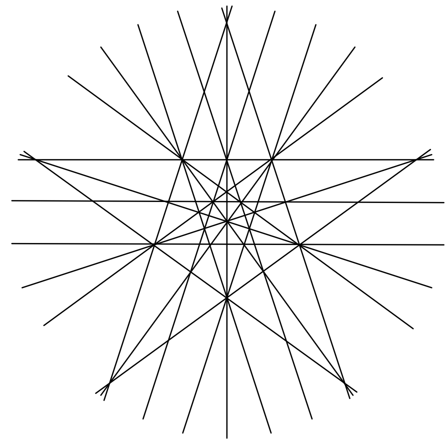
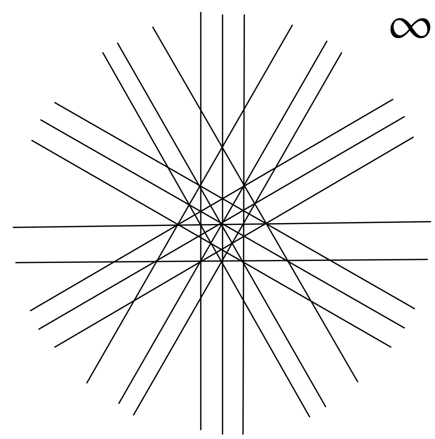
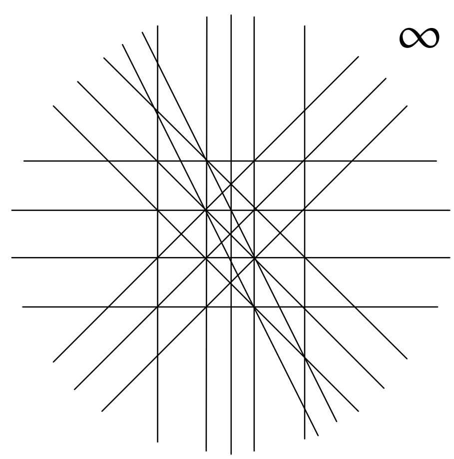


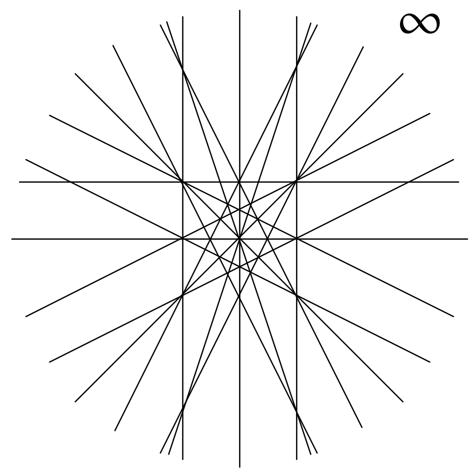
$\mathcal{A}(18, 5)$



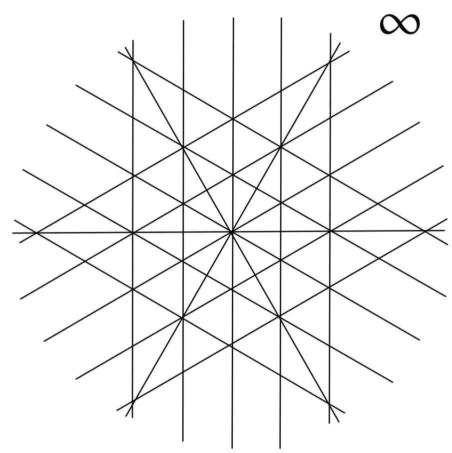
$\mathcal{A}(18, 6)$



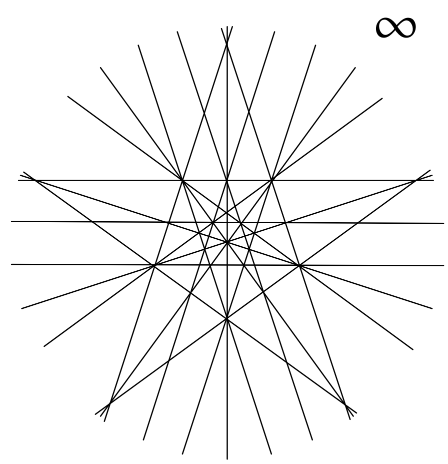
$\mathcal{A}(18, 7)$



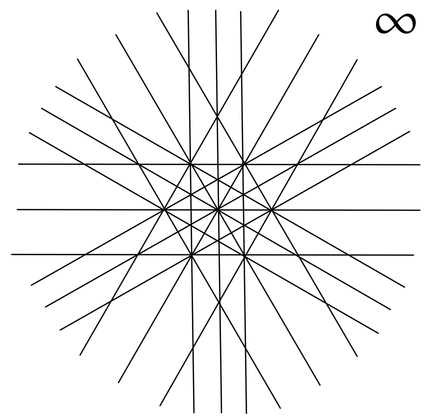
$\mathcal{A}(18, 8)$



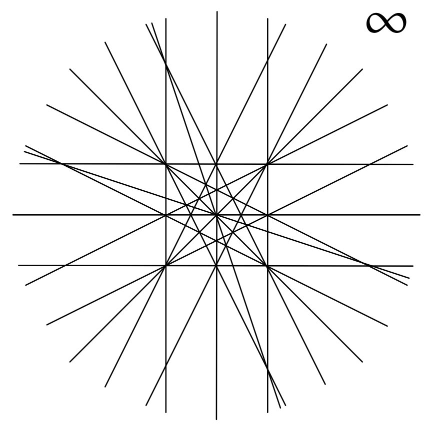
$\mathcal{A}(19, 1)$



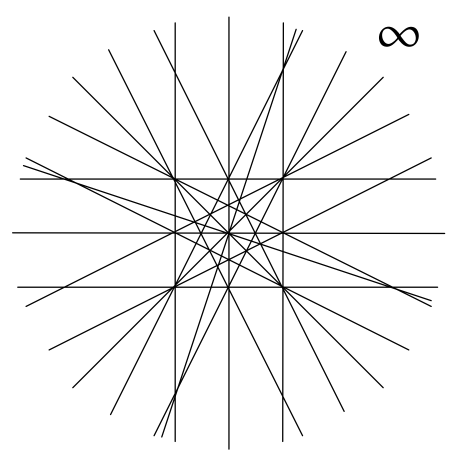
$\mathcal{A}(19, 2)$



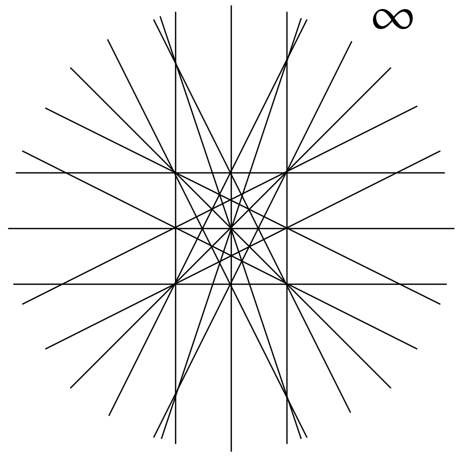
$\mathcal{A}(19, 3)$



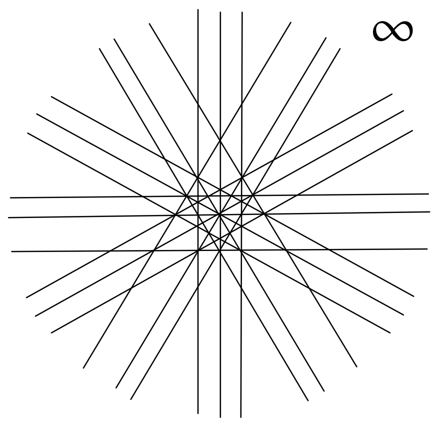
$\mathcal{A}(19, 4)$



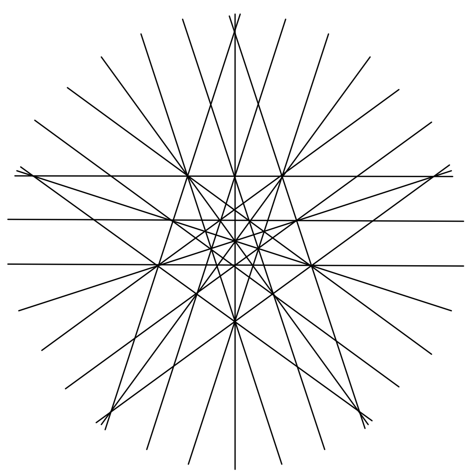
$\mathcal{A}(19, 5)$



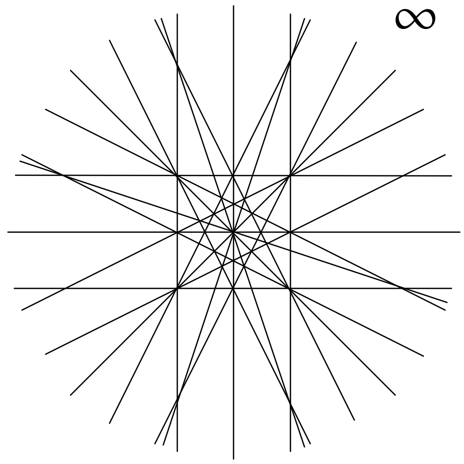
$\mathcal{A}(19, 6)$



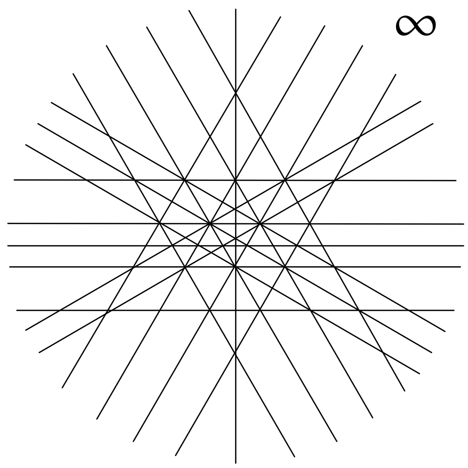
$\mathcal{A}(19, 7)$



$\mathcal{A}(20, 2)$



$\mathcal{A}(20, 3)$



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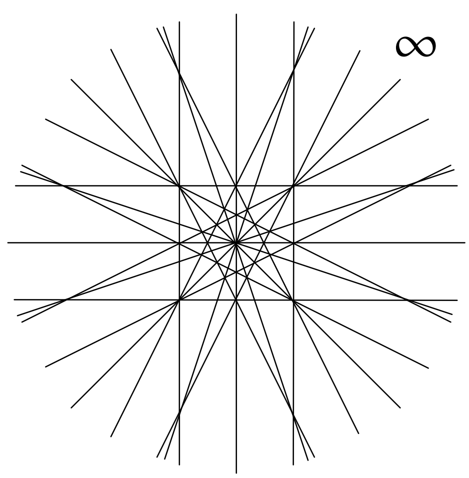
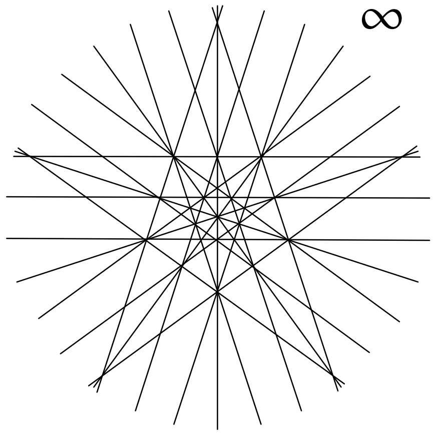
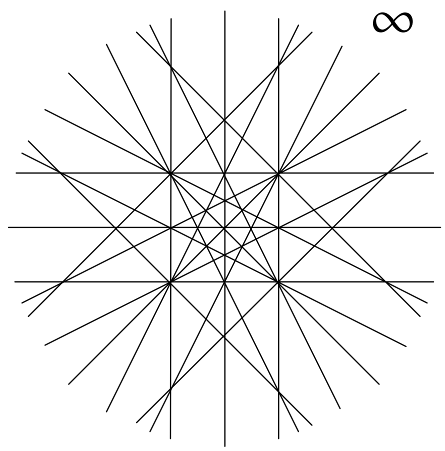


$\mathcal{A}(20, 5)$



$\mathcal{A}(21, 2)$



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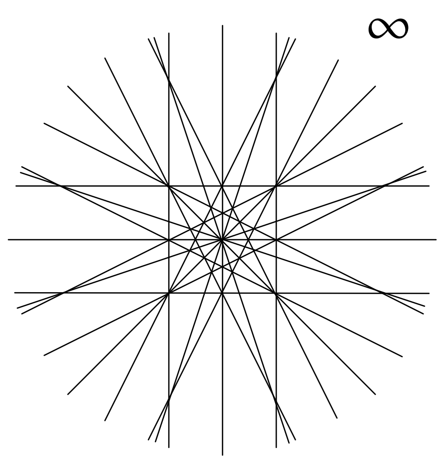

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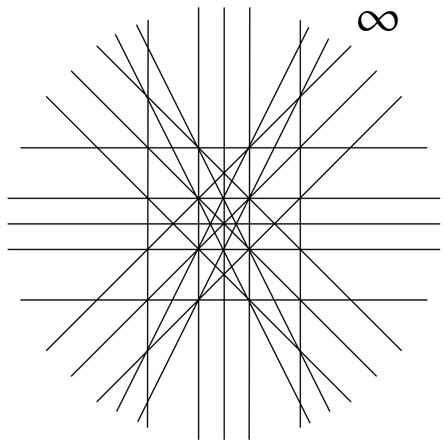

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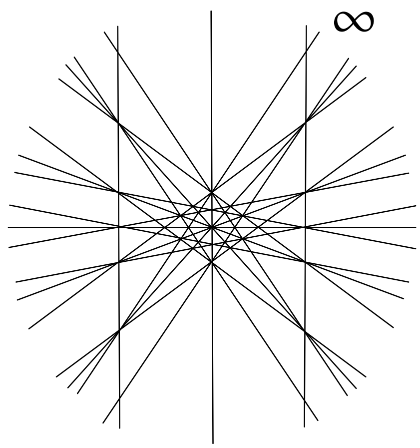

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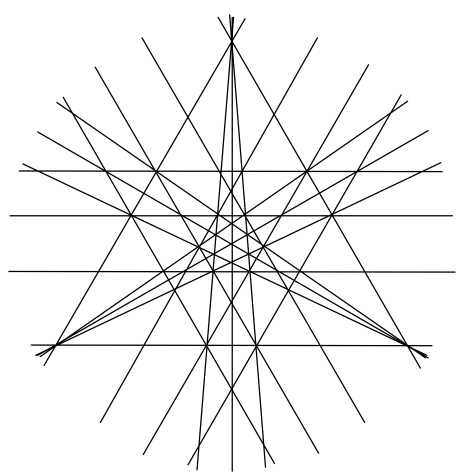

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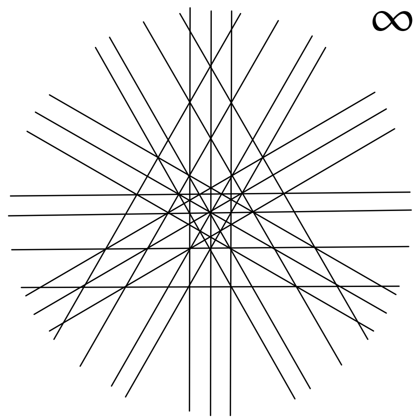

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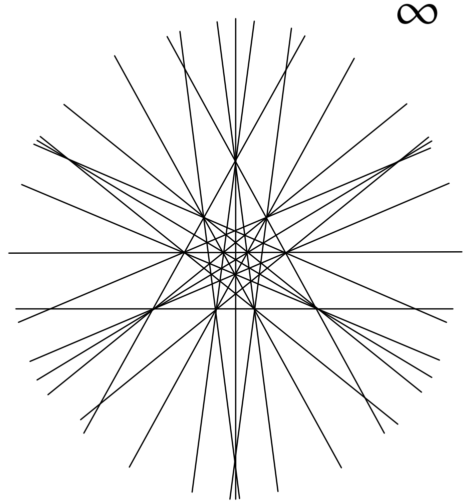

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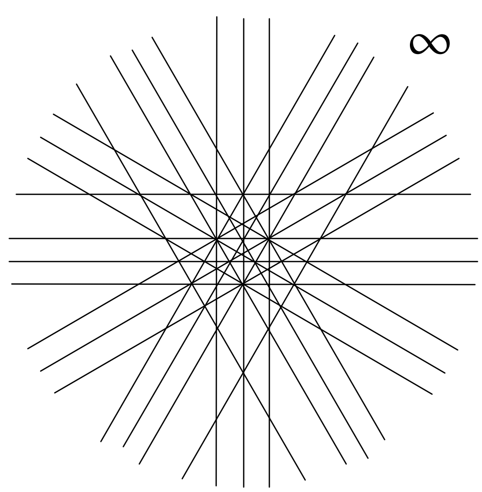

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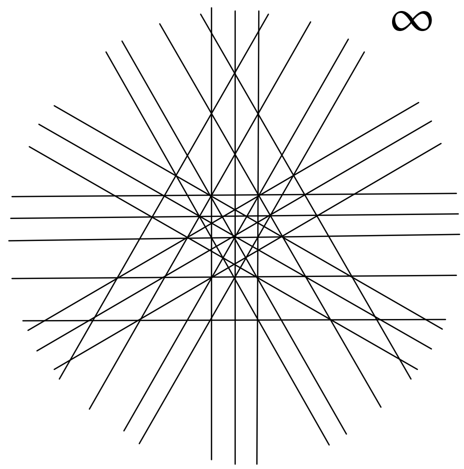
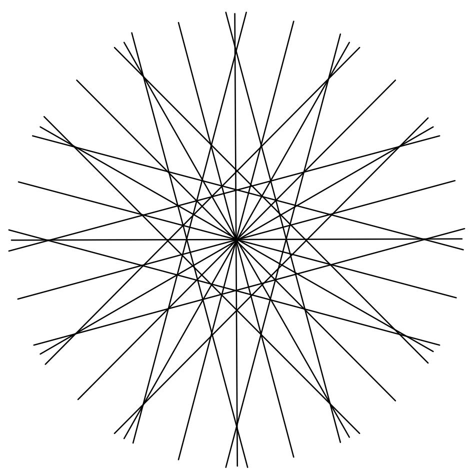
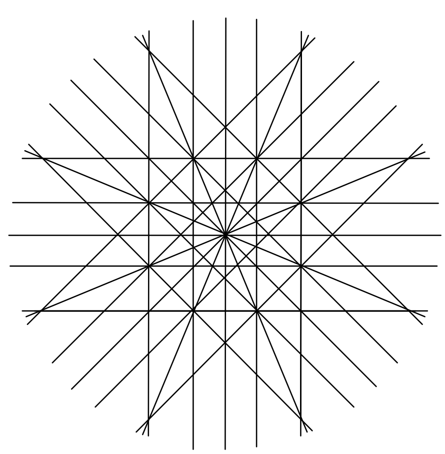


$\mathcal{A}(23, 1)$



$\mathcal{A}(24, 1)$



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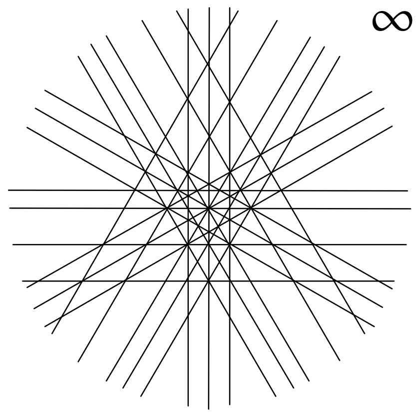

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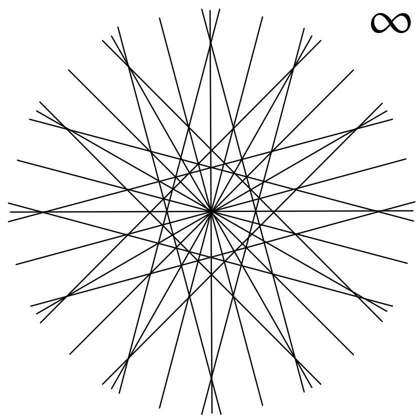
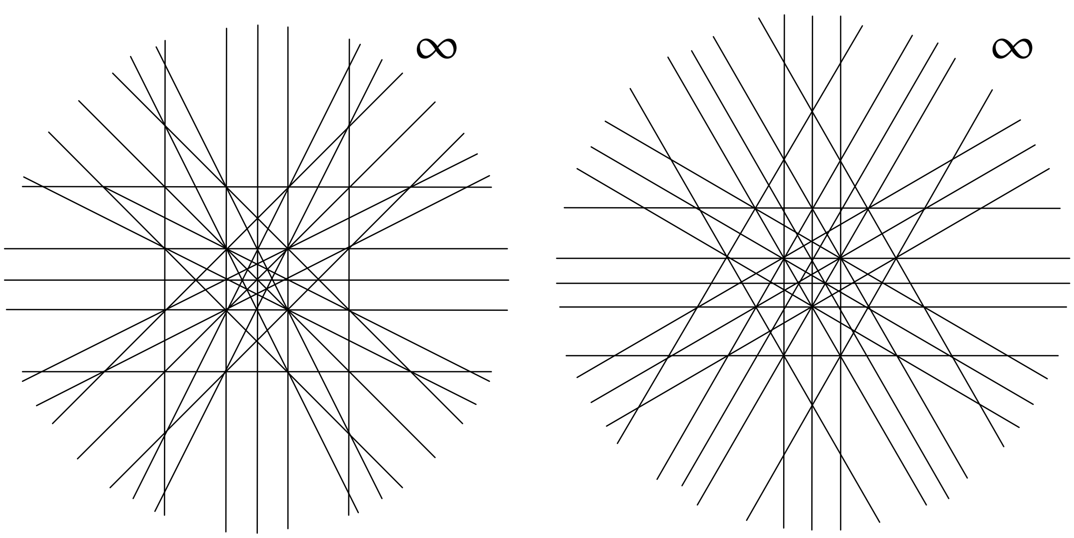


$\mathcal{A}(25, 1)$



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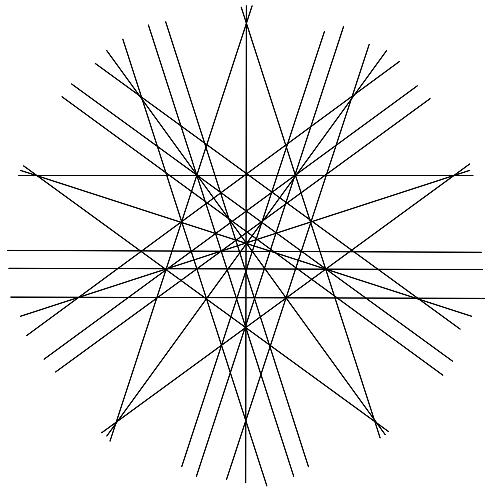

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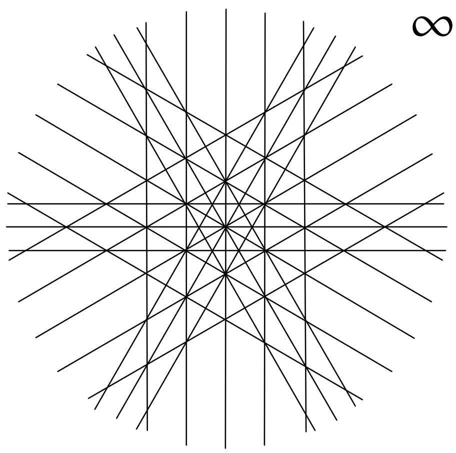
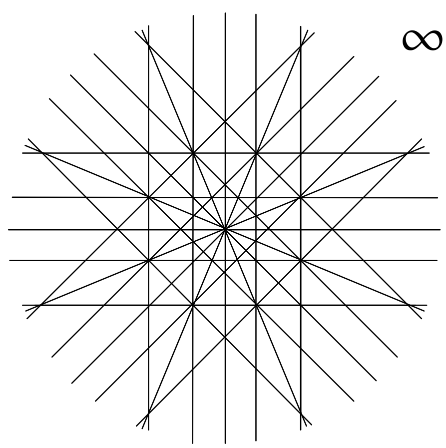
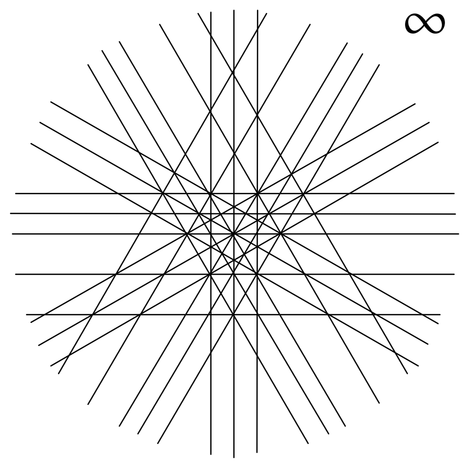


$\mathcal{A}(25, 4)$



$\mathcal{A}(25, 5)$



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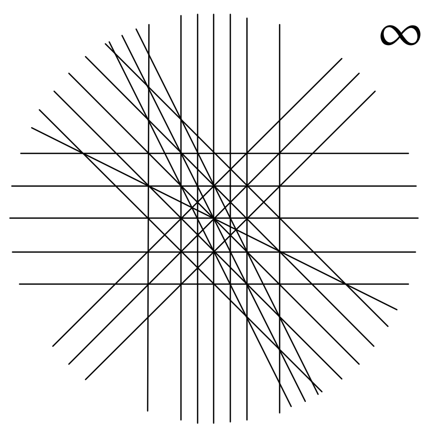

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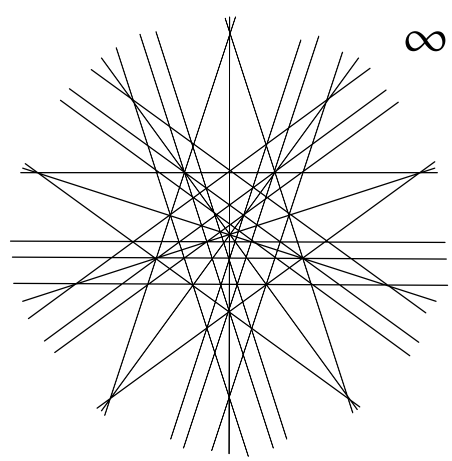

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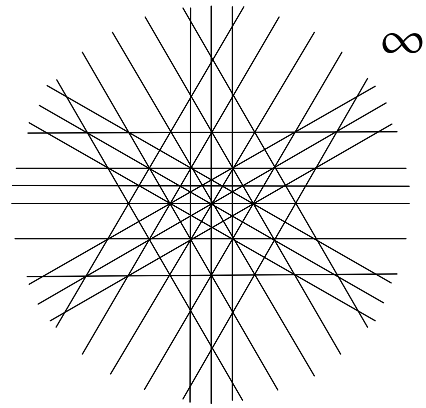

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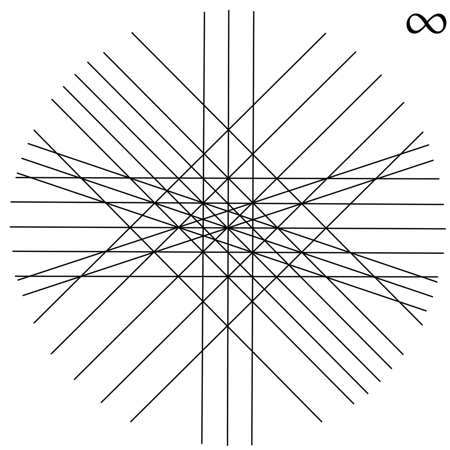

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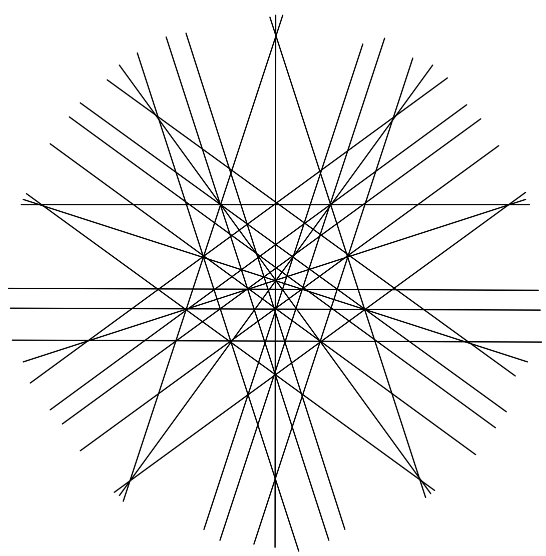

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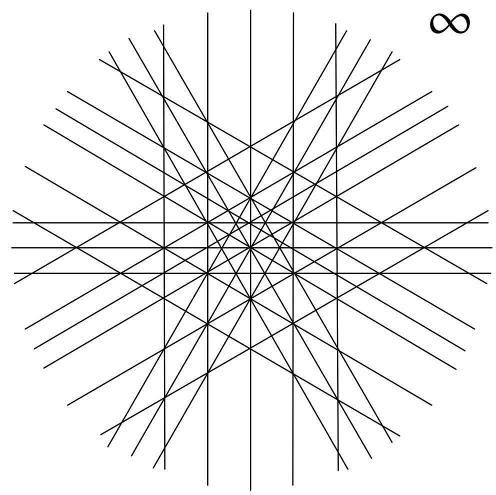

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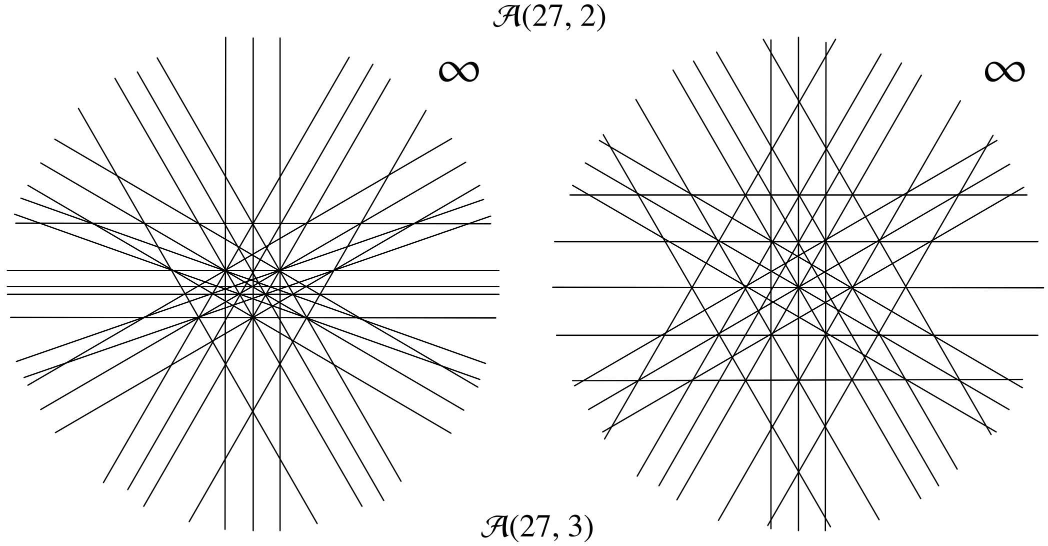

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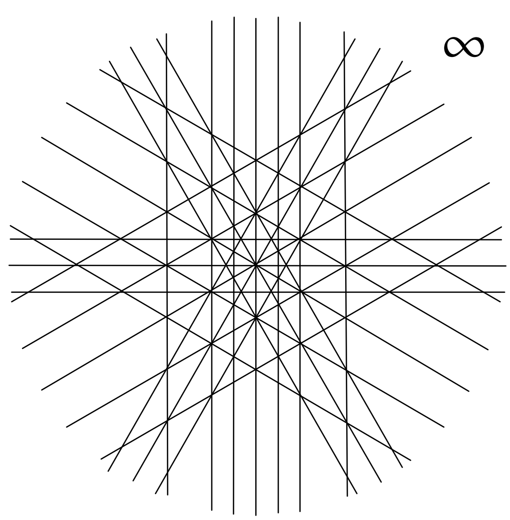

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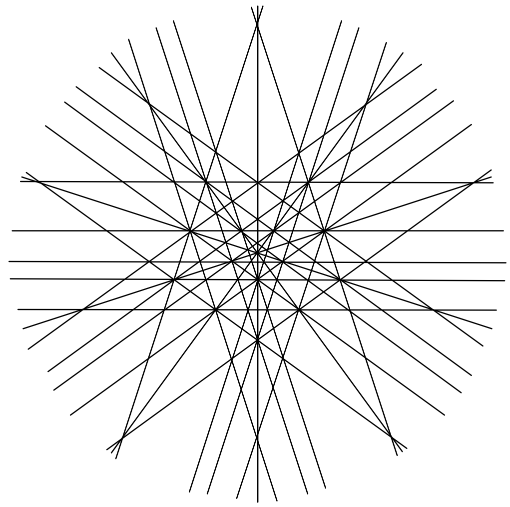

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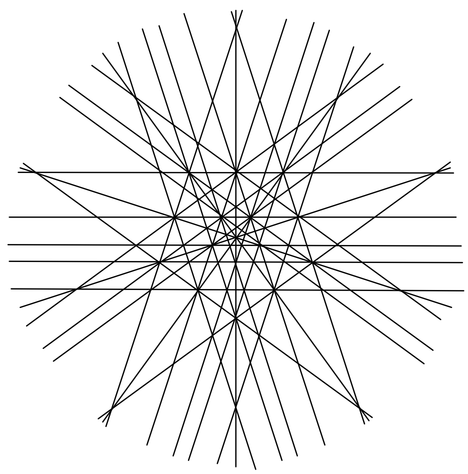

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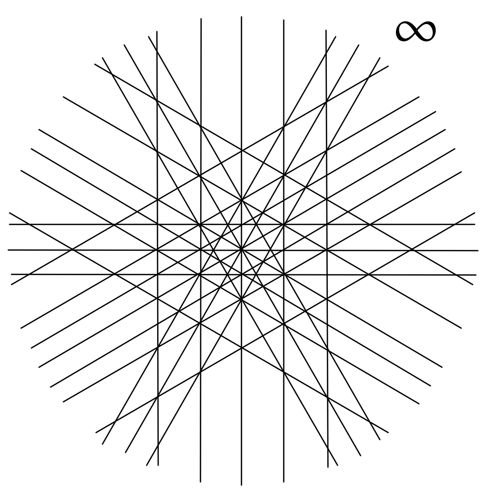

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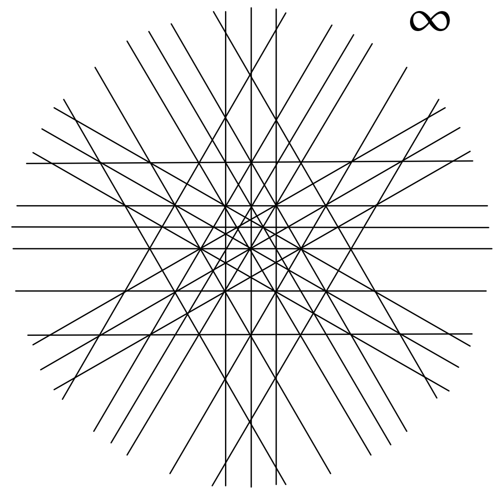
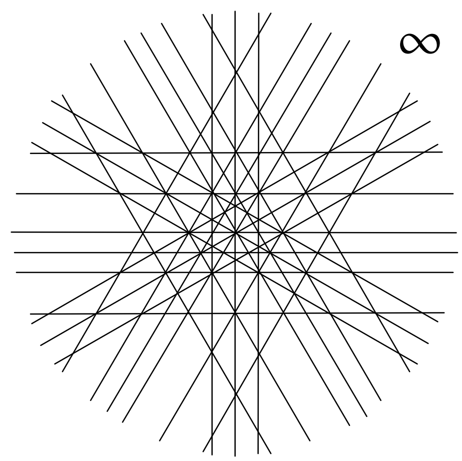
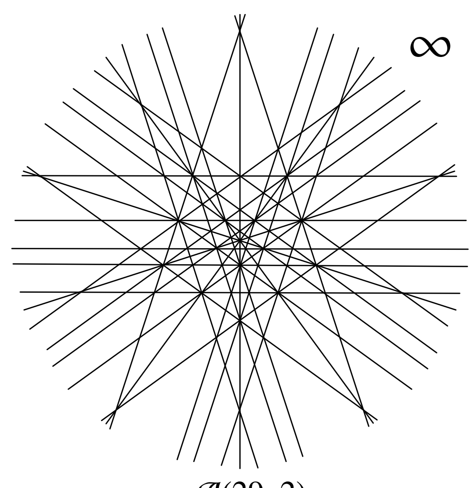


$\mathcal{A}(28, 5)$



$\mathcal{A}(28, 6)$



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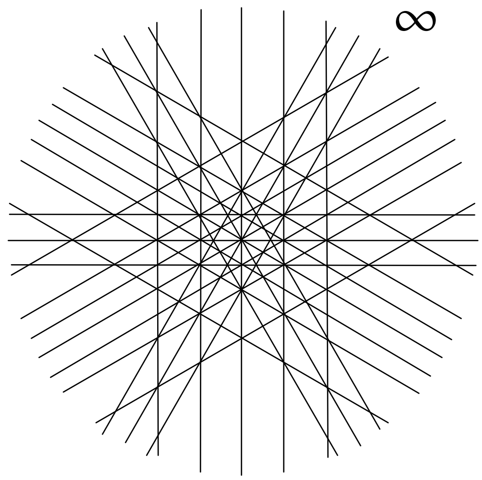

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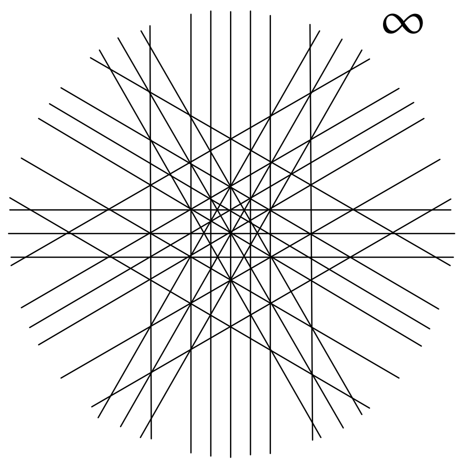

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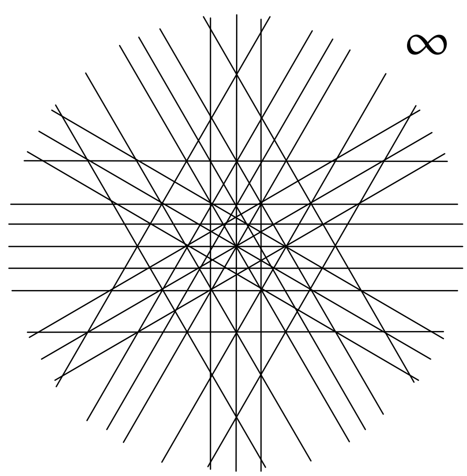

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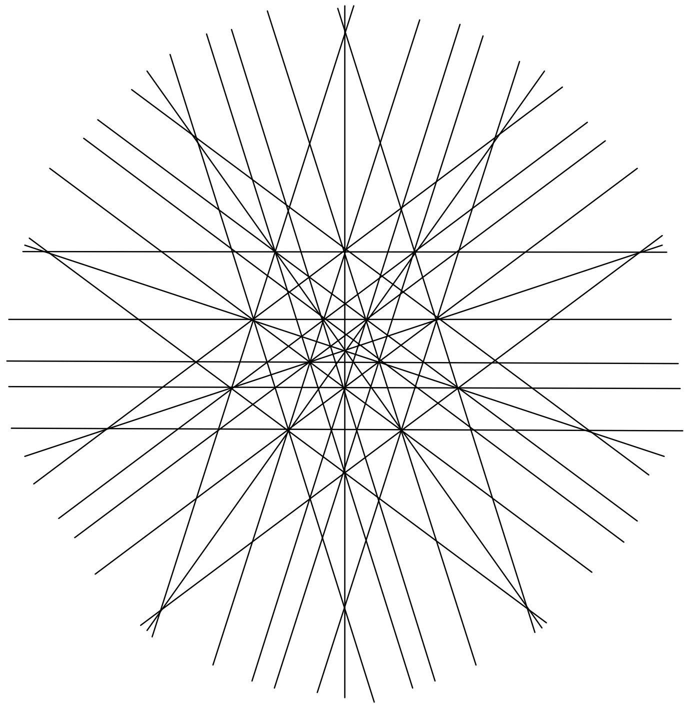

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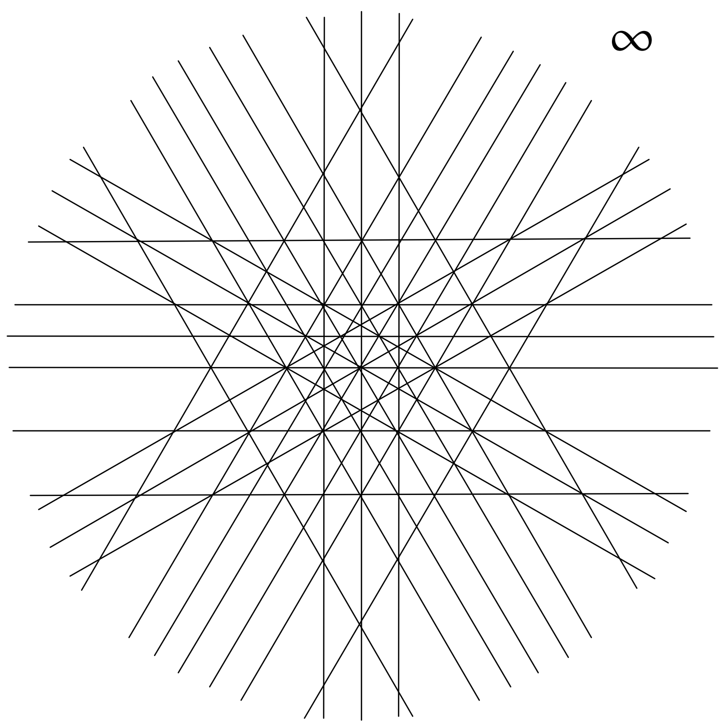

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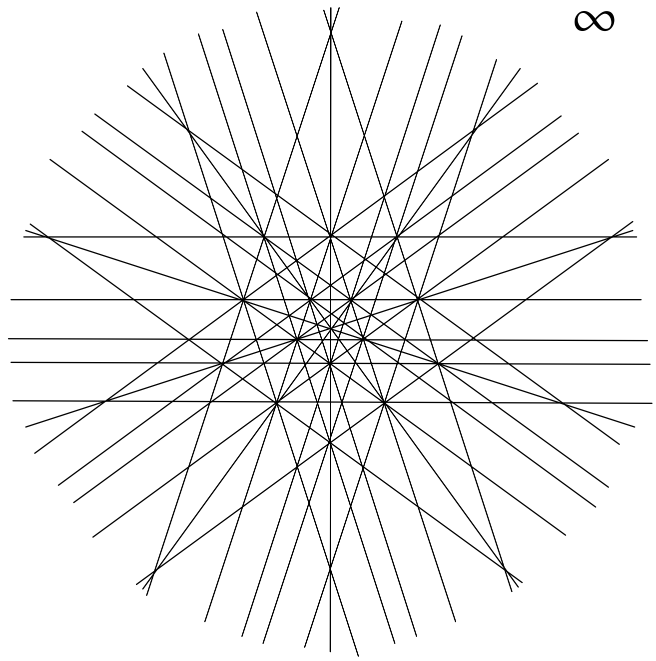

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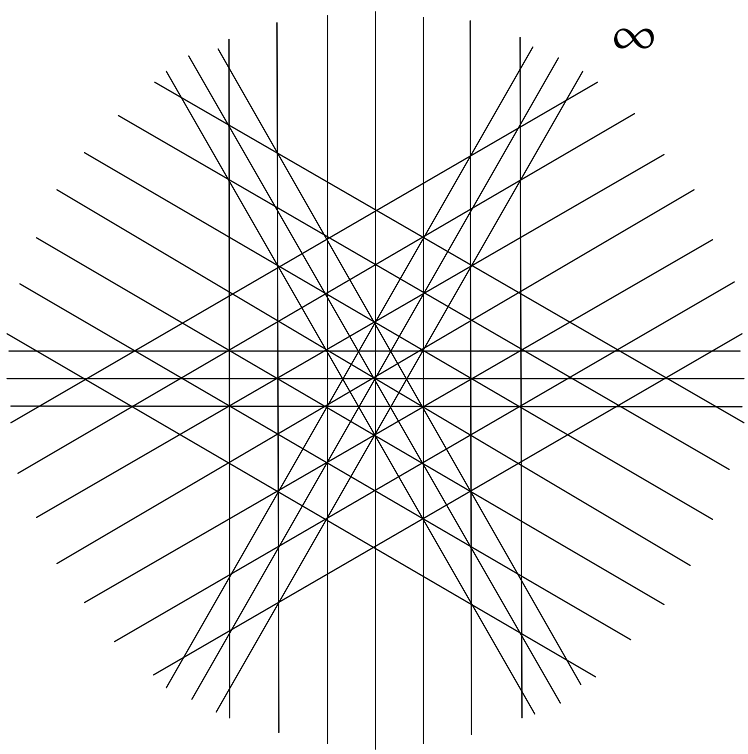

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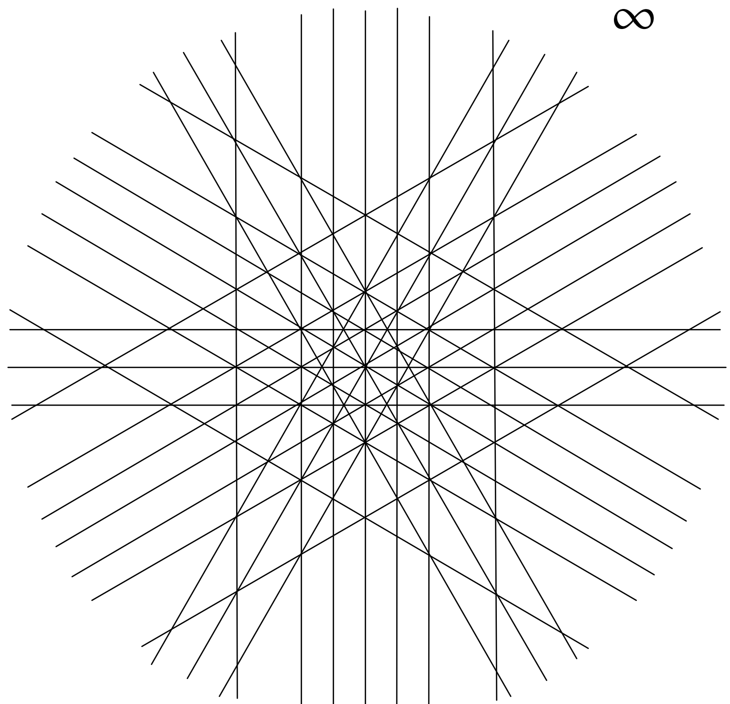

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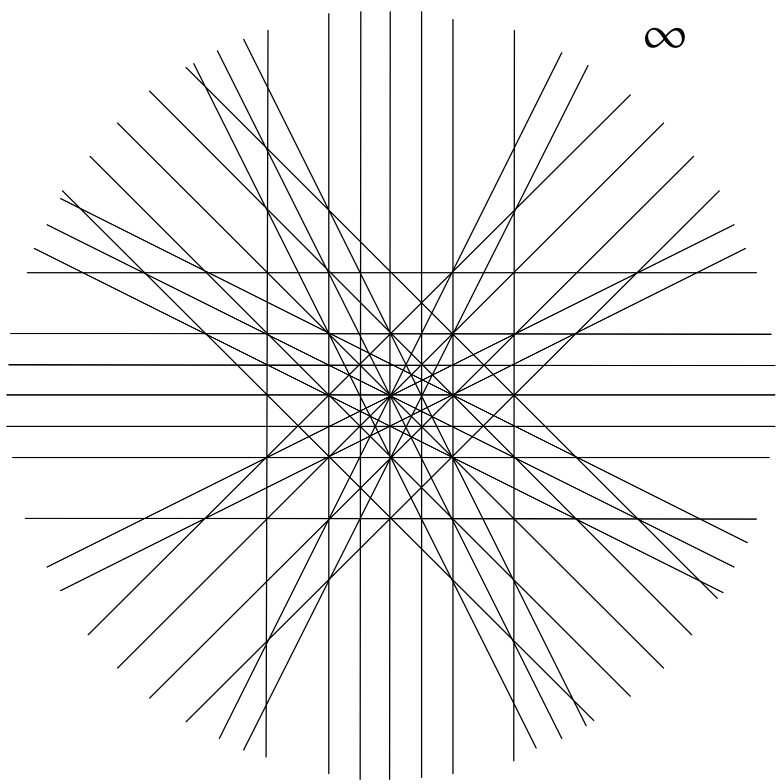

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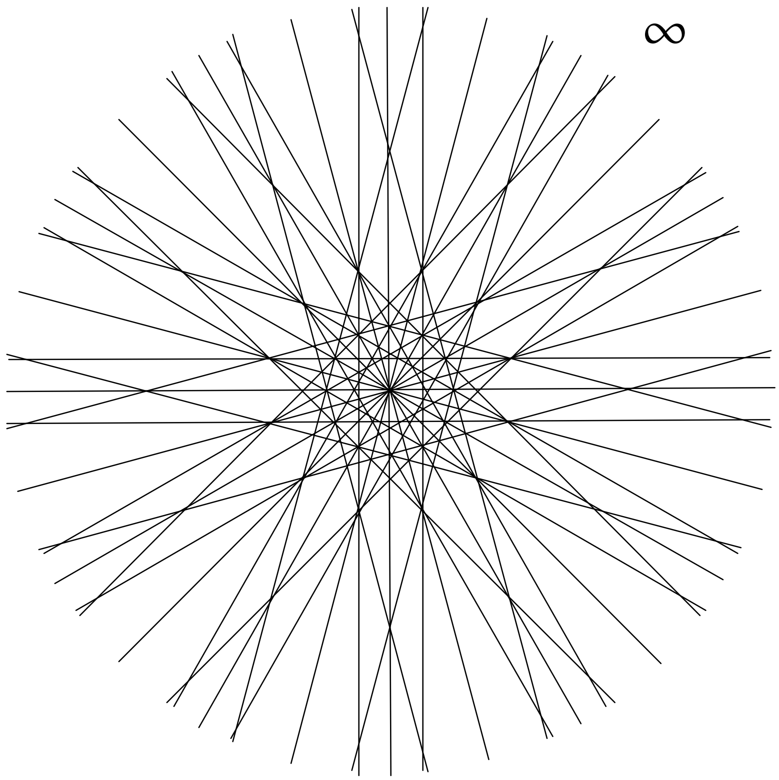

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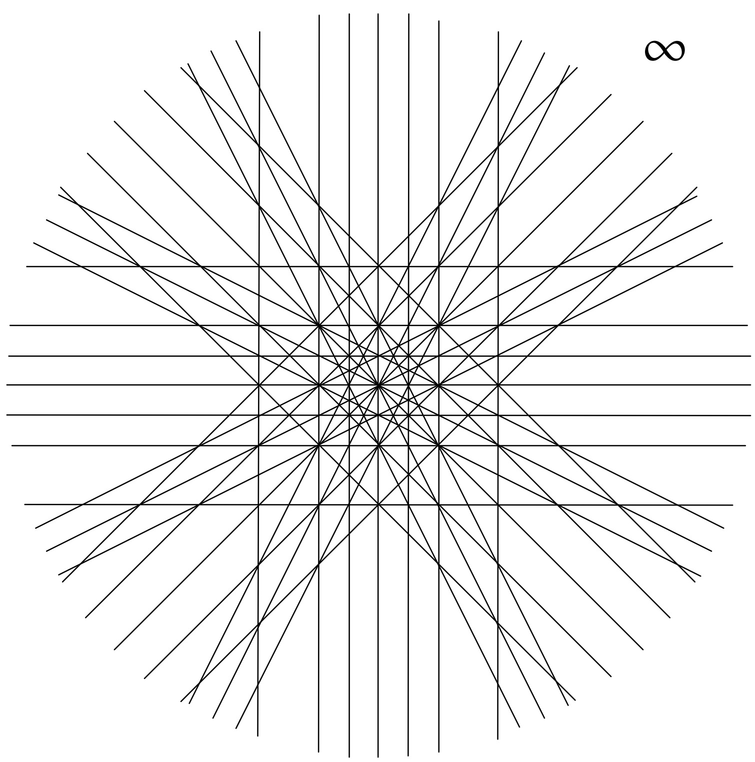

$\mathcal{A}(37, 3)$