CHAPTER 4. OTHER CONFIGURATIONS.

4.0 OVERVIEW

We devoted long chapters each to 3-configurations and to 4-configurations. In contrast, this short chapter covers all other configurations. The reason for this difference in extent of coverage is a direct and inevitable consequence of the paucity of knowledge about configurations that are neither 3- nor 4-configurations.

Despite the generality of the definition of configurations proposed a century and a quarter ago, strikingly little effort was devoted to the study of the k-configurations for $k \geq 5$ and the related unbalanced configurations.

In Section 4.1 we review the information that is available about 5-configurations. The first images are barely a decade olds, and there is still great uncertainty concerning what is possible regarding 5-configurations, and what is not possible.

Section 4.2 nominally deals with all k-configurations for $k \geq 6$. In fact, it is mostly devoted to 6-configurations. I am indebted to L. Berman for permission to include the recently found (and not previously published) images of (110_6) and (120_6). These are the first 6-configurations to appear in print anywhere.

The unbalanced configurations $(12_4,16_3)$ and $(16_3,12_4)$ have enjoyed a measure of popularity, but other [4,3]- and [3,4]-configurations have fared much less well. The material about these is presented in Section 4.3.

Unbalanced $[k_1,k_2]$-configurations with $\{k_1,k_2\} \neq \{3,4\}$ are considered in Section 4.4.

Section 4.5 deals with a recently discovered class of configurations, the "floral" configurations. They are characterized by their hierarchical construction, rather than by the particular incidence parameters.

In Section 4.6 we collected results on topological configurations. These have been investigated in some detail only very recently, and the topic abounds in open questions.

The topics presented in Section 4.7 are several kinds of unconventional configurations. We briefly touch on configurations of circles, and on two kinds of configurations involving infinite sets of points and lines.

The concluding Section 4.8 presents a few open problems that have not been mentioned in the earlier sections.

4.1 5-CONFIGURATIONS

The history of 5-configurations is even shorter than that of 4-configurations, and the knowledge is also much skimpier. However, there are several interesting aspects that do not appear in 3- and 4-configurations.

From the obvious necessary conditions it follows that any (n_5) configuration must satisfy $n \geq 21$. The build-up of a combinatorial configuration (21_5) using the "greedy" approach (as for (7_3) in Table 2.2.2 and for (13_4) in Table 3.1.1) can probably be carried out without undue effort. However, it seems more interesting to note that (21_5) is the cyclic configuration based on (0,3,4,9,11). As noted by Gropp [G8], while it is obvious that this cyclic basis works for all $n \geq 2 \cdot 11 + 1 = 23$, its validity for $n = 21$ is unexpected but easily verified. The configuration is presented in Table 4.1.1. Gropp [G32] seems also to be the first to discover that (0,1,4,9,11) is a cyclic basis for (n_5) for all $n \geq 23$ as well; but it does not yield (21_5). Gropp establishes a connection of these bases with the "Golomb rulers" — combinatorial objects interesting in their own right; for some details see [G18], [G4], [G5][1].

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	0	1	2
4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	0	1	2	3
9	10	11	12	13	14	15	16	17	18	19	20	0	1	2	3	4	5	6	7	8
11	12	13	14	15	16	17	18	19	20	0	1	2	3	4	5	6	7	8	9	10

Table 4.1.1. The cyclic combinatorial configuration (21_5) generated by the basis (0,3,4,9,11). This basis work also for each $n \geq 23$ to yield a configuration (n_5).

So far we avoided mentioning the configuration (22_5). It is a particularly interesting one because — in contrast to the situation we encountered for 3- and 4-configurations — this configuration does not exist even combinatorially. The proof of this requires tools that are outside the scope of this text.

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[1] Much additional information can be found on the Internet. See, for example, [C4], [S13], and, in particular, "Golomb ruler" in the Wikipedia.

Except for the existence of two non-isomorphic cyclic combinatorial configurations (23_5) there seems to be no information available regarding the numbers of distinct (n_5). It is easy to construct, for all $n \geq 25$, additional cyclic bases such as (0,3,4,10,12), (0,1,4,10,12), or (0,1,6,10,12); but neither their number, nor possible isomorphisms, nor the existence of non-cyclic configurations seem to have been investigated.

For 4-configurations we have seen in Section 3.4 that one needs to increase the number of points only slightly from the minimal value n = 13 to reach values for which topological or geometric configurations exist — n = 17 for the former, and n = 18 for the latter. Moreover, all these are best possible values. In case of 5-configurations the information available is far less satisfactory.

It is obvious that the configuration LC(5) (see definition in Section 1.1) is geometrically realizable; however, with $5^5 = 3125$ points and lines there is no intelligible realization. The first description of a graphically presentable 5-configuration appeared in [G50]; it is a (60_5) that is 3-astral in the extended Euclidean plane, and is also shown in [G46] and as Figure 4.1.1 below. (By the convention adopted in Section 1.5, we may call such 3-astral 5-configurations **astral**.) The construction is based on the idea that many 4-configurations have quadruplets of points aligned on diameters and are such that these diameters are parallel to quadruplets of lines. Then the addition of the diameters gives [5,4]-configurations, for which the addition of points at infinity results in 5-configurations. This construction is also illustrated in Figure 4.1.2 in the case of a (50_5) configuration, which is a smallest such configuration known. Another (50_5) configuration is shown in [G50].

All these configurations are symmetric only in the extended Euclidean plane, since they include points at infinity. Switching to their polars is no remedy due to the lines through the center. Allowing a slight larger size enables one to construct 5-configurations with dihedral symmetry by a slightly different process, starting with 5-astral 4-configurations. An example appears in Figure 4.1.3. It is a (54_5) configuration with d_9 symmetry, that is 6-astral in the extended Euclidean plane.

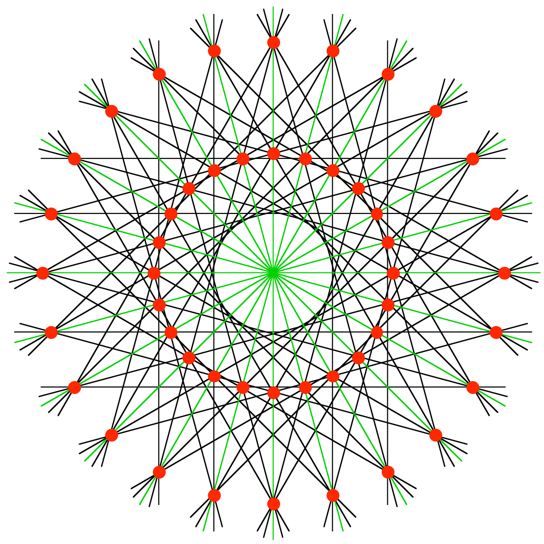


Figure 4.1.1. Deleting the 12 lines (green) through the center yields the astral (48_4) configuration (2) 12#(5,4;1,4). With these lines it is a (48_5, 60_4) configuration. Adding 12 points at infinity, in the directions of the ten green lines, results in a (60_5) configuration that is astral in the extended Euclidean plane.

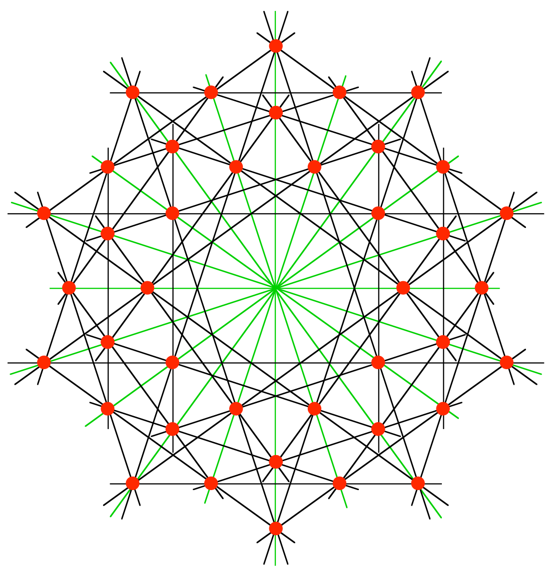


Figure 4.1.2. Deleting the ten lines (green) through the center yields the 4-astral configuration 10(#4,3,2,3,1,2,1,2). With these lines it is a (40_5, 50_4) configuration. Adding ten points at infinity, in the directions of the ten green lines, results in a (50_5) configuration that is 5-astral in the extended Euclidean plane.

The smallest 5-configuration discovered so far is the (48_5), found by L. Berman and shown in Figure 4.1.4. It has cyclic symmetry c_{12}; moreover, it is 4-astral in the Euclidean plane.

As mentioned above, the configuration (60_5) illustrated in Figure 4.1.1 has the advantage of being *astral* — but only in the *extended Euclidean* plane E^{2+}. One of the long-standing conjectures (see [G46], [B6]) is:

Conjecture 4.1.1. There are no 5-configurations 3-astral in the *Euclidean* plane E^2.

The existence of certain types of astral 5-configurations in the Euclidean plane has been ruled out in the recent paper [B11], but the more general question is still open.

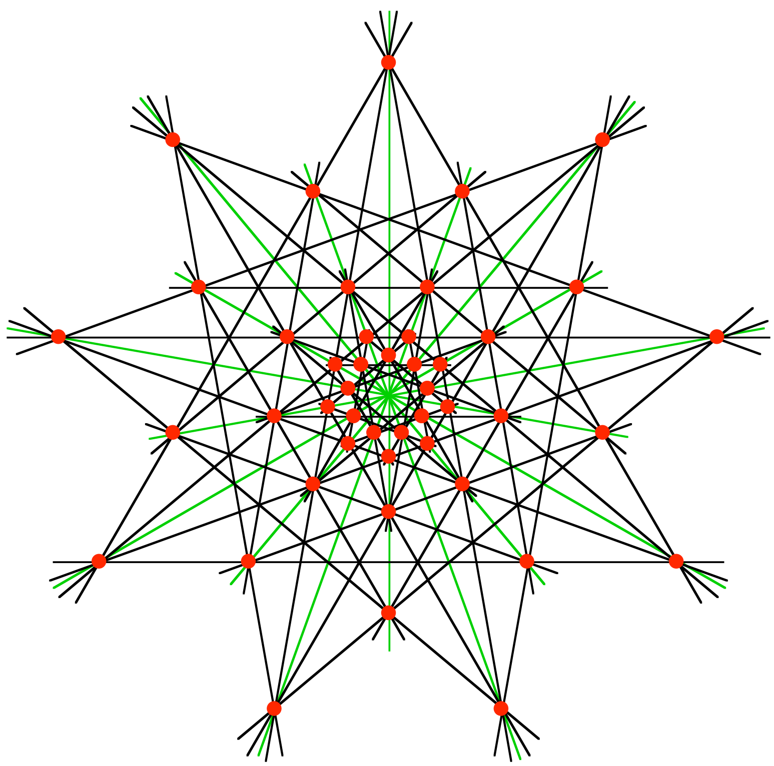


Figure 4.1.3. The addition of 9 diameters (green) to the 5-astral configuration 9#(3,4;1,3;2,3;4,1;3,2) together with the inclusion of 9 points at infinity in the direction of quintuplets of parallel lines, yields a 6-astral (54_5) configuration.

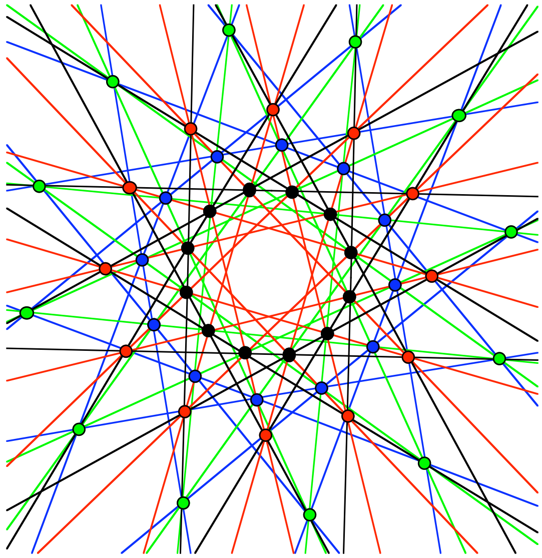


Figure 4.1.4. The smallest 5-configuration known is this 4-astral (48_5). (L. Berman, private communication)

One of the basic differences in the knowledge about 5-configurations compared to 3- and 4-configurations is our ignorance whether geometric configurations (n_5) exist for **all** n that are greater than some fixed bound. On the other hand, a similarity appears to exist: Among the known 5-configurations, there are topological ones that are smaller than the smallest known geometric configuration. One of several topological (42_5) configurations is shown in Figure 4.1.5. This is to be compared with the result mentioned in the proof of Theorem 3.2.1 to the effect that any topological (n_5) must satisfy $n \geq 25$. Although the gap from 25 to 42 is still large, it is not unexpected: There has been no investigation of 5-configurations — topological or geometric — till very recently, and no systematic approaches have been developed so far.

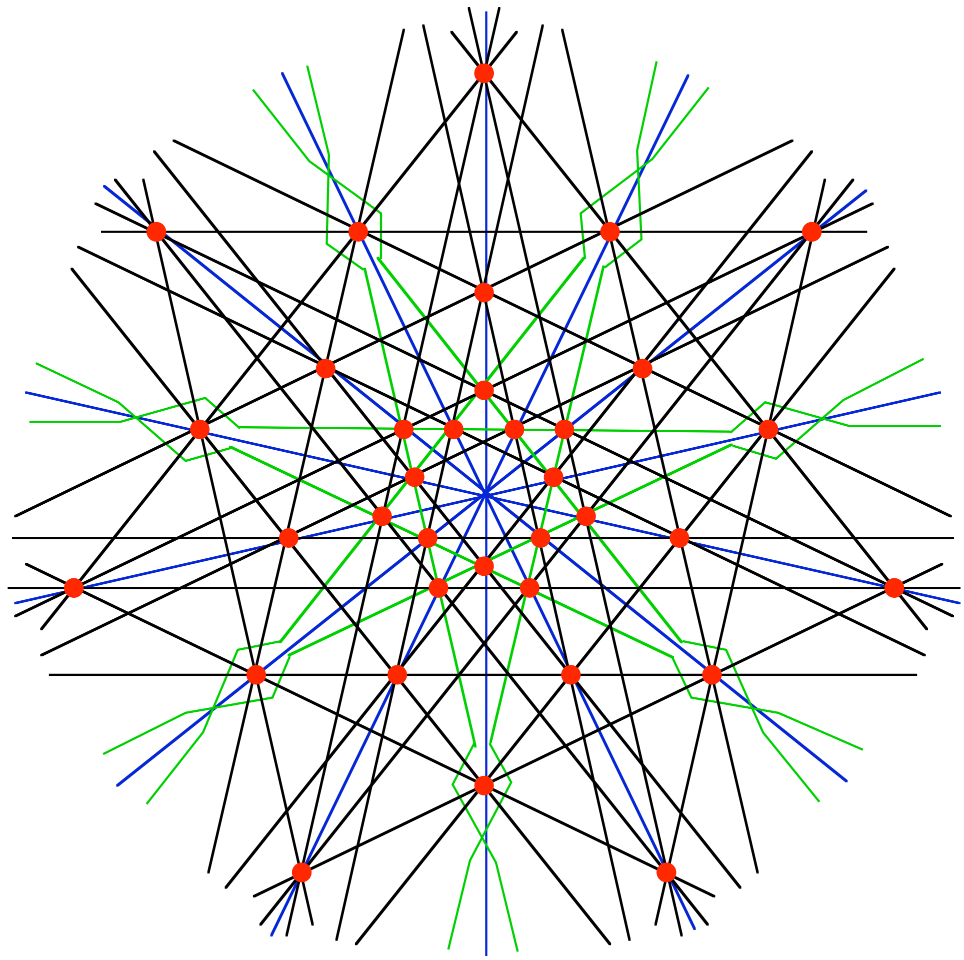


Figure 4.1.5. The geometric configuration 7#(2,1;2,1;3,2;1,2;1,3) has unintended incidences, and is just a prefiguration. If these incidences are avoided by using pseudolines we obtain a topological (35_4) configuration formed by the black lines and green pseudolines. Adding the seven blue lines yields a (35_5, 42_4) configuration, and adding also the seven points at infinity (in the directions of the quintuplets of lines/pseudolines) results in a topological (42_5) configuration.

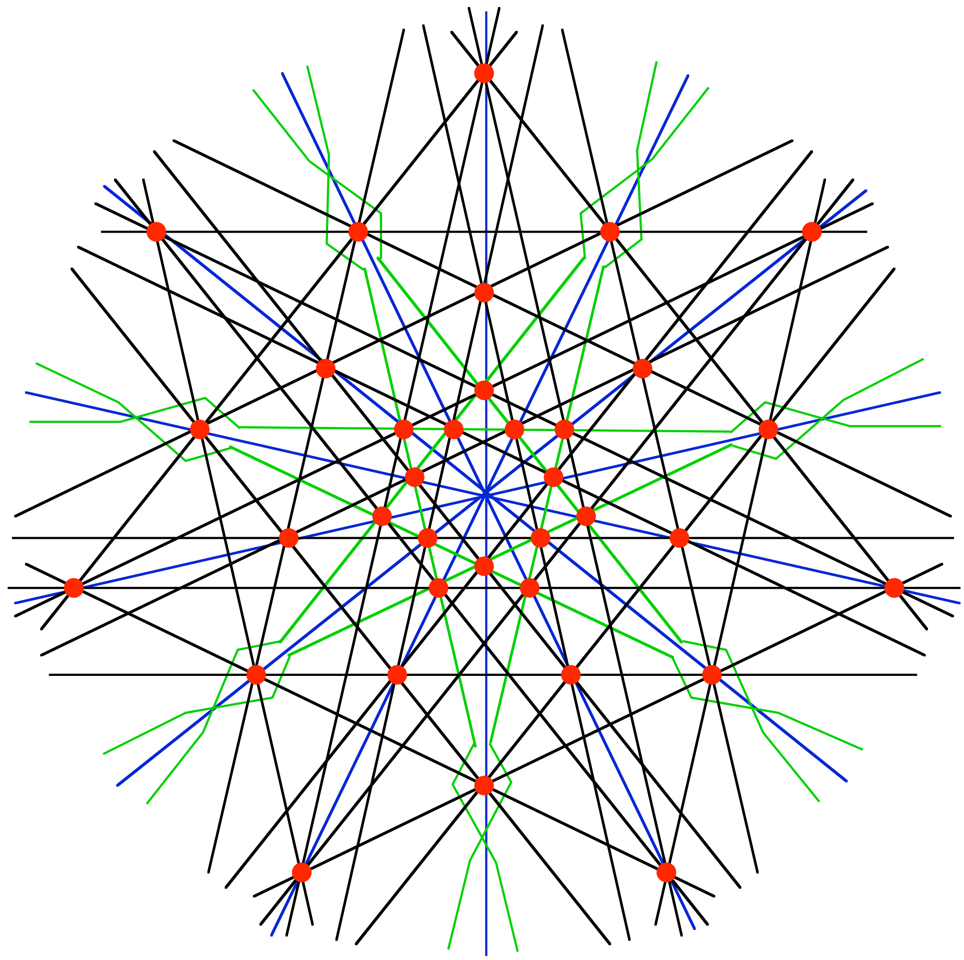


Figure 4.1.5. The geometric configuration 7#(2,1;2,1;3,2;1,2;1,3) has unintended incidences, and is just a prefiguration. If these incidences are avoided by using pseudolines we obtain a topological (35_4) configuration formed by the black lines and green pseudolines. Adding the seven blue lines yields a $(35_5, 42_4)$ configuration, and adding also the seven points at infinity (in the directions of the quintuplets of lines/pseudolines) results in a topological (42_5) configuration.

Exercises and problems 4.1.

1. Determine the 4-configurations that can be turned into 5-configurations by adding lines, and points at infinity. It seems that all 4-astral 4-configurations can be used, admitting duplication if necessary. Are there any others?

2. The configuration in Figure 4.1.1 was constructed in an obvious way from two copies of the astral (24_4) configuration. Can this method be applied to all astral 4-configurations?

3. Are there any 4-astral 5-configurations in the *Euclidean* plane that have dihedral symmetry?

4. Decide whether any of the configurations in Figures 4.1.1 to 4.1.3 is selfpolar.

5. Decide whether there are geometric (n_5) configurations for any $n < 48$.

6. Decide whether there are topological (n_5) configurations for any $n < 42$.

7. Find a useful and convenient way of encoding symmetric 5-configurations.

8. Show that the 4-astral configuration 10#(4,3;1,2;3,4;2,1) can be used to construct a configuration (50_5). Determine all 3-astral configurations (40_4) that can be used for that purpose.

9. The constructions we have seen can be generalized. Determine criteria on 4-astral configurations $((4m)_4)$ that make it possible to obtain configurations $((5m)_5)$. Similarly, for $((5m)_4)$ configurations to yield $((6m)_5)$ configurations.

4.2 k-CONFIGURATIONS FOR $k \geq 6$

As justification for general existence statements for k-configurations with $k \geq 6$ we recall the configurations LC(k) introduced in Section 1.1. They illustrate the possibility of geometric configurations $((k^k)_k)$ for all k. Naturally, one may be interested in smaller examples, and there are systematic ways to find them, even though they yield configurations that are neither stimulating to look at, nor very small.

The first such construction, during the "prehistory period" of configurations, is due to Cayley [C2*] in 1846. Reflecting the spirit of the times, Cayley writes (in French, in a paper published in a German journal!):

> " *sans recourrir à aucune notion métaphysique à l'égard de la possibilité de l'éspace à quatre dimensions*, ..."

and proceeds to define configurations of flats of various dimensions spanned by families of points in general position; intersecting these with suitable planes he devises (for $k \geq 2$) configurations (n_{k+1}) where $n = (2k+1)!/k!(k+1)!$ Thus, what Cayley describes are geometric configurations (35_4), (126_5), (462_6), (1716_7), and so on. He also describes various unbalanced configurations, about which we shall report in Section 4.3.

Although Cayley's constructions yield smaller configurations than the LC(k), there are better construction methods that are easy generalizations of the ones we detailed in Section 3.3, considered there for the 4-configurations.

The (5m) construction which in Section 3.3 led from a configuration (m_3) to $((5m)_4)$ generalizes immediately: From any (m_k) configuration, taking k+1 copies that all intersect at the same points of a suitable line, and then adding m appropriate lines connecting corresponding points in these copies, we obtain a $(((k+2)m)_{k+1})$ configuration. Using the smallest configurations available, this construction leads from (9_3) to (45_4), from (18_4) to (108_5), from (48_5) to (336_6), from (110_6) to (880_7), and so on. Except for the last one, these are not the known minimal configurations — but in the last case this is the best available. Carrying out only the first step, with only k copies of the starting configuration, leads to a (k,k+1)-configuration. Taking a stack of k configurations and adding the lines connecting the corresponding points leads to a (k+1,k)-configuration. Some of the other methods in Section 3.3 generalize as well.

For 6-configurations we can do better than for general k. Figure 4.2.1 shows a 10-astral (110_6) configuration, and Figure 4.2.2 a 4-astral configuration (120_6); both were discovered by L. Berman (private communication).

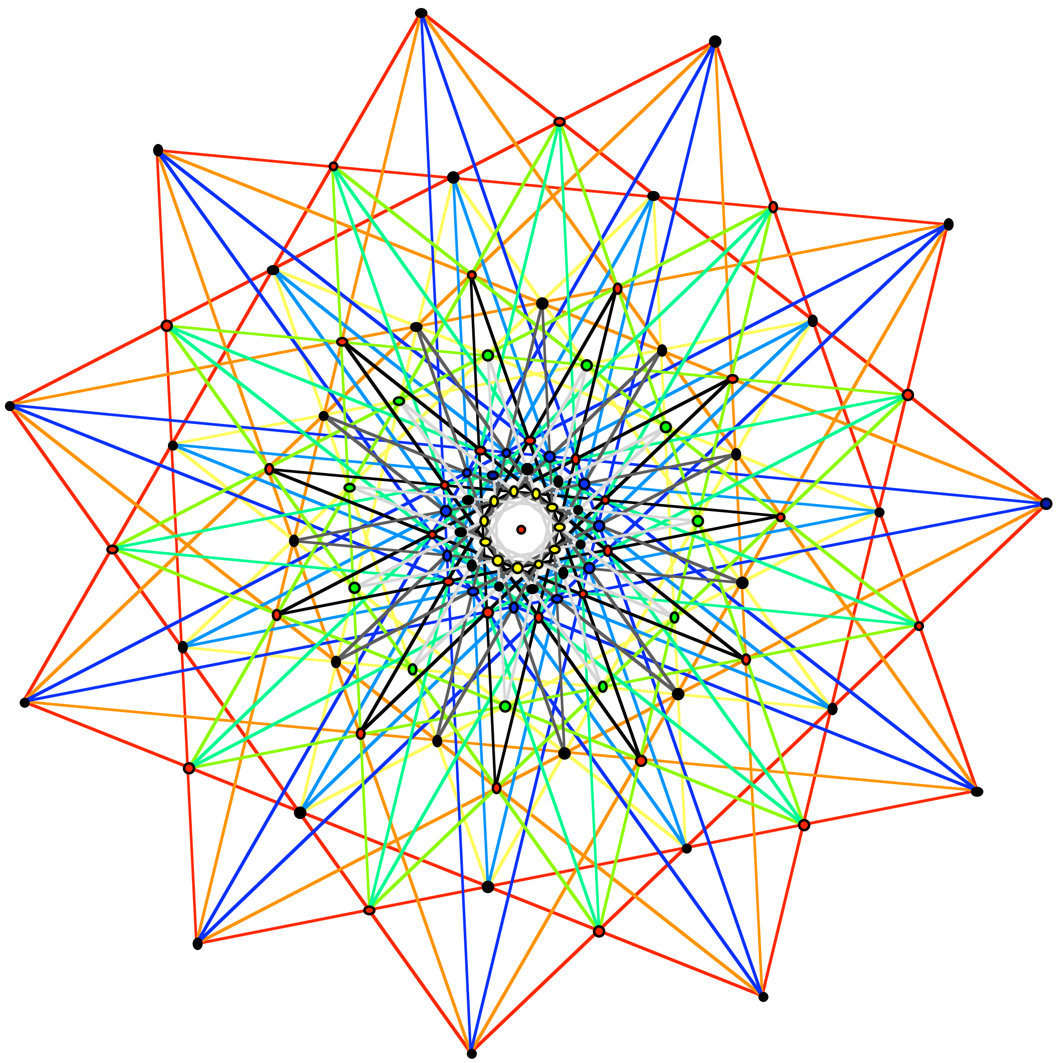


Figure 4.2.1. A 10-astral (110_6) configuration with symmetry group d_{11}, found by L. Berman.

On the other hand, there is negative information available concerning astral (that is, 3-astral) 6-configurations. As proved by Berman [B5], no such configurations exist, nor do any astral [2k,2h]-configurations for $k \geq 3, h \geq 3$.

The paucity of information on the topic of this section is clearly evidenced by its brevity, and the absence of references beyond [C2*] and [B5]. Notice that these are separated by more than a century and a half!

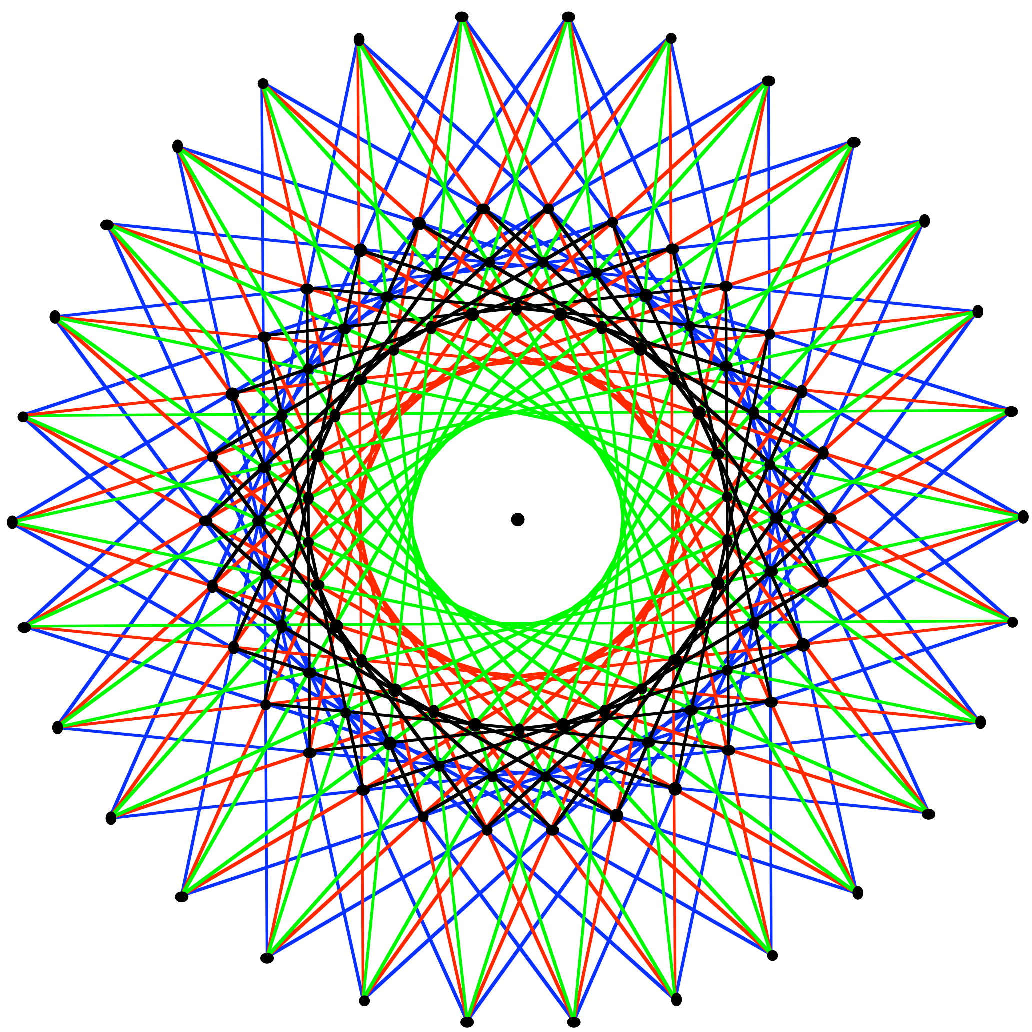


Figure 4.2.2. A 4-astral (120_6) configuration with symmetry group d_{12}, found by L. Berman.

Exercises and problems 4.2.

1. Decide whether there exist any other 4-astral 6-configurations.

2. Find some "small" topological 6-configurations.

3. Is there some systematic construction for 6-configurations that is analogous to the passage from 4-configurations to 5-configurations mentioned in Exercise 9 of section 4.1.

4. Find a visually intelligible 7-configuration.

4.3 [3, 4]- AND [4, 3]-CONFIGURATIONS

In the present section we shall survey the known facts concerning combinatorial and geometric [3, 4]- and [4, 3]-configurations.

The parameters of any combinatorial (p_3, n_4) or (n_4, p_3) configuration must satisfy the conditions $3p = 4n$, $p \geq 1 + 3 \cdot 3 = 10$ and $n \geq 1 + 4 \cdot 2 = 9$. Thus p must be divisible by 4 and n must be divisible by 3, so that the only possible configurations are those of the form $((4r)_3, (3r)_4)$ or $((3r)_4, (4r)_3)$, respectively, for $r = 3, 4, 5, \dots$. For combinatorial as well as geometric configurations, the existence of $((4r)_3, (3r)_4)$ implies by duality resp. polarity the existence of $((3r)_4, (4r)_3)$, and conversely. Hence it is sufficient in the following result to limit attention to one of the two cases.

Theorem 4.3.1. For each integer $r \geq 3$ there exists a combinatorial configuration $((4r)_3, (3r)_4$; topological and geometric $((4r)_3, (3r)_4)$ configurations exists for each $r \geq 4$.

Proof. We start with a combinatorial $(12_3, 9_4)$ configuration, given by the following configuration table.

1	2	3	4	5	6	7	8	9
A	A	A	L	L	L	M	M	M
B	G	K	B	G	K	B	G	K
C	F	J	J	C	F	F	J	C
D	E	H	E	H	D	H	D	E

Table 4.3.1. A configuration table for a $(12_3, 9_4)$ configuration.

In order to complete the proof in case $r = 3$, we have to prove that no combinatorial configuration $(12_3, 9_4)$ can be realized by points and lines. For that we recall the result known as "Sylvester's problem", which we mentioned in Section 2.1 as Lemma 2.1.1.

To apply the Sylvester result to the question at hand, we note that in any combinatorial configuration $(12_3, 9_4)$ the 36 pairwise intersections of the 9 lines have to occur in

12 triplets — three intersections at each of the 12 points of the configuration. However, since (by Sylvester) in every topological or geometric configuration at least one such intersection is an “ordinary” one (which is therefore not a point of the configuration), there are not enough pairwise intersections to form 12 triplets.

On the other hand, it is possible to give a geometric realization of the dual configuration, but with two of the “lines” neither straight lines nor pseudolines. An example is shown in Figure 4.3.1.

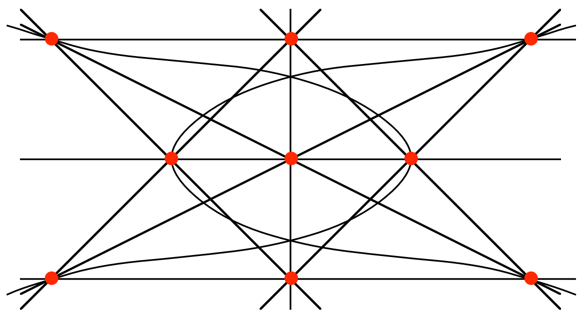


Figure 4.3.1. A realization of a $(9_4,12_3)$ configuration, dual to the one in Table 4.3.1; two of the “lines” are not straight. With a slight modification these two "lines" could have been chosen as circles.

For the remaining part of the proof of Theorem 4.3.1 we only have to exhibit appropriate geometric configurations of points and lines. The literature contains a number of papers devoted to the $(16_3, 12_4)$ configurations, or to the $(12_4, 16_3)$ configurations dual to them; several examples of the former kind are shown in Figure 4.3.2.

There appears to be no published mention of geometric $((4r)_3, (3r)_4)$ configurations with $r \geq 5$. However, examples of such configurations are very easy to produce. One method (see Figure 4.3.3) starts by placing $2r$ points equidistributed on a circle. Each of these points is connected to the one diametrally opposite to it, as well as to the two points separated from it by two other points. Adjoining the $2r$ triple intersections (whose existence is clear by the symmetry of the diagram) yields a $((4r)_3, (3r)_4)$ configuration, as required.

Other $((4r)_3, (3r)_4)$ configurations may be constructed by slight variations of this method; several are shown in Figure 4.3.4. In all these cases, the geometric existence of the configurations is an obvious consequence of the high degree of symmetry involved.

Although the configurations $(16_3, 12_4)$ and/or $(12_4, 16_3)$ have been studied for at least 150 years (starting not later than Hesse [H2] in 1848, in the "prehistoric" era of configurations), there still are many unresolved questions. It has been shown (or claimed – there seems to have been no independent verification) that there are precisely 574 combinatorial configurations $(12_4, 16_3)$, see Gropp [G14], [G16]. The large number of such configurations helps explain why there is no clarity on the question which (or, whether all) configurations $(124_4, 16_3)$ have geometric realizations in the Euclidean plane. Two additional aspects probably contribute to the lack of clarity: On the one hand, most of the relevant papers have been published in journals that are not well known nor widely available, many in Czech which is not too widely spoken; a large number of references is listed below. On the other hand, from the very beginning, these configurations have been

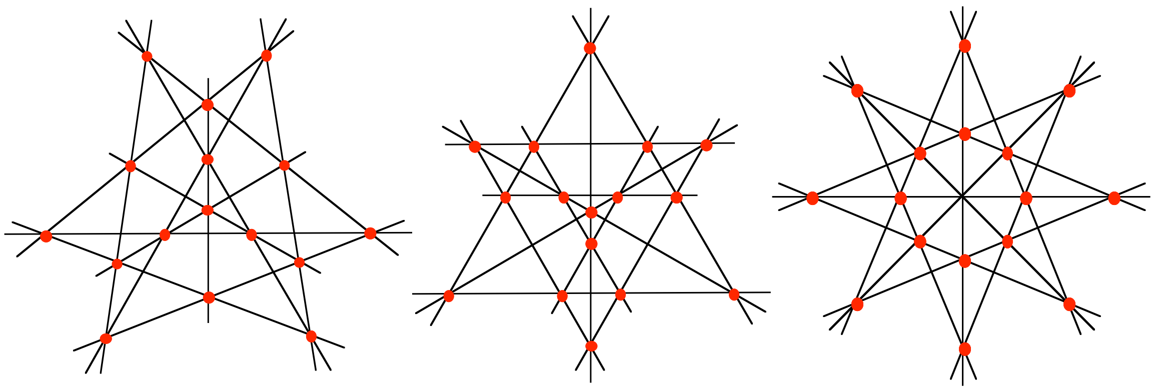


Figure 4.3.2. Three examples of configurations $(16_3, 12_4)$.

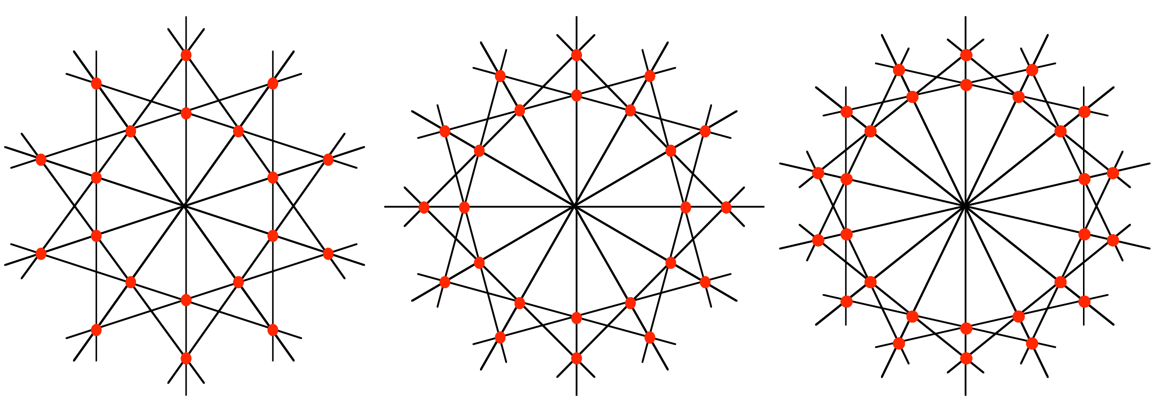


Figure 4.3.3. Examples of configurations $(20_3, 15_4)$, $(243_3, 18_4)$ and $(28_3, 21_4)$.

studied in close connection with the theory of cubic curves. This connection, in turn, is not too well known these days, and also makes it hard to know which parts of the claims of possibility of realization rely on configurations in the complex plane, and which claims of impossibility are due to the restriction of attention to configurations with vertices on cubic curves. See below for some relevant ideas.

From the duality in the projective plane it follows that geometric configurations $((3r)_4, (4r)_3)$ exist if and only if $r \geq 4$. One example of a $(12_4, 16_3)$ configuration is shown in Figure 4.3.5. In contrast to the very symmetric diagrams representing the $(16_3, 12_4)$ configurations, the diagrams of the $(12_4, 16_3)$ configurations shown in most publications are far from symmetric.

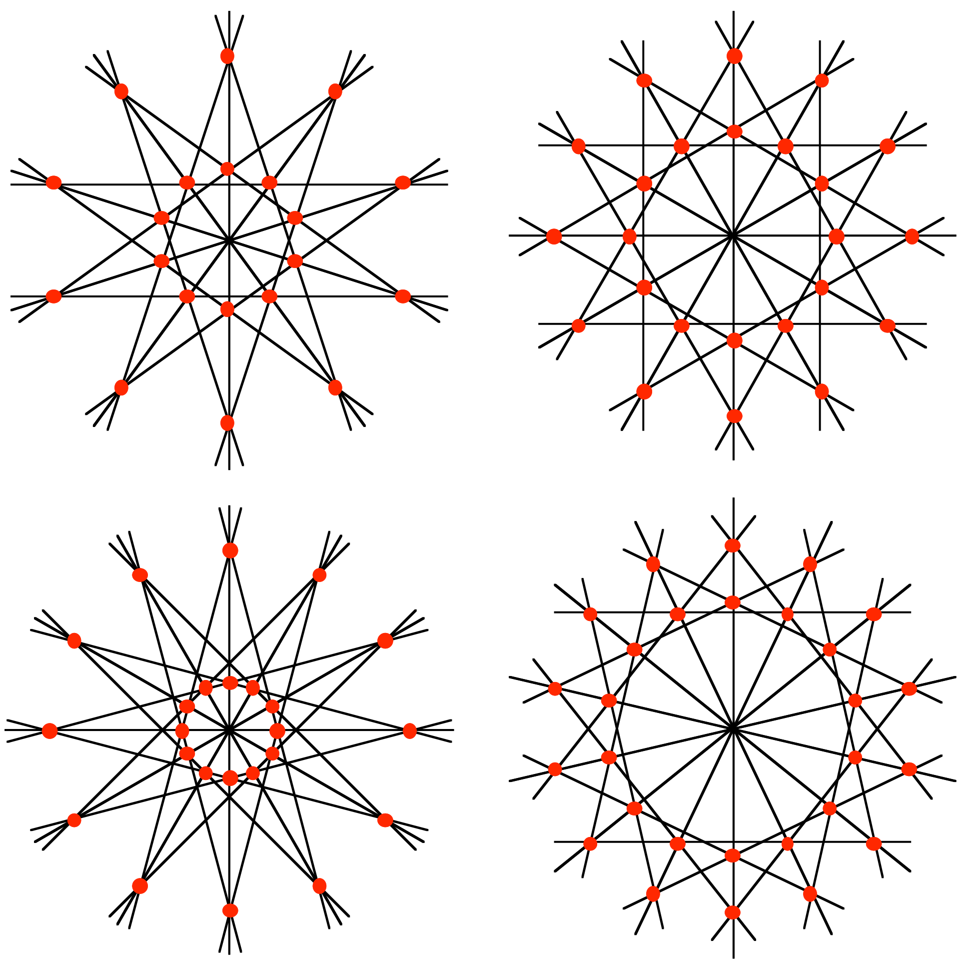


Figure 4.3.4. Additional examples of configurations $(20_3, 15_4)$, $(24_3, 18_4)$ and $(28_3, 21_4)$.

The reason for the difference is that projective duality does not in general preserve Euclidean symmetries — unless one considers the configurations in the extended Euclidean plane. In particular, all examples in Figures 4.3.3 and 4.3.4 have lines passing through the center of symmetry (taken at the origin) which have to be mapped to "ideal points" in order to preserve symmetry. If this is accepted, then it is easy to produce very symmetric $(16_3, 12_4)$ configurations, such as the one in Figure 4.3.6.

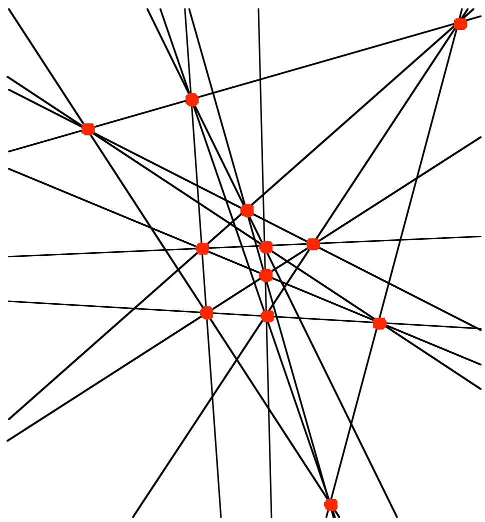


Figure 4.3.5. An example of a geometric configuration $(12_4,16_3)$.

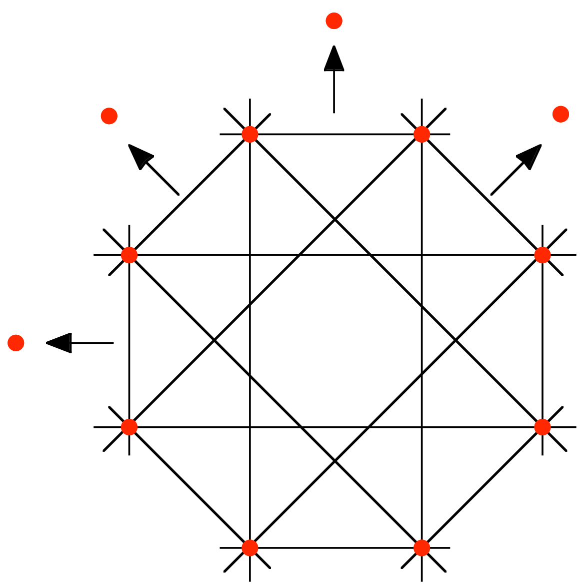


Figure 4.3.6. An example of a geometric configuration $(12_4,16_3)$ that is astral in the extended Euclidean plane.

Additional examples of quite symmetric $(12_4, 16_3)$ configurations are shown in Figure 4.3.7. These have vertices on cubic curves.

In order to give a feeling for the relation of cubic curves to configurations, we show another example in Figure 4.3.8. This is a geometric configuration $(12_4,16_3)$ on a cubic curve, from the paper by V. Metelka [M17]. The equation of this cubic curve in homogeneous coordinates (x,y,z) is

$$z(x^2 + y^2) + x(x^2 - 3y^2) = 0$$

and the points are:

M = (1,1,1)	N = (0,1,0)	O = (1,-1,1)	P = (1,-t,2)
Q = (1,t,2)	R = (t,1,0)	S = (-t,1 0)	T = (1,t-2,1-t)
U = (1,2-t,1-t)	V = (1,t+2,t+1)	W = (1,-t-2,t+1)	X = (1,0,-1)

where $t = \sqrt{3}$.

As is well known, an easy way to see whether three points given in homogeneous coordinates are collinear is by checking whether the determinant formed by their coordinates is 0. Thus the assertions about which triplets are collinear (as indicated by Figure 4.3.8) can be algebraically verified.

As Metelka observed (this is the reason he considered the configuration "special") there are three additional lines that pass through three of the points. These three lines are indicated by the dashed lines in Figure 4.3.9. It is worth noting that the maximal number of collinear triplets determined by 12 point is 19 – this is one of frequently raised "orchard problems"; see more details at [B33].

As a clarification of what was briefly mentioned above regarding the use of cubic curves in looking for construction of configurations and related objects, in Figure 4.3.10 we show a diagram of a cubic curve on which are marked several values of the "degree" parameter. The following explanations are taken from the old paper [B33], from which the curve in Figure 4.3.10 was copied as well. References to texts that establish the properties in question are given in [B33]; the notation is the one that seems traditional in the literature.

A suitable projective image of each real non-singular cubic curve has an equation of the form

(1) $$y^2 = 4x^3 - g_2x - g_3$$

where g_2 and g_3 are real constants. The curve C given by equation (1) may be parametrized by

(2) $$x = \wp(u),\ \ y = d\wp(u)/du,$$

where $\wp(u)$ is the Weierstrass elliptic function defined by

$$u = \int_{\wp(u)}^{\infty} (4x^3 - g_2x - g_3)^{-1/2}\, dx.$$

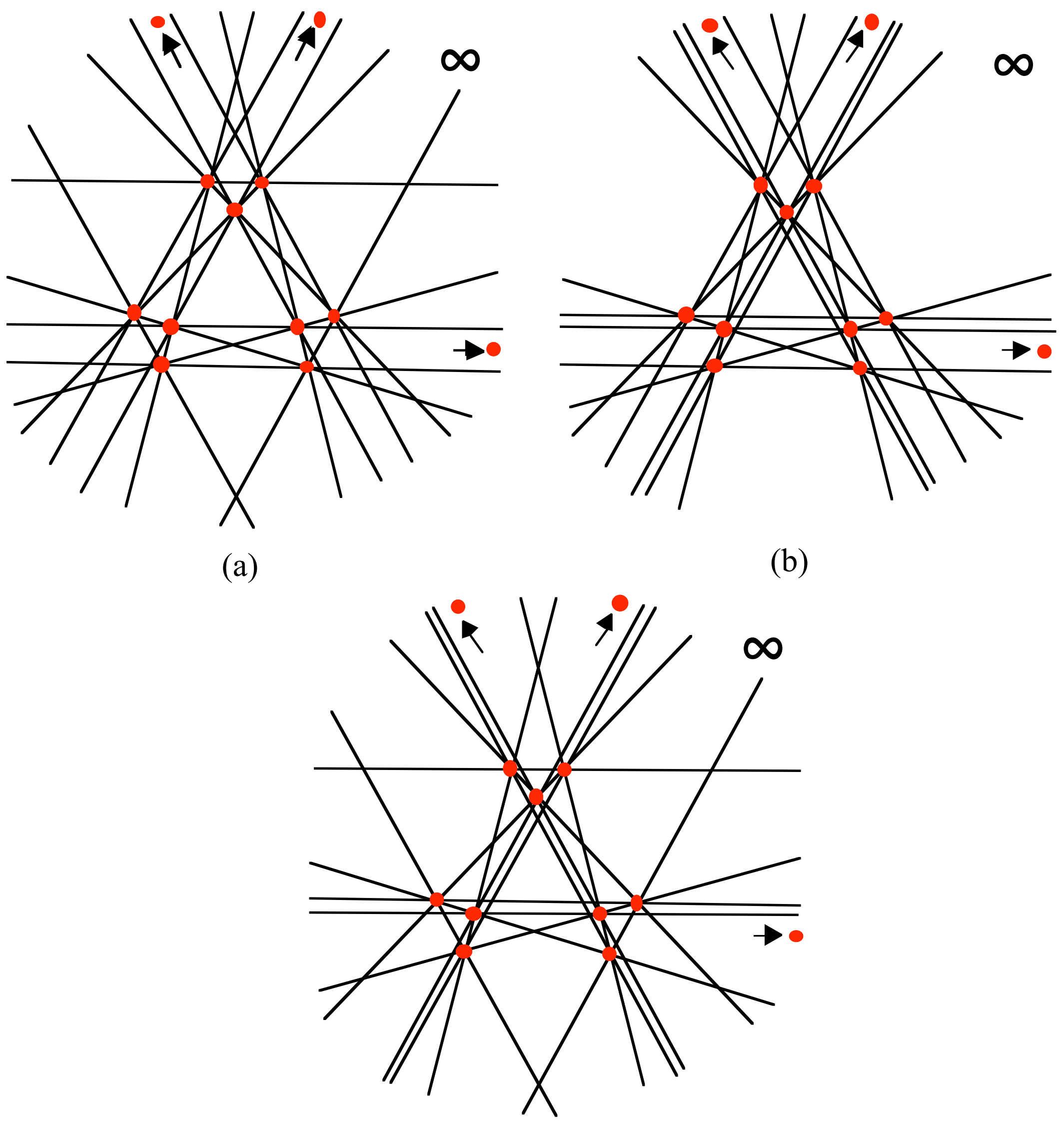


Figure 4.3.7. Three examples of quite symmetric configurations $(12_4, 16_3)$. The ∞ symbol is meant to indicate that the line at infinity is a line of the configuration.

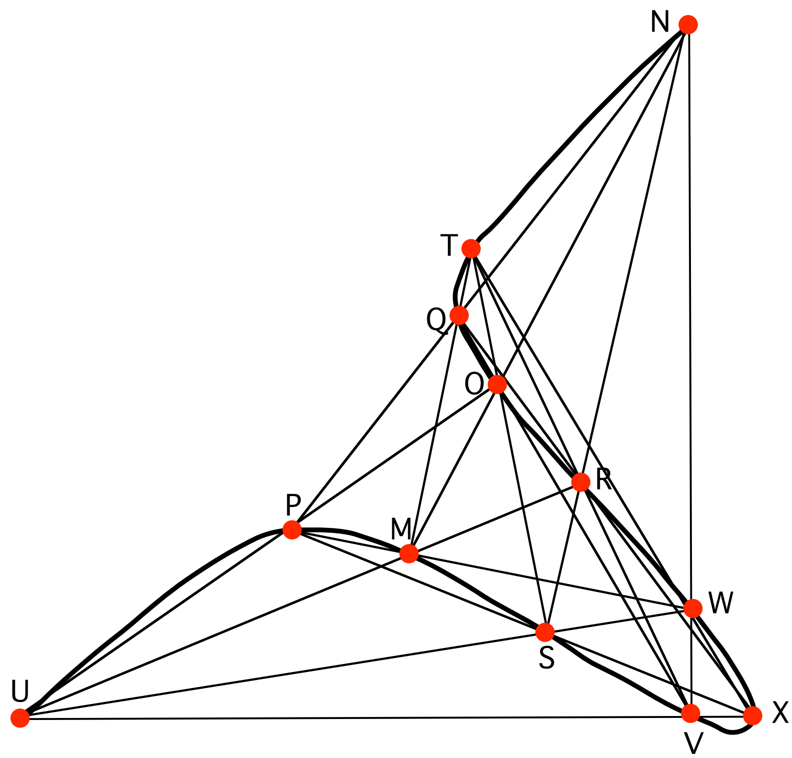


Figure 4.3.8. A configuration $(12_4, 16_3)$ with points on a cubic curve.

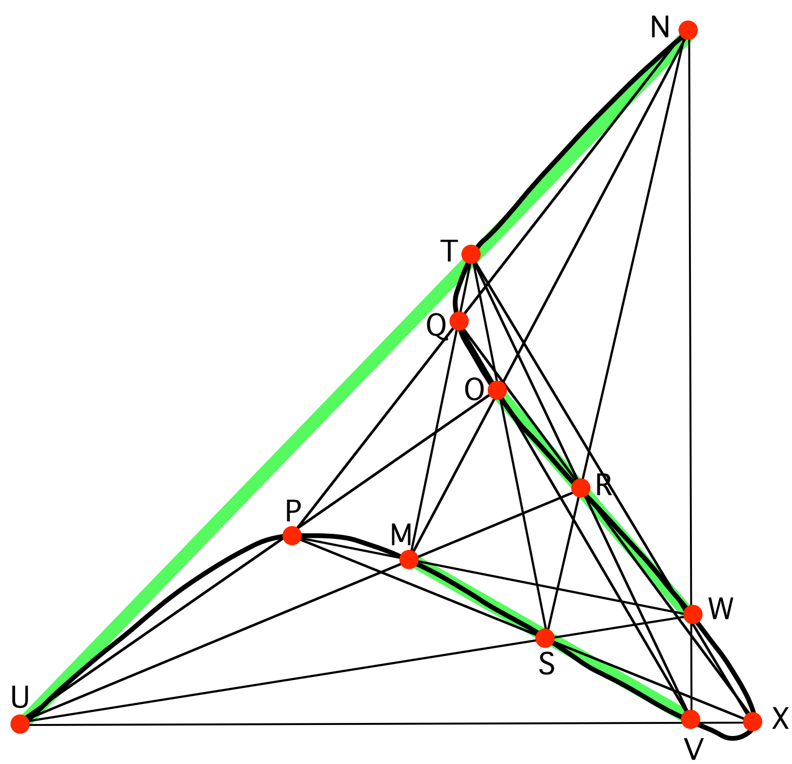


Figure 4.3.9. The points of the $(12_4, 16_3)$ configuration in Figure 4.3.8 determine three additional lines each incident with three of the points.

The Weierstrass elliptic function ℘(u) is a doubly-periodic meromorphic function of the **complex** variable u, and for real g_2, g_3 it has a real period that we shall denote 2ω (as well as a purely imaginary period $2\omega'$). The parametrization (2) yields for real u the "odd circuit" (branch) of the cubic C. In case $D = g_2^3 - 27g_3^2 < 0$ this is the only real part of the curve C ("unipartite cubic"), while in case D > 0 the curve C has also an "even circuit" corresponding to the values $u = v + \omega'$, where v is real. (We shall be interested only in the "odd circuit".)

The importance of cubic curves for the present concerns is based on the following result of N. H. Abel:

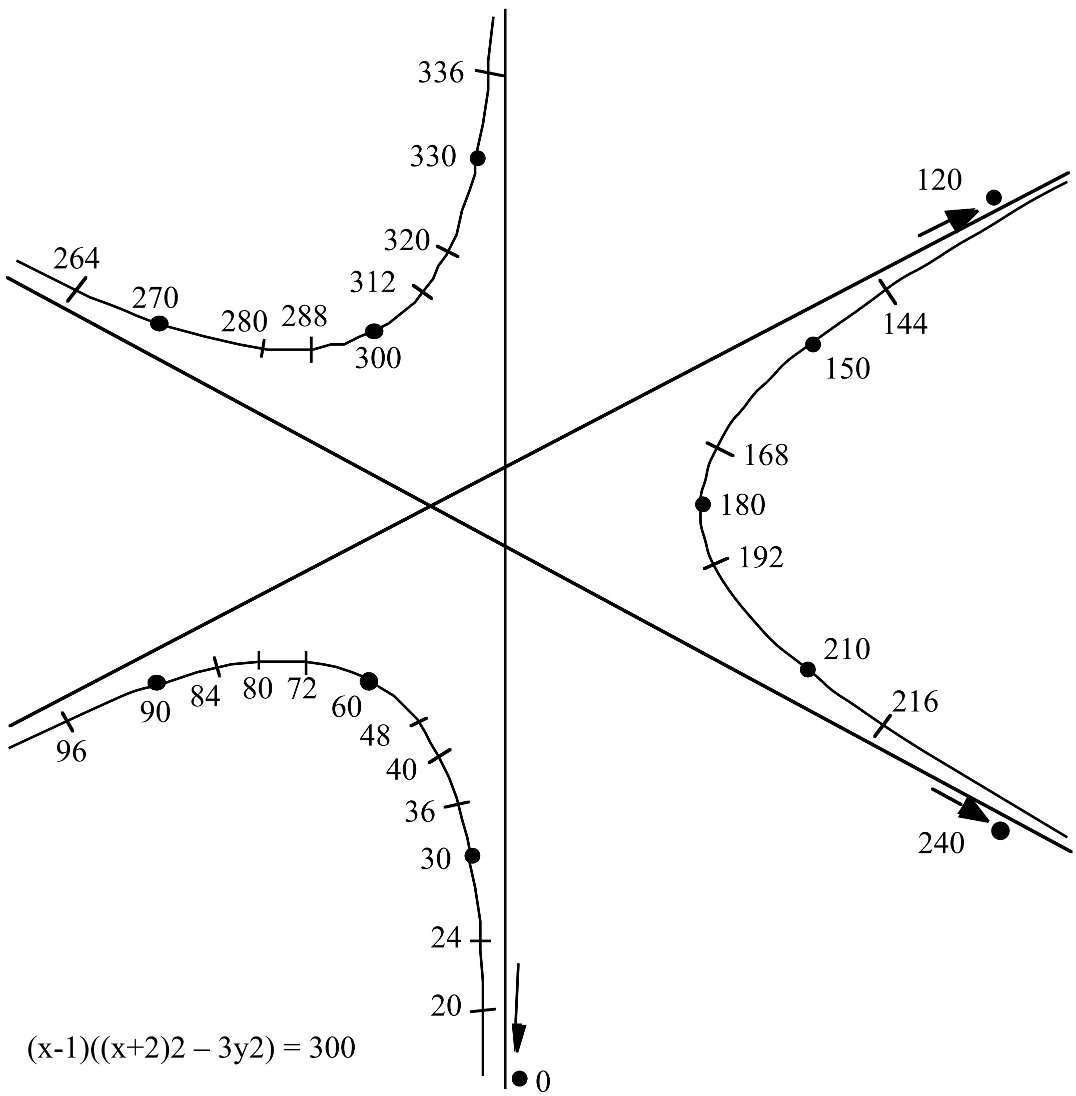


Figure 4.3.10. A cubic curve, with a parametrization derived from the Weierstrass ℘(u) function, as explained in the text.

Denoting by P(u) = ($\wp$(u), d$\wp$(u)/du) the point on the cubic C given by (1), (2) and corresponding to the real parameter u, a necessary and sufficient condition for the collinearity of the points P(u), P(u'), P(u") on the odd circuit of C is

$$u + u' + u'' \equiv 0 \pmod{2\omega}.$$

The curve we use is given by the equation $y^2 = 4x^3 - 1$, and by consulting appropriate tables or software we find that $\omega = 1.529954037...$. As in much of the numerical work on the elliptic functions, we replace 2ω by 360°; in Figure 4.3.10 we denote the points simple by their parameter-value in "degrees".

A practical weakness of the method is an inconvenient bunching of the points of interest. The situation can be improved by using a suitable projective transformation of the curve C; this goes back to W. K. Clifford in 1865. The "odd circuit" of C contains three collinear points of inflection P(0), P(2ω/3), P(4ω/3). If we choose the line determined by these points as the "ideal line", and the points themselves to be in equiinclined directions, there results a very convenient and symmetric representation of C.

We are using the curve C with equation $y^2 = 4x^3 - 1$, for which the Clifford transformation may be achieved by

$$x = (2x^* + 1)/(2x^* - 2), \qquad y = 3y^*/(x^* - 1).$$

This results (on omitting the asterisks) in the equation

$$(x - 1)(3y^2 - (x + 2)^2) = \text{const.}$$

For better visibility we choose the constant as –300, yielding the curve in Figure 4.3.10. This curve is used in some of the exercises below.

The following is an extensive list of papers I am aware of that deal with $(16_3, 12_4)$ or $(12_4, 16_3)$ configurations. Some of them contain additional references to earlier papers. [B34], [D5], [G14], [H2], [M9], [M10], [M11], [M12], [M13], [M14], [M15], [M16], [M17], [M5], [M6], [M7], [R6], [Z1], [Z2], [Z3], [Z6].

A configuration $(15_4, 20_3)$ was described by Cayley [C2*] in 1846. There seems to be no other discussion in the literature of $((4r)_3, (3r)_4)$ configurations with $r \geq 5$ (or their duals).

*   *   *   *   *   *

The introduction of k-astral configurations helped develop the study of 3- and 4-configurations. It seems reasonable that investigations of [3,4]-configurations and similar objects would be advanced by moving from the concentration on the smallest cases to more general situations. As examples capable of various generalizations we show in Figure 4.3.11 and 4.3.12 configurations $(20_3,15_4)$ and $(15_4,20_3)$ with cyclic symmetry group c_5, and in Figure 4.3.12 a configuration $(18_4,24_3)$ with symmetry group c_6. It is clear that such configurations fit into infinite families for which the systematic investigation and notation still need to be developed.

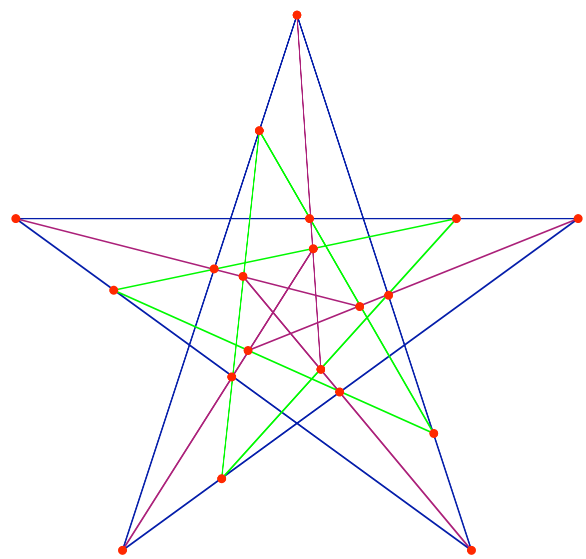


Figure 4.3.11. A [4,3]-astral $(20_3,15_4)$ configuration with symmetry group c_5.

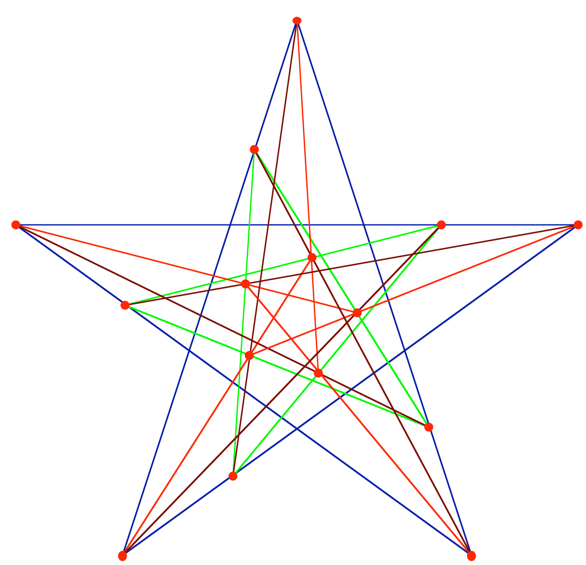


Figure 4.3.12. A [3,4]-astral $(15_4,20_3)$ configuration with symmetry group c_5.

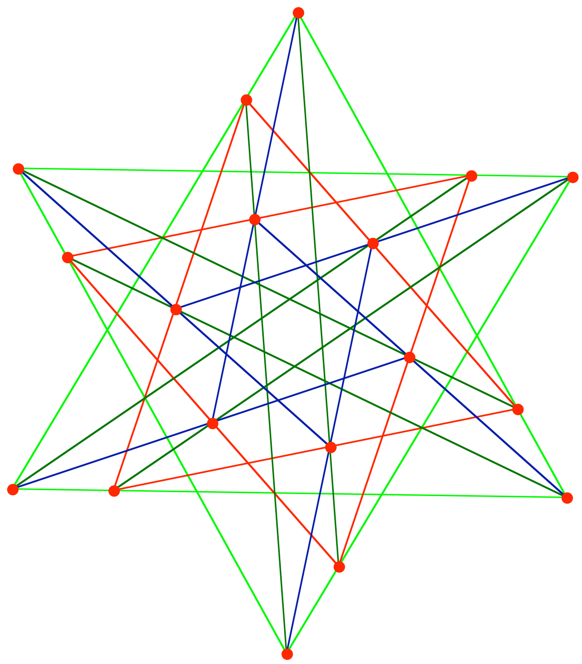


Figure 4.3.13. A [3,4]-astral $(18_4, 24_3)$ configuration with symmetry group c_6.

Exercises and problems 4.3.

1. Show that each of the permutations (described by their cycle decompositions) (A)(L)(M)(BGK)(CFJ)(DEH) and (ABCD)(LGJHMKFE) maps the combinatorial configuration $(12_3, 9_4)$ of Table 4.3.1 onto itself. Deduce that the automorphisms of the configuration act transitively on its points as well as on its lines. Decide whether the configuration is flag-transitive? (**Flag** = pair consisting of a "point" and a "line" incident with it.)

2. Decide whether all combinatorial $(12_3, 9_4)$ configurations are isomorphic, that is, whether the configuration $(12_3, 9_4)$ is unique. (Hint: Delete a line and all its points.)

3. Prove that any geometric realization of the $(12_3, 9_4)$ configuration must contain at least two "lines" that are not straight.

4. Set up the configuration table of the configuration $(9_4, 12_3)$ dual to the configuration in Table 4.3.1. Decide whether this configuration can be geometrically realized with straight lines or with pseudolines.

5. Describe the configuration table of the $((4r)_3, (3r)_4)$ configuration constructed in the proof of Theorem 4.3.1.

6. Show that the two $(20_3, 15_4)$ configurations shown in Figures 4.3.3 and 4.3.4 are not isomorphic.

7. Decide whether any among the three configurations $(16_3, 12_4)$ in Figure 4.3.2 are isomorphic, and whether the two configurations $(24_3, 18_4)$ in Figure 4.3.4 are isomorphic.

8. Starting with 12 points equidistributed on a circle, how many $(24_3, 18_4)$ configurations can you construct that have different appearance? Are any two among them isomorphic?

9. For general r, starting with 2r points equidistributed on a circle, how many $((4r)_3, (3r)_4)$ configurations can you construct that have different appearance? Are any among them isomorphic?

10. Draw symmetric realizations in the extended Euclidean plane of the polars of the configurations in Figure 4.3.2.

11. Decide whether any among the three configurations $(12_4, 16_3)$ in Figure 4.3.7 are isomorphic.

12. Draw the polar configurations of the configurations in Figure 4.3.7.

13. Verify that those triplets shown as collinear in Figures 4.3.8 and 4.3.9 that contain the point U are, in fact, collinear.

14. Find in Figure 4.3.9 a configuration $(12_4,16_3)$ that contains the dashed lines, and decide whether it is isomorphic with the configuration in Figure 4.3.8.

15. Determine the group of automorphisms of the configuration in Figure 4.3.8.

16. On the cubic curve in Figure 4.3.10, find a configuration $(9_2,6_3)$, and a configuration (12_3). Can you find any other configurations?

17. Decide whether the configurations in Figures 4.3.11 and 4.3.12 are duals of each other? If so, find a duality map. If not, find their duals.

18. Find the dual of the configuration in Figure 4.3.13.

19. Develop a theory — similar to the ones in Chapters 2 and 3 — of the [4,3]-configurations.

4.4 UNBALANCED [q,k]-CONFIGURATIONS WITH [q,k] ≠ [3,4]

Very little has been published about geometric [q,k]-configurations with $q \neq k$ and $\{q,k\} \neq \{3,4\}$. As mentioned in Section 4.2, a few results were found by Cayley [C2*], using planar sections of configurations of flats of various dimensions generated by families of points in general position. Specific instances will be mentioned below. Some of these methods have been used (mostly in special cases) by later writers.

There is more information about the corresponding combinatorial configurations, much of it due to H. Gropp. Here is a survey of what is known.

For combinatorial [3,5]-configurations (p_3, n_5) the necessary conditions for existence are $3p = 5n$, $p \geq 13$, and $n \geq 11$. Therefore we must have $p = 5r$ and $n = 3r$ for some integer r, so that we are looking at $((5r)_3, (3r)_5)$ configurations with $r \geq 4$. A combinatorial configuration $(20_3, 12_5)$ is shown in Table 4.4.1. From results on the "orchard problem" (see [B33]) it is known that 12 lines determine at most 19 triple points; it follows that no geometric $(20_3, 12_5)$ configuration is possible. Unfortunately, I do not know of any simple proof of the orchard problem result.

1	1	1	2	2	3	3	4	4	5	5	8
2	6	10	6	9	7	13	8	9	6	7	13
3	7	11	14	10	11	17	12	11	10	12	15
4	8	12	15	16	14	18	14	18	17	15	16
5	9	13	18	19	16	19	17	20	20	19	20

Table 4.4.1. A $(20_3, 12_5)$ combinatorial configuration.

There are interesting connections between combinatorial configurations $(12_5, 20_3)$ and Steiner triple systems S(2,3,13). We recall that a Steiner triple system S(2,3,v) is a collection of triplets from a v-element set, such that each pair of elements occurs in one and only one triplet. It is well known that a Steiner triple system S(2,3,v) exists if and only if v º 1 or 3 (mod 6). (For general information about Steiner triple systems see, for example, [B29, Section 10.3] or [R5, pp. 388 – 390].) The unique system S(2,3,7) is one of the incarnations of the combinatorial configuration (7_3), (the Fano plane) which we discussed in Section 2.1. There is unique system S(2,3,9). There are two (and only two) non-isomorphic systems S(2,3,13), which are of interest here. Information about them is presented in Tables 4.4.2 and 4.4.3, taken from [M4]. For a history of the S(2,3,13) see Gropp [G12].

1	1	1	1	1	1	2	2	2	2	2	3	3
2	4	6	8	10	12	4	5	8	9	11	4	5
3	5	7	9	11	13	6	7	10	12	13	8	12

3	3	3	4	4	4	5	5	5	6	6	7	7
6	7	9	7	10	11	6	8	9	8	9	8	10
10	11	13	9	13	12	13	11	10	12	11	13	12

Orbit: {1,2,3,4,5,6,7,8,9,10,1,12,13}. Automorphisms group has order 39.
Generators: (1 2 13 5 3 11 6 12 7 9 8 10 4) (1 2 3) (4 10 7) (9 13 12)

Table 4,4.2. The Steiner triple system $S(2,3,13)_1$.

1	1	1	1	1	1	2	2	2	2	2	3	3
2	4	6	8	10	12	4	5	8	9	11	4	5
3	5	7	9	11	13	6	7	10	12	13	8	12

3	3	3	4	4	4	5	5	5	6	6	7	7
6	7	9	7	10	11	6	8	9	8	9	8	10
13	11	10	9	13	12	10	11	13	12	11	13	12

Orbits: {1, 2, 5, 6, 8, 13} {3, 9, 10} {4, 11, 12} {7}. Automorphisms group has order 6.
Generators: (1 2 8) (3 10 9) (4 11 12) (5 13 6) (1 5) (2 6) (3 10) (8 13) (11 12)

Table 4.4.3. The Steiner triple system $S(2,3,13)_2$.

One interesting property of Steiner systems S(2,3,13) is that the deletion of one point and the triplets containing it yields a combinatorial configuration $(12_5, 20_3)$. It is clear that the deletion of different points from the same orbit yields isomorphic configurations. As it happens, deleting points from different orbits of the Steiner systems S(2,3,13) yields non-isomorphic configurations. Hence there are five such configurations $(12_5, 20_3)$. This result is due to Novak [N2]; see also Gropp [G17].

* * * * * *

Concerning values of $r \geq 5$ we shall show that there exist geometric $((5r)_3, (3r)_5)$ configurations for all $r \geq 5$. By duality and polarity, the same is true for configurations $((3r)_5, (5r)_3)$.

Theorem 4.4.1. There exist geometric $((5r)_3, (3r)_5)$ configurations for all $r \geq 5$; moreover, they can be chosen as astral in the extended Euclidean plane.

Proof. The validity of this statement follows at once from the family of configurations illustrated in Figure 4.4.1; clearly, analogous configurations exist for all $r \geq 5$.

Additional examples of geometric [3,5]-configurations are shown in Figures 4.4.2 and 4.4.3.

Cayley [C2*] described a $(21_5,35_3)$ configuration.

*　　*　　*　　*　　*　　*

For combinatorial [3,6]-configurations (p_3, n_6) the necessary condition for existence are $p = 2n$ and $n \geq 13$. A combinatorial configuration $(26_3, 13_6)$ is shown in Table 4.4.4. It can also be shown (see [G20]) that combinatorial configurations $((2n)_3, n_6)$ exist for all $n \geq 13$. Gropp [G17] states that there are exactly 787 distinct $(28_3, 14_6)$ combinatorial configurations.

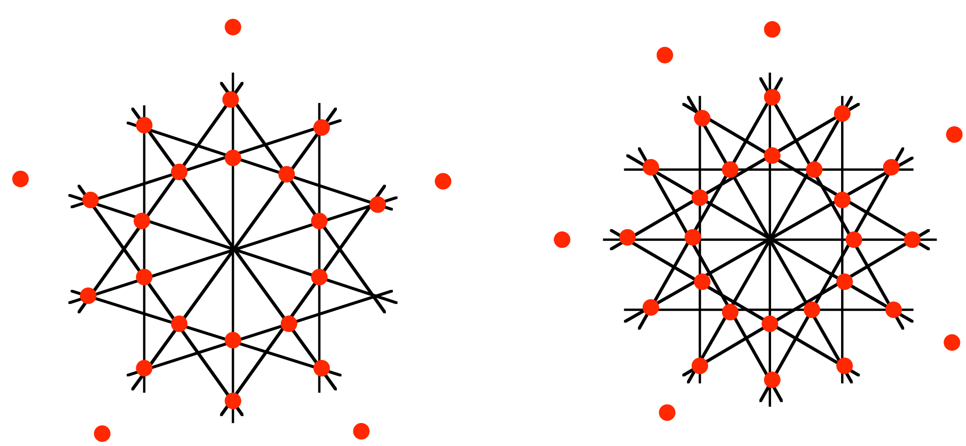


Figure 4.4.1. Typical examples of [3,5]-configurations astral in the extended Euclidean plane. The two examples correspond to r = 5 and r = 6.

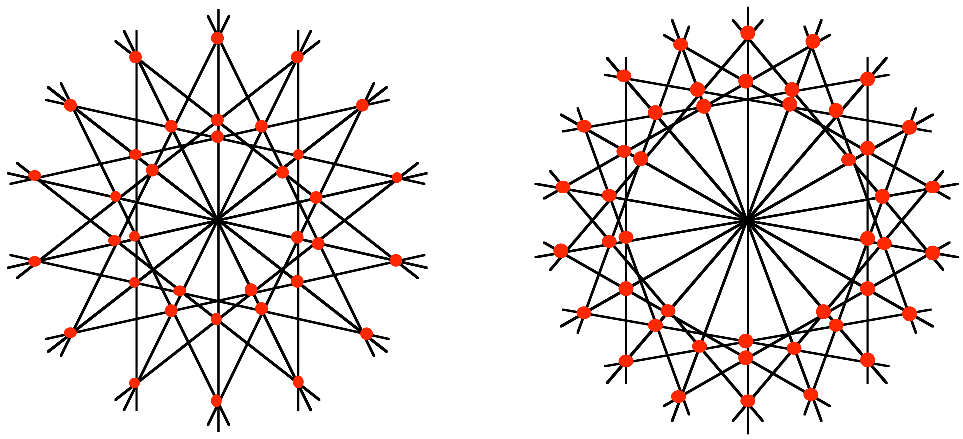


Figure 4.4.2. Examples of astral [3,5]-configurations in the Euclidean plane. These are clearly representatives of an infinite family, and several variants are possible.

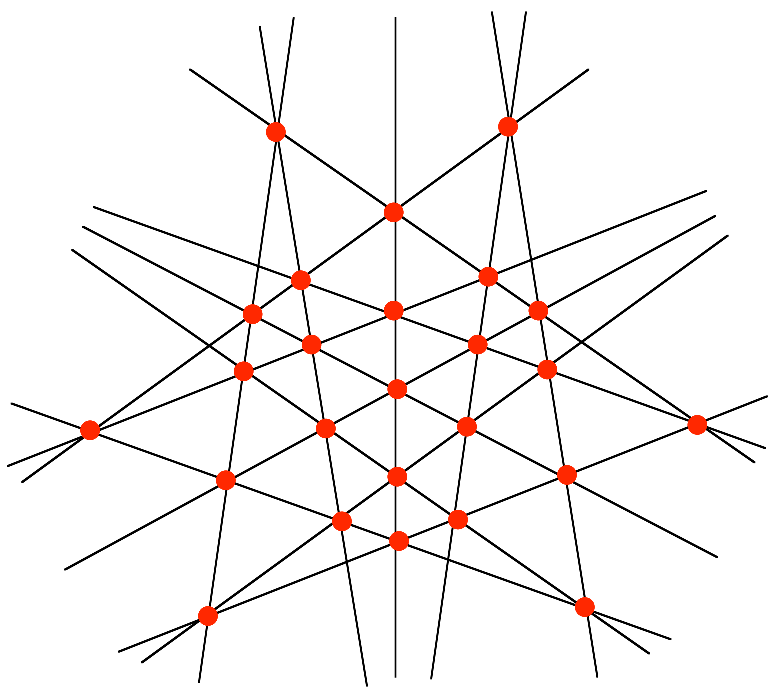


Figure 4.4.3. Another $(25_3, 15_5)$ configuration.

There seems to be no geometric $(26_3, 13_6)$ configuration, but I am not aware of any proof. Also, there is a large difference between the case of [3,6]-configurations and the [3,5]-configurations considered above. In the latter case, for all values of n that satisfy the necessary conditions and are beyond a certain limit (in fact $n \geq 15$), an astral

1	1	1	2	2	3	3	4	4	5	5	6	6
2	7	12	7	8	7	8	9	10	10	11	9	11
3	8	13	12	16	15	13	14	12	16	13	15	14
4	9	14	17	20	23	17	17	22	19	18	18	21
5	10	15	18	25	24	19	20	24	21	20	19	22
6	11	16	21	26	26	22	23	25	23	24	25	26

Table 4.4.4 A $(26_3, 13_6)$ combinatorial configurations (found in 1999 by Xin Chen, at the time a student in one of my classes). The number of distinct $(26_3, 13_6)$ combinatorial configurations seems not to be known.

configuration is possible in the extended Euclidean plane. For [3,6]-configurations this is not the case. We have:

Theorem 4.4.2. For all $r \geq 5$ there exist astral $((6r)_3, (3r)_6)$ geometric configurations in the Euclidean plane.

Proof. In Figure 4.4.4 we show the two typical configurations of this kind for r = 6 and 7. The only known configuration $(30_3, 15_6)$ is not typical; it is shown in Figure 4.4.5, and we have already seen it in Figure 1.6.8. ¨

Thus, except for small values of n, there exist geometric configurations $(2n_3, n_6)$ for all n that are multiples of 3. For no other values of n are any geometric [3,6]-configurations known.

It is clear that geometric [3,k]-configurations and [k,3]-configurations can be constructed for all $k \geq 7$ in analogy to the configurations in Figures 4.4.1, 4.4.2, and 4.4.4. As these are not really interesting, and no additional information seems available, we shall not pursue this topic any farther. Instead, we turn now to [4,k]-configurations and their duals.

For [4,5]-configurations (p_4, n_5) the necessary conditions are $p \geq 17$, $n \geq 16$, $5n = 4p$. Therefore the configurations are necessarily of the form $((5r)_4, (4r)_5)$ for $r \geq 4$. According to Gropp [G20], combinatorial configurations with these parameters exist for

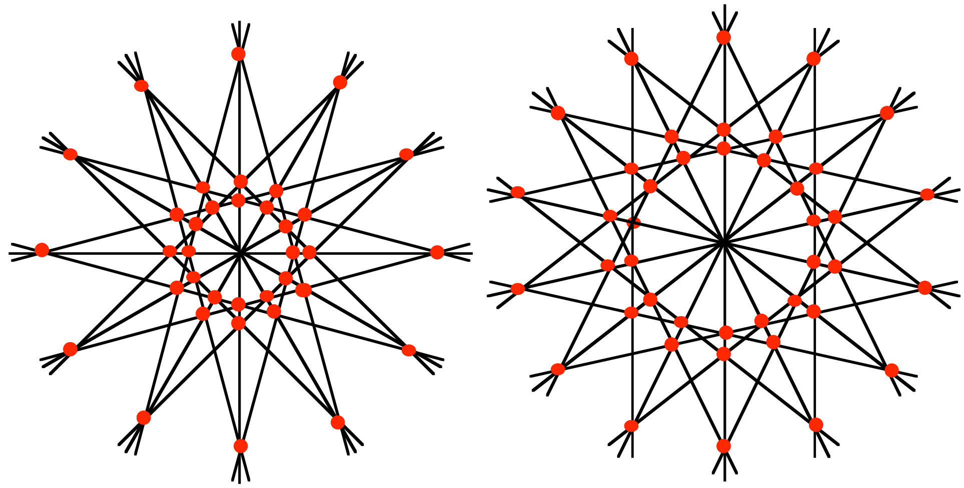


Figure 4.4.4. Configurations $(36_3, 18_6)$ and $(42_3, 21_6)$.

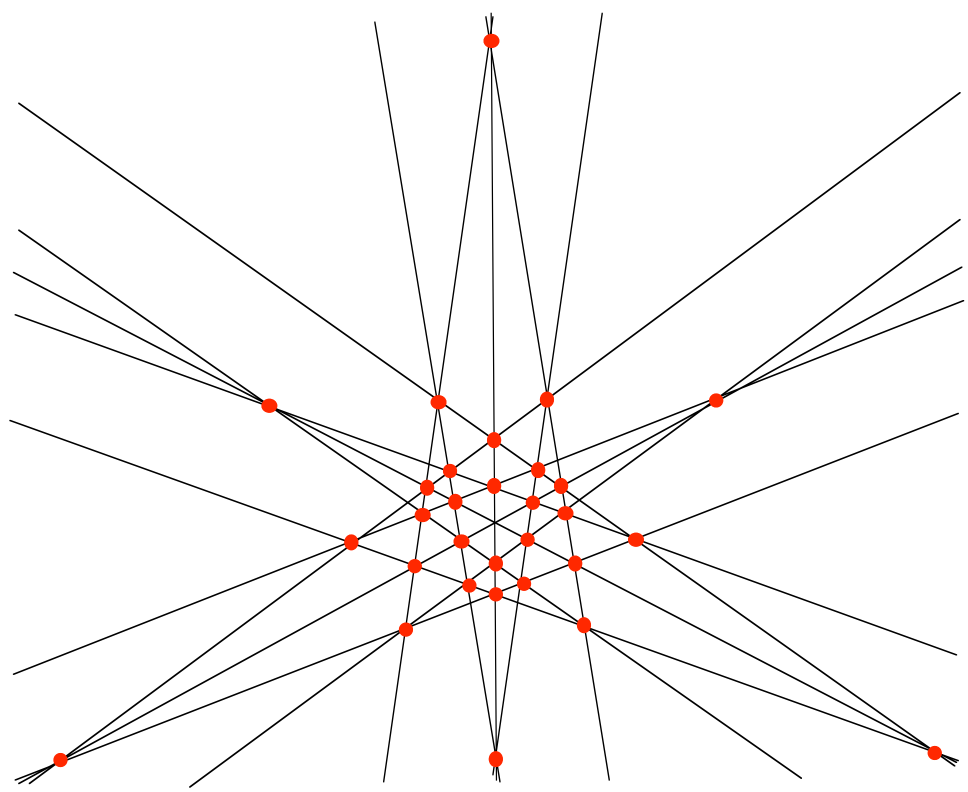


Figure 4.4.5. The only known $(30_3, 15_6)$ configuration.

all r ≥ 4. There seems to be no information available concerning the number of distinct configurations for each value of r.

Concerning topological or geometric [4,5]-configurations, there is an elegant family of geometric configurations $((5r)_4, (4r)_5)$ for $r \geq 9$. (I do not know whether or not there are any for $r \leq 8$.) Its two smallest members are shown in Figure 4.4.6, and their construction can be explained as follows: Starting for $r \geq 9$ from the 4-astral 4-configurations $((4r)_4)$, such as the ones denoted in Section 3.8 by 9#(3,1;2,4;3,2;3,2) or r#(3,1,2,4,1,3,4,2), additional r points are added at-infinity (in the directions of the quadruplets of parallel lines of the 4-configuration). This yields a $((5r)_4, (4r)_5)$ configuration with five orbits of points and four orbits of lines. Polars of these configurations are [5,4]-configurations. Other [5,4] configurations can be obtained by adding r mirrors to the same 4-configurations $((4r)_4)$, but only for odd $r \geq 9$. The two cases analogous to the ones in Figure 4.4.6 are illustrated in Figure 4.4.7. We used a combination of these methods in Section 4.1.

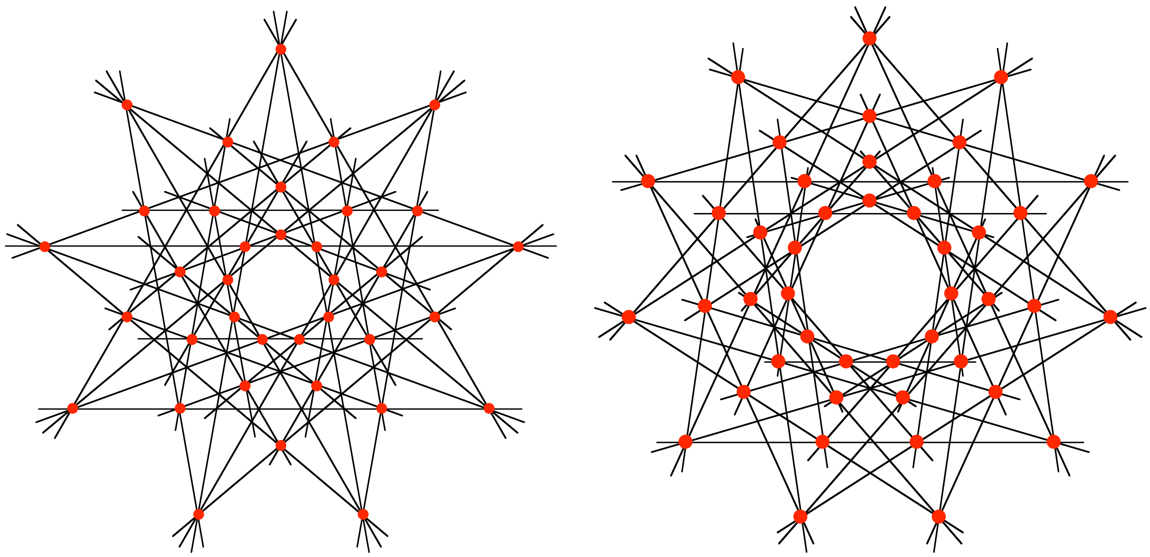
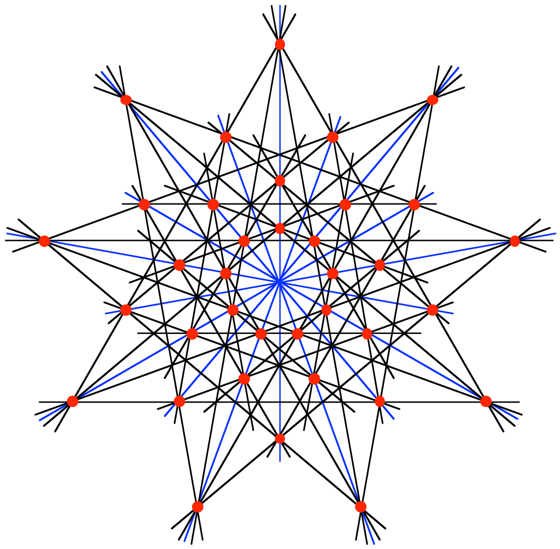


Figure 4.4.6. Typical $((4r)_4)$ configurations with symbols 9#(3,1;2,4;3,2;3,2) and 10#(3,1,2,4,1,3,4,2). Addition of r points-at-infinity to each yields configurations $((5r)_4, (4r)_5)$. Addition of r mirrors gives $((4r)_5, (5r)_4)$



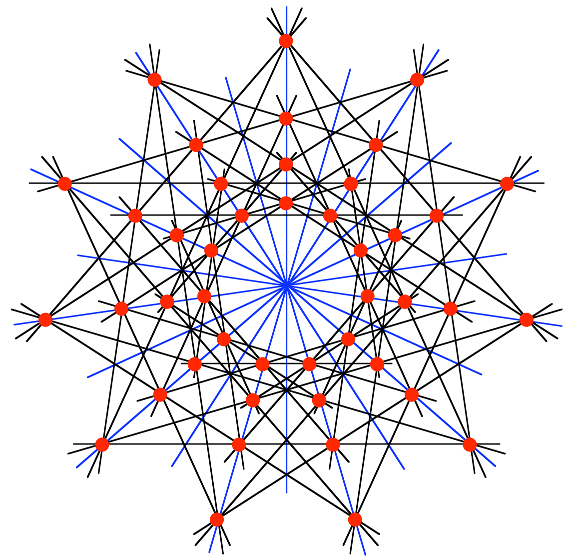
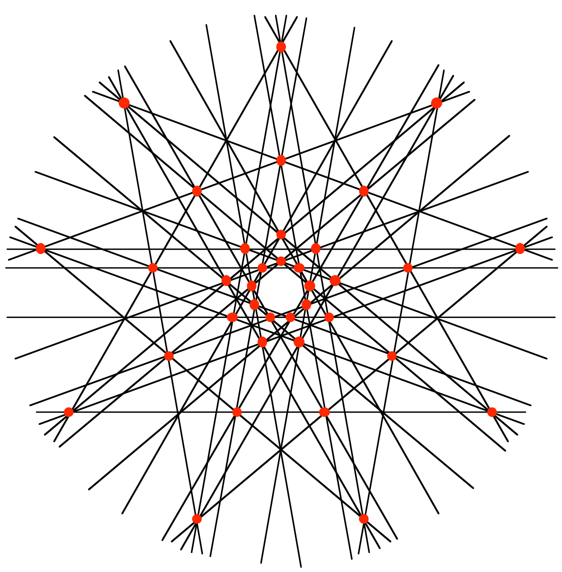


Figure 4.4.7. For odd r, adding r mirrors to 4-configurations such as 9#(3,1;2,4;3,2;3,2) and 11#(3,1,2,4,1,3,4,2) yields [5,4]-configurations $((4r)_5, (5r)_4)$.

There is very little information available about small [q,k]-configurations with still larger values of q and k. Some examples, similar to those above and possible for some particular parameter values, are shown in Figures 4.4.8 and 4.4.9.



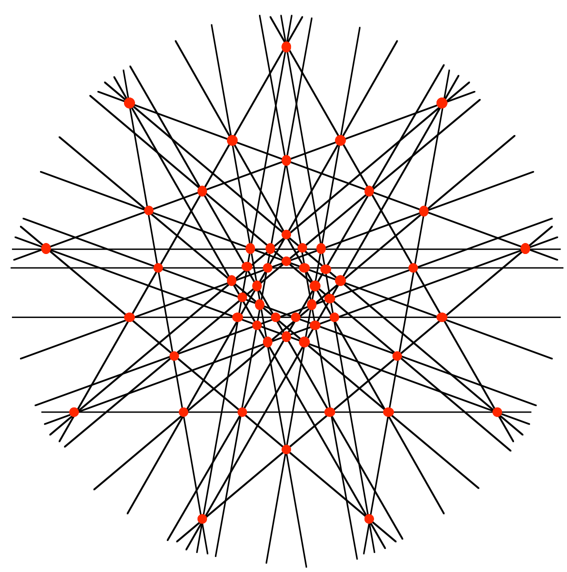


Figure 4.4.8. At left, a 4-configuration (36_4) with symbol 9#(3,1,4,2,1,3,2,4). Adding 18 points yields a (4,6)-configuration $(54_4, 36_6)$ shown at right.

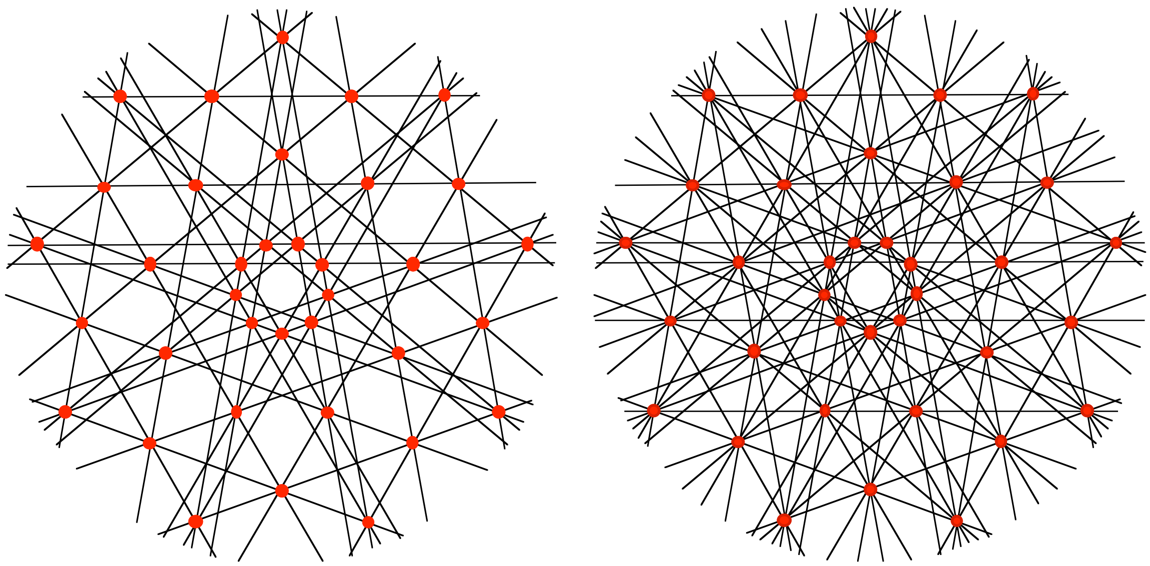
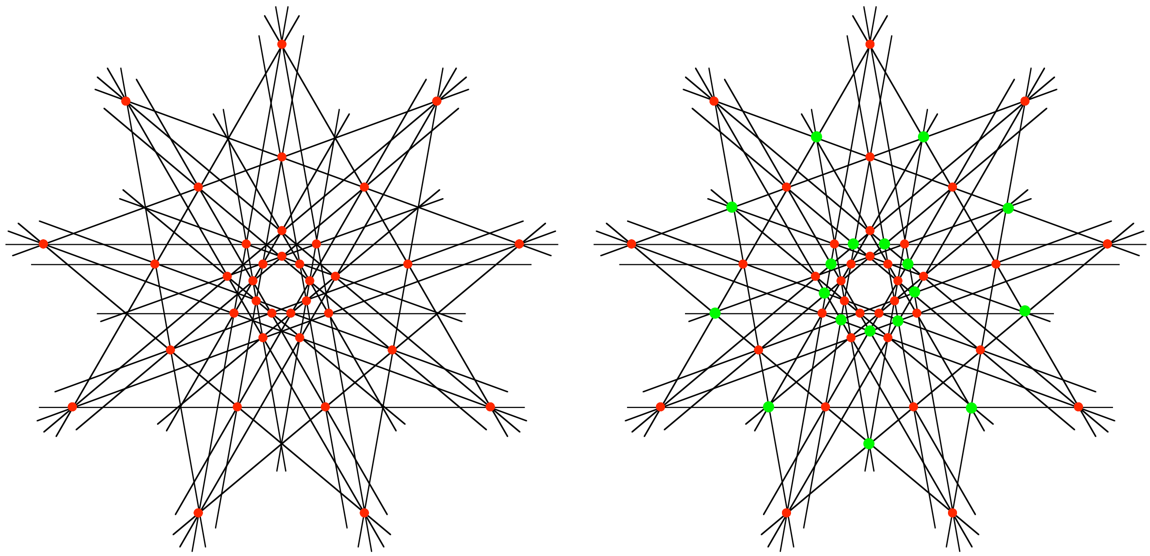


Figure 4.4.9. At left, a 4-configuration (36_4) with symbol 9#(4,3;1,4;2,1;3,2). Adding 18 lines yields a [6,4]-configuration $(36_6, 54_4)$ shown at right. Adding to it the nine lines of mirror symmetry yields a $(36_7, 63_4)$ configuration. Adding instead the nine points at infinity leads to a configuration $(45_6, 54_5)$.

Many other 4-astral 4-configurations can be used in constructions similar to the ones illustrated in Figures 4.4.7 to 4.4.10.

A complete determination of astral [6,4]-configurations (and their polars) was carried out by L. Berman [B2]. These are configurations in which each point is on six lines and each line contains four point, there being two orbits of points and three orbits of lines. As demonstrated in [B5] there are precisely five connected astral $(60_6,90_4)$ configurations, and no other connected astral [6,4]-configurations. One of these is shown in Figure 4.4.11. This configuration can be understood as superposition of three astral (30_4) configurations: the sporadic 30#(12,10;6,10), and the systematic 30#(12,10;3,9) and 30#(10,6;3,9). Similarly for the other four. Some other results on [q,k]-configurations can also be found in [B5].

The material we have presented in this section exhausts the knowledge available to us. As in most other sections, there are lots of obvious questions and open problems for which we have no guesses as to the correct answers. The hope is that some readers will take it as a challenge to enlarge the compass of known facts.



(a) (b)

Figure 4.4.10. (a) The 4-astral configuration 9#(3,1;4,2;1,3;2,4) has quadruplets of lines concurrent at points that are not configuration points. (b) Adding these 18 points (green) yields a $(54_4, 36_6)$ configuration. Adding nine points at infinity (in the direction of quadruplets of parallel lines) yields a $(63_4,36_7)$. Adding instead the nine mirrors results in a $(54_5,45_6)$ configuration.

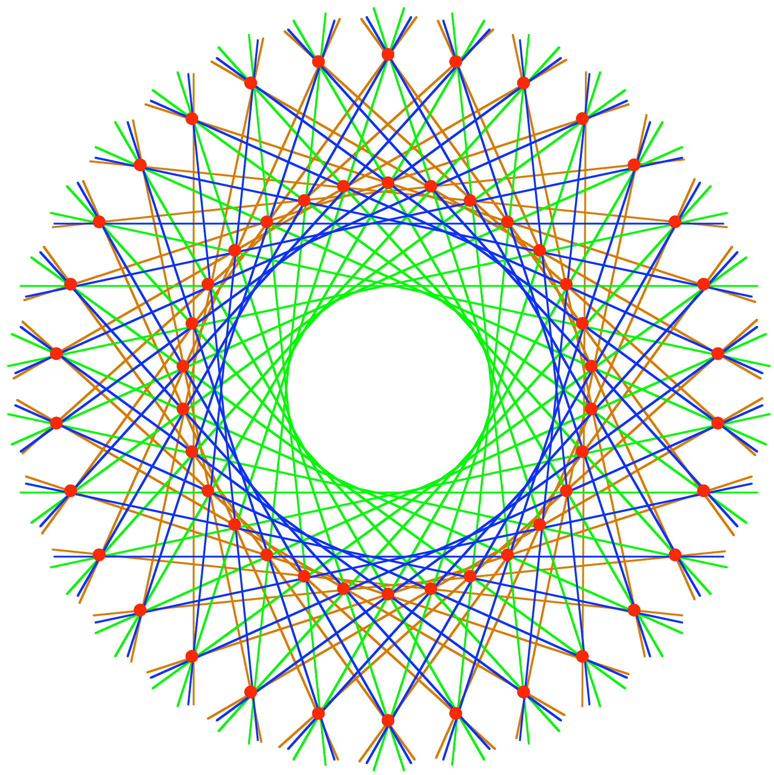


Figure 4.4.11. An astral $(60_6, 90_4)$ configuration, taken from [B5].

Exercises and problems 4.4

1. Decide whether any combinatorial configuration $(26_3, 13_6)$ can be realized geometrically or topologically.

2. Do there exist any geometric configurations $(2n_3, n_6)$ with n not a multiple of 3?

3. Draw a configuration $(45_4, 36_5)$.

4. Draw a configuration $(54_4, 36_6)$.

5. Draw a configuration $(49_3, 21_7)$.

6. Draw as small a configuration of type (p_8, n_3) as you can find.

7. By consulting the lists in Section 3.6, describe the other four astral $(60_6, 90_4)$ configurations.

8. Find configuration $(60_6, 90_4)$ that are not astral.

4.5 FLORAL CONFIGURATIONS

Floral configurations provide a means of visualizing some rather large configurations in a pleasing and visually accessible way. The topic was initiated by J. Bokowski late in 2006, in an email message asking whether the configuration attached to the message has been already found by anybody. The configuration in question is shown in Figure 4.5.1. It was a completely new type of configuration, and the curiosity it engendered quickly led to a wealth of configurations analogous in some sense. Collectively they became known as "floral configurations". The results of the early investigations of these configurations have been presented in [B12]; most will be reviewed here, together with new developments. Many of the latter arose in discussions with the coauthors of [B12], and I owe them sincere gratitude.

Loosely speaking, a floral configuration is a connected configuration that has a number of parts, called florets, arranged within the configuration in a symmetric way.

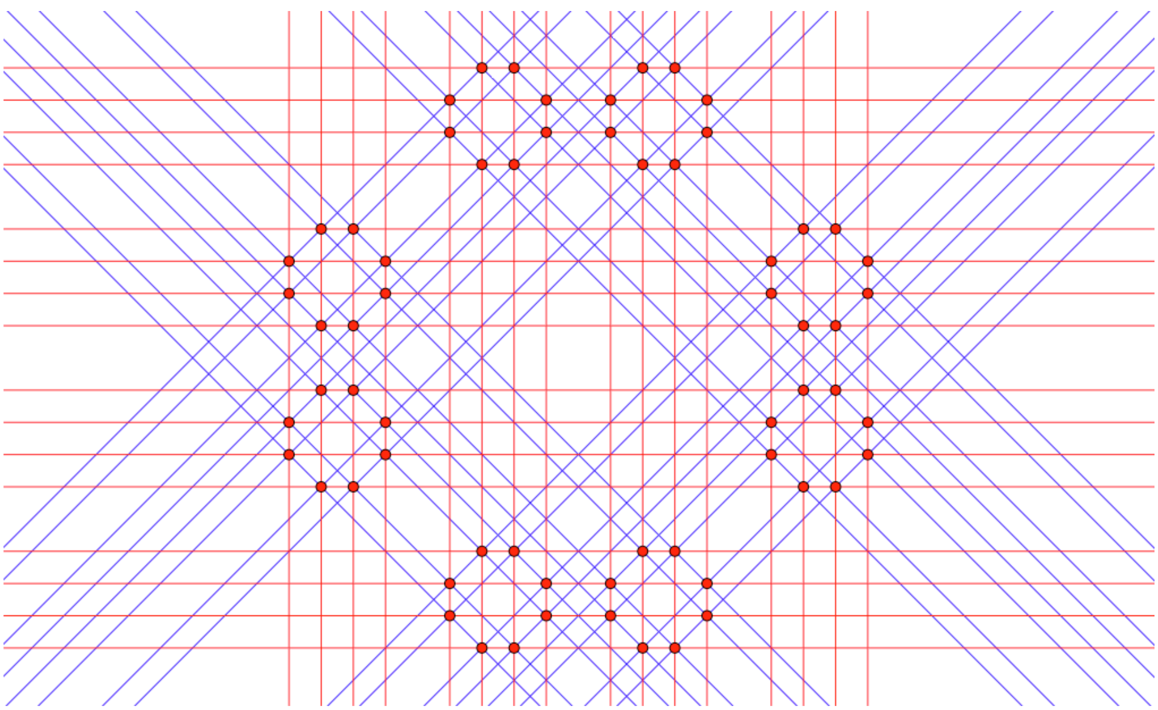


Figure 4.5.1. The first "floral" configuration, from an email by J. Bokowski on October 28, 2006.

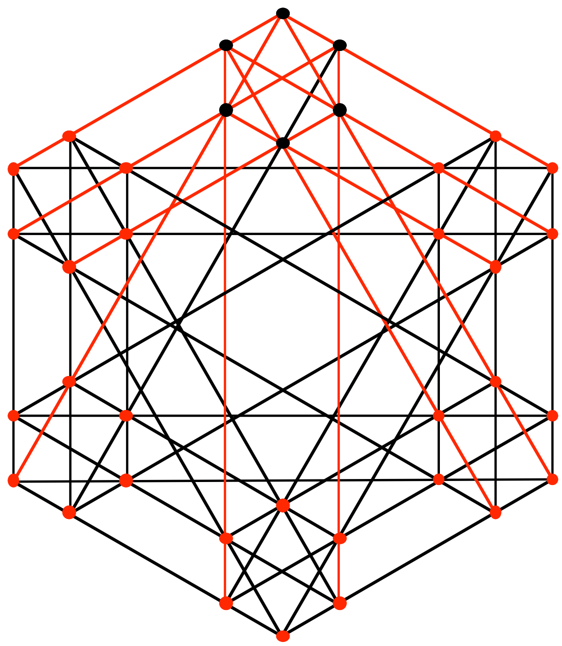
Reasonable people may (and do) differ regarding what level of generality is reasonable in the context of floral configurations. For our purposes the following approach seems most appropriate; it is more general than the approach in [B12], with which we shall compare it near the end of this section.

Definition 4.5.1. A **floret** is a collection of points and lines, with prescribed incidences. A **floral configuration** is a configuration that consists of a collection of florets, such that the symmetry group of the configuration acts transitively on the florets.

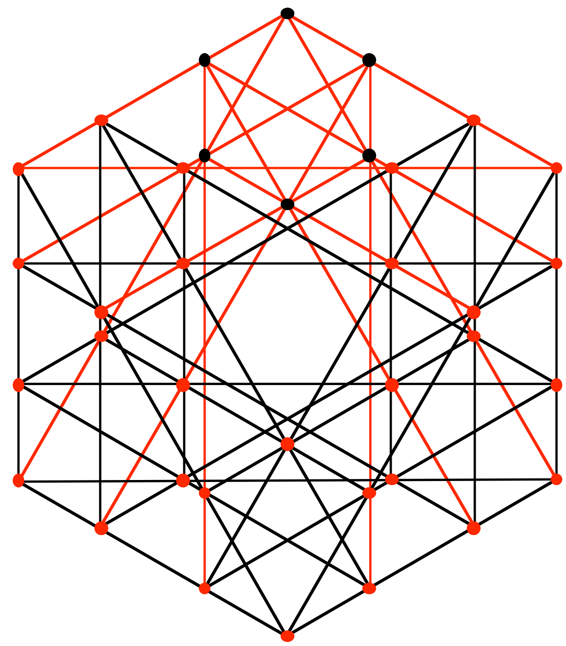
In view of the generality of the concept, it is not surprising that one may distinguish several varieties of floral configurations. To begin with, there is the question of what symmetry group is being considered. It turns out that dihedral groups are much more productive in this context, and we shall devote most of the section to them. The cyclic groups will be considered briefly afterwards.

Even before discussing methods for the construction of floral configurations, a limitation of their appeal needs to be discussed. It is quite clear that the florets in Bokowski's configuration are easily picked out, as are the ones in parts (a) and (b) of Figure 4.5.2. However, this becomes increasingly more difficult in the other parts of that illustration, and the question arises whether it is appropriate to call all of them "floral". One may wish to restrict consideration to only those configurations in which each floret is contained in a single sector determined by the mirrors of the symmetry group, or in two such sectors – but there really is no obvious and natural delimitation. Hence we shall not make any such restriction, although we shall endeavor to present examples in which the florets can be readily discerned.

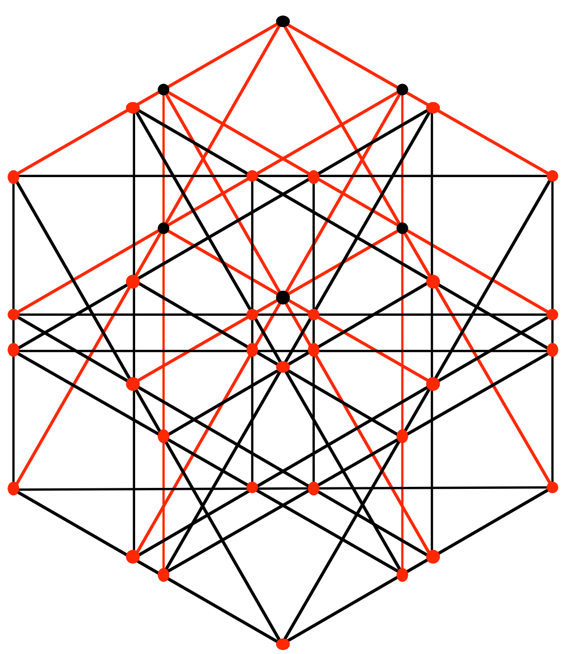
Four construction methods seem to furnish all known examples of floral configurations. Since they depend on the **m**irrors of the dihedral groups, we denote them as constructions ($\mathcal{M}_1$), ($\mathcal{M}_2$), ($\mathcal{M}_3$) and ($\mathcal{M}_4$).



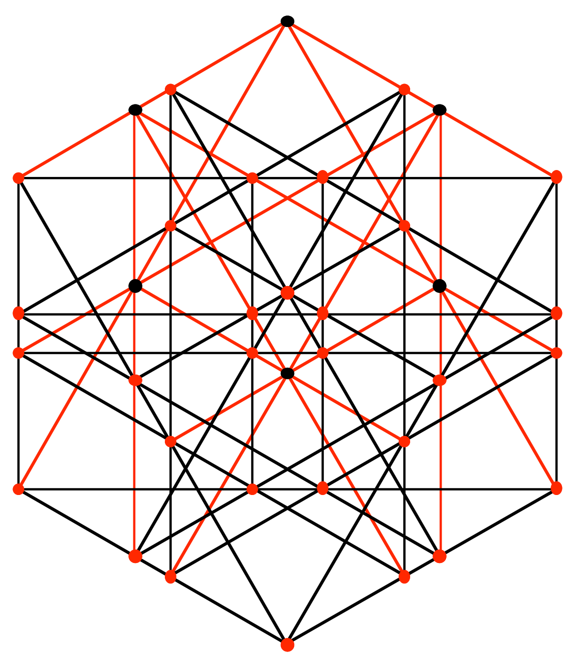
(a)



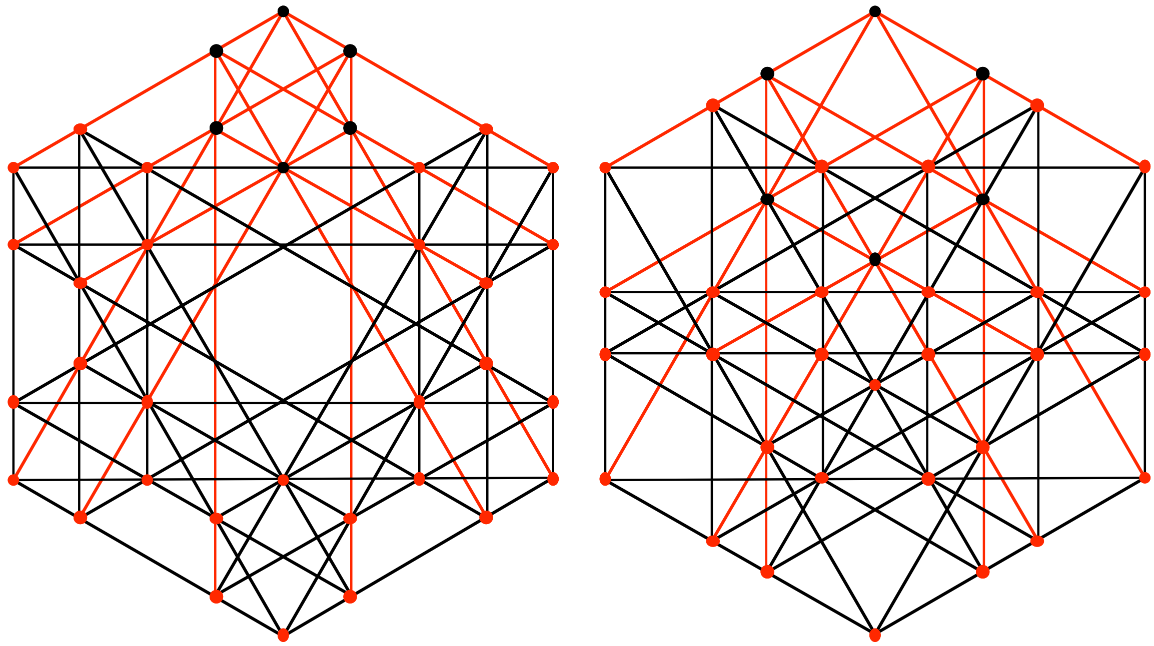
(b)



(c)



(d)



(e) (f)

Figure 4.5.2. The first four floral (36_4) configurations are isomorphic, and differ only in the size of the florets. The florets of the configuration are clearly distinguishable in (a) and (b), but less easily in (c) and especially in (d). In fact, in all four cases the situation is complicated by the fact that two different sets of florets can be picked out. When constructing the configurations, the top florets in (a), (b), and (c) had their top three points on the upper half of the top sides of the overall hexagon, while in (d) they reached beyond the top half. However, with a bit of contemplation it is easy to reverse the perception in all four of the configurations. The diagrams in (e) and (f) show two representations of the same configuration – with some unintended incidences.

Construction ($\mathcal{M}_1$)**.** Let S be a family of s concurrent lines, equiinclined to each other, so that they represent the s mirrors of a dihedral group d_s. Let the **protofloret** F be a [q,t]-configuration such that each of the lines in F is perpendicular to one of the mirrors in S, and no mirror in S is a mirror for F. Then images of the protofloret F under all reflections in the s mirrors of S create *in general* a floral (q,2t)-configuration with 2s florets.

The "in general" part refers to two possibilities of failure of the construction:

- The resulting configuration may be disconnected, hence cannot qualify as a floral configuration;
- There may be some accidental incidences, which make this a representation of the underlying combinatorial configuration – but not a realization of it

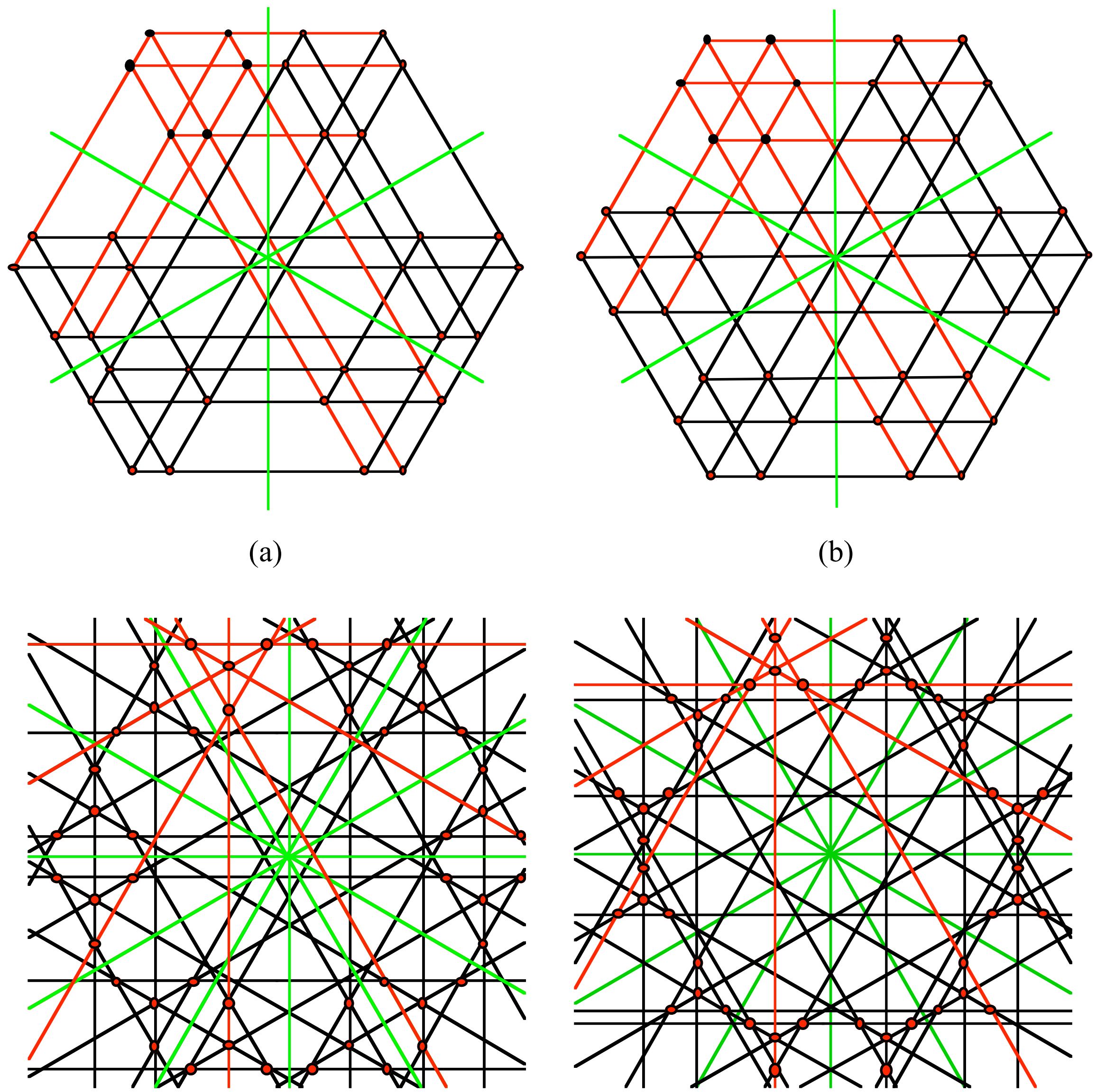
It should be stressed that neither the points of an individual floret, nor its lines, are required to have any non-trivial symmetries – although in many cases they do have them.

An illustration of the construction ($\mathcal{M}_1$) is provided in Figure 4.5.3. There s = 3 (in parts (a) and (b)) or s = 6 (in parts (c) and (d)), with q = 3 and t = 2. The lines of S are shown green. The protofloret F is shown with black points and red lines, while the other points are red and lines are black. Bokowski's original floral configuration, shown in Figure 4.5.1, can also be obtained by the ($\mathcal{M}_1$) construction.

The color-coding in Figure 4.5.3 will be used throughout the present section for all constructions with method ($\mathcal{M}_1$).

It should also be stressed that all floral configurations obtained by the construction ($\mathcal{M}_1$) have at least two degrees of freedom. This is most easily seen by observing that if the distance of the center of symmetry from the centroid of the protofloret is kept fixed, the size of the protofloret can be changed continuously, as can its position with respect to the mirrors. The configuration in Figure 4.5.2 shown earlier was constructed by the ($\mathcal{M}_1$) method as well, and the influence of the size of the protofloret can be discerned easily.

Construction ($\mathcal{M}_2$)**.** As before, let S be a family of s concurrent lines, equiinclined to each other, so that they represent the s mirrors of a dihedral group d_s. Let the **protofloret** F be a [q,t]-configuration such that each of the lines in F is perpendicular to one of the mirrors in S, and precisely one mirror M in S is a mirror for F. Moreover, let no line of F be perpendicular to M. Then images of the protofloret F under all reflections in the s mirrors of S create in general a floral [q,2t]-configuration with s florets.



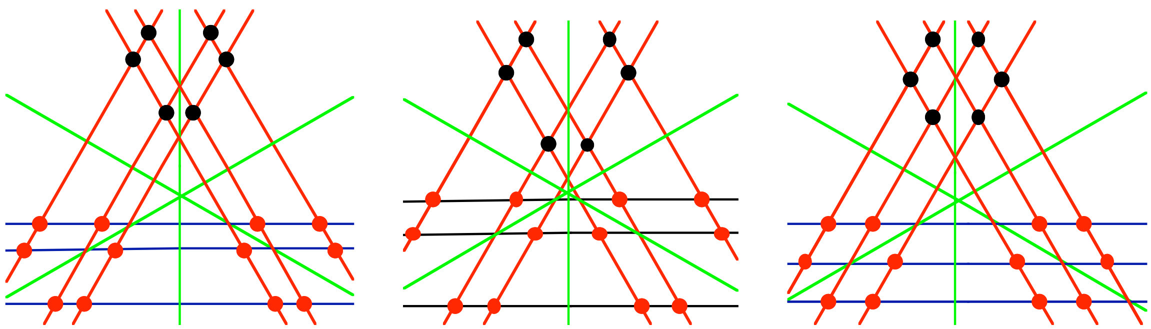
(a) (b)

(c) (d)

Figure 4.5.3. Four floral (3,4)-configurations formed using construction ($\mathcal{M}_1$). In (a) and (b) the protofloret F is a $(6_3, 9_2)$ configuration, with points at the vertices of an isogonal hexagon. In each case the result is a $(36_3, 27_4)$ configuration with symmetry group d_3; the two configurations are isomorphic. In (c) and (d) the configurations are $(48_3, 36_4)$ with symmetry group d_6, and the protofloret F consists of the vertices of an equilateral triangle and its center, the sides of the triangle, and its mirrors; hence it is a $(4_3, 6_2)$ configuration. In all parts the lines incident with F are shown in red, the mirrors in S are shown green.

An example of construction ($\mathcal{M}_2$) is provided in Figure 4.5.4. There s = 3, q = 2, and t = 2. The result is a floral [2,4]-configuration (18_2, 9_4). Figure 4.5.2 also shows configurations obtained by method ($\mathcal{M}_2$). As with ($\mathcal{M}_1$), unintended incidences may occur. This happened, for example, in Figure 4.5.2, parts (e) and (f).

The floral configurations constructed using ($\mathcal{M}_2$) have at least one degree of freedom for continuous changes — the size of the protofloret relative to its distance from the center of symmetry. This is illustrated in parts (a) and (b) of Figure 4.5.4. In some cases the floret itself may have continuous changes in shape; this is shown in part (c).



(a) (b) (c)

Figure 4.5.4. Three isomorphic floral configurations (18_2, 9_4) obtained by method ($\mathcal{M}_2$). The florets in (a) and (b) differ only in size, those in (c) differ in shape.

Construction ($\mathcal{M}_3$) starts with a floral [q,k]-configuration C constructed by method ($\mathcal{M}_2$). Assuming, for ease of formulation, that the mirror M of the protofloret F is vertical, we look for the uppermost points on florets F' and F" symmetric with respect to M. If each of these two florets has a single highest point X' and X", respectively, we focus on the corresponding points Y' and Y" in the protofloret F, and on a pair of lines, symmetric with respect to M, which go to points Z' and Z" on F that are lower that Y' and Y". We create a new protofloret F* by deleting from F the two lines just mentioned, and introducing a horizontal line H containing the two points Z' and Z". The protofloret F* is not a configuration, since it has two points (Y' and Y") that are incident with only q–1 lines. Mirroring the changes we made to get F* from F to all the florets of C, we now utilize the presence of a degree of freedom in floral configurations constructed like C by method ($\mathcal{M}_2$), and change the size of the protofloret F* until the line H passes through the highest points on F' and on F" — if this is possible. Then, if there are no unintended incidences, this renders the whole collection of points and lines into a floral [q,k]-configuration.

An illustration of the ($\mathcal{M}_3$) construction is shown in Figure 4.5.5. In part (a) we have a (49_4) floral configuration constructed by the ($\mathcal{M}_2$) method, and in the other parts are three floral configurations obtained from it by ($\mathcal{M}_3$). The newly introduced lines in these configurations are shown in blue. In distinction from the other floral configurations we have seen, configurations obtained by ($\mathcal{M}_3$) have lines that are incident with three florets each. It also should be noted that although in many cases the florets F' and F" are the ones nearest to F, this is not always the case. In Figures 4.5.6 and 4.5.7 we show floral configurations (56_4) in which the points of the protofloret are seven of the eight vertices of a regular octagon. In the second of these the special lines go to non-adjacent florets.

Construction ($\mathcal{M}_4$) is analogous to ($\mathcal{M}_3$), but it starts with a floral [q,k]-configuration C constructed by either method ($\mathcal{M}_1$) or method ($\mathcal{M}_2$). Assuming, for ease of formulation, that a mirror M mapping the protofloret F to its adjacent floret F° is vertical, we look for the uppermost points on florets F' and F" symmetric with respect to M, and adjacent to F and F°, respectively. If each of these two florets has a single highest point X' and X", respectively, we focus on the corresponding points Y' and Y" in the protofloret F and the floret F°, and on a pair of lines, symmetric with respect to M, which go to points Z' and Z" on F and F° that are lower than Y' and Y". We create a new protofloret F* by deleting from F the line just mentioned and omitting its companion from F°, and introducing a horizontal line H containing the two points Z' and Z". The protofloret F* is not a configuration, since it has a point (namely Y') that is incident with only $q-1$ lines. Mirroring the changes we made to get F* from F to all the florets of C, we now utilize the presence of a degree of freedom in floral configurations constructed like C by method ($\mathcal{M}_1$), and change the size of the protofloret F* until the line H passes through the highest points on F' and on F" — if this is possible. Then, if there are no unintended incidences, this renders the whole collection of points and lines into a floral [q,k]-configuration.

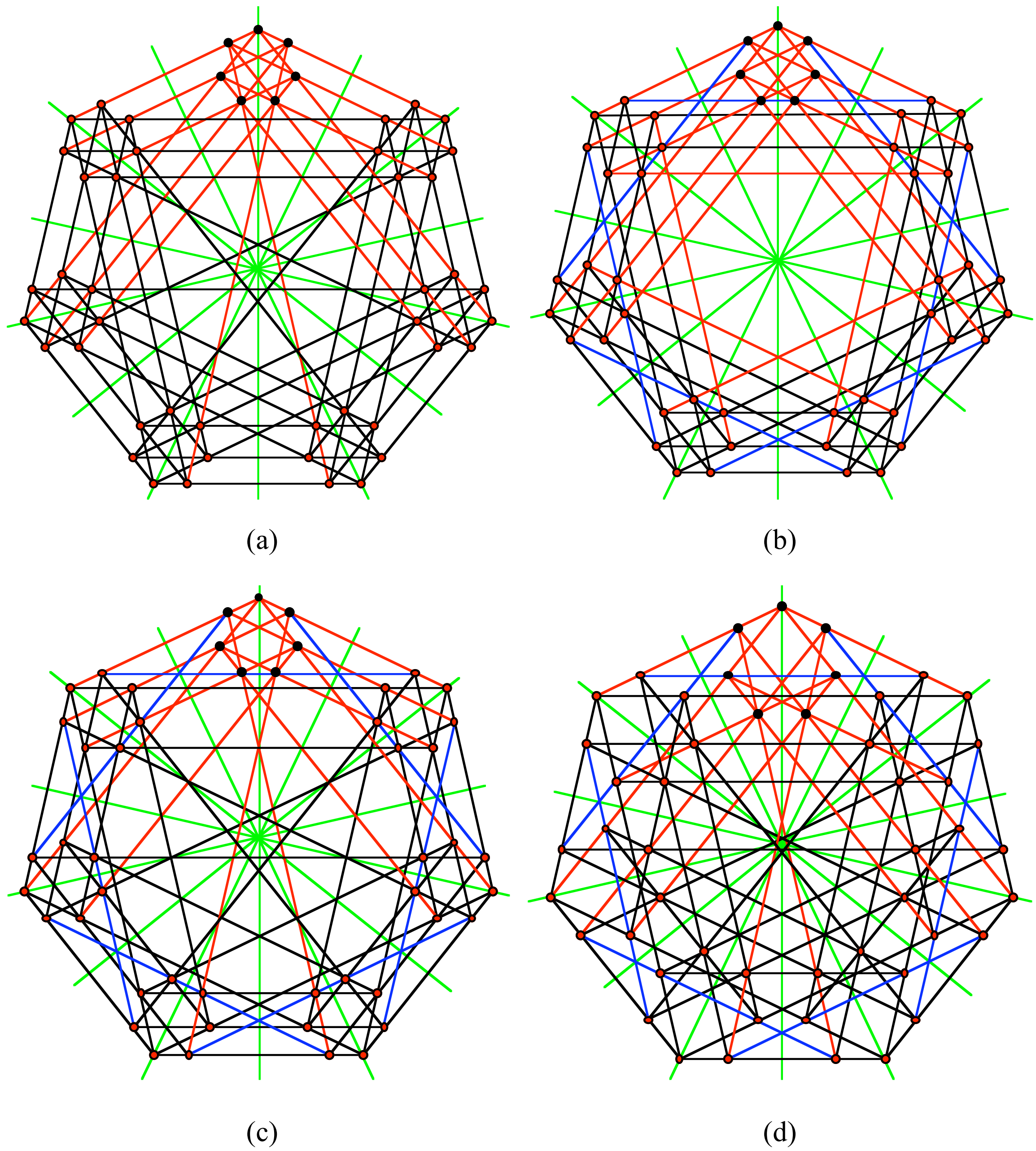


Figure 4.5.5. A floral configuration (49_4) obtained using $(\mathcal{M}_2)$ is shown in (a). From it three distinct configurations (49_4) are obtained by construction $(\mathcal{M}_3)$. The special lines resulting from the construction are shown in blue.

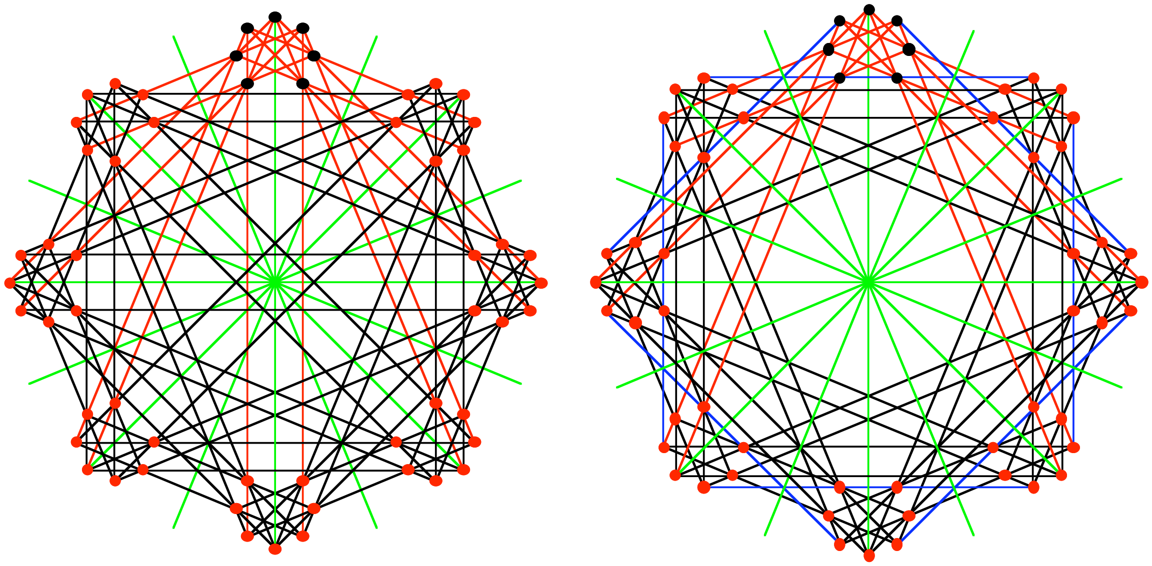


Figure 4.5.6. A floral (56_4) configuration obtained by the $(\mathcal{M}_2)$ construction, and another (56_4) configuration resulting from it by the $(\mathcal{M}_3)$ method. The points of the protofloret are seven of the eight vertices of a regular octagon. The protofloret has d_1 symmetry, the configurations have d_8 symmetry.

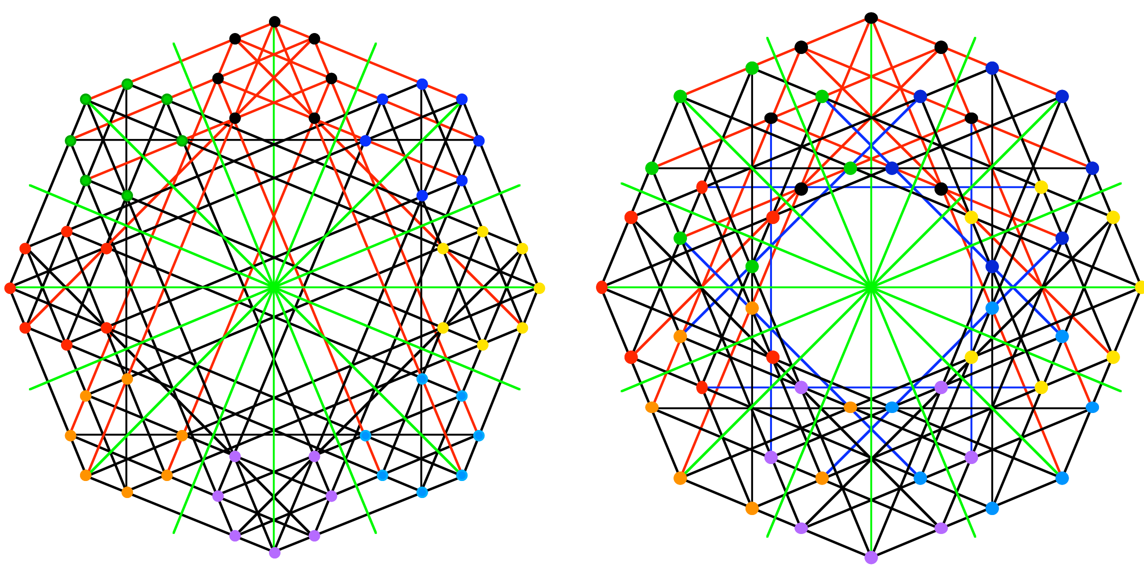


Figure 4.5.7. The same procedure as in Figure 4.5.6, except that the special line determined by two points of the protofloret was aimed not at adjacent florets but at a pair of more distant florets. To achieve the intended incidence, the size of the protofloret had to be increased, leading to a situation similar to that in Figure 4.5.2(d). As a visual aid, the points of each floret were given distinct colors, matched in the two parts.

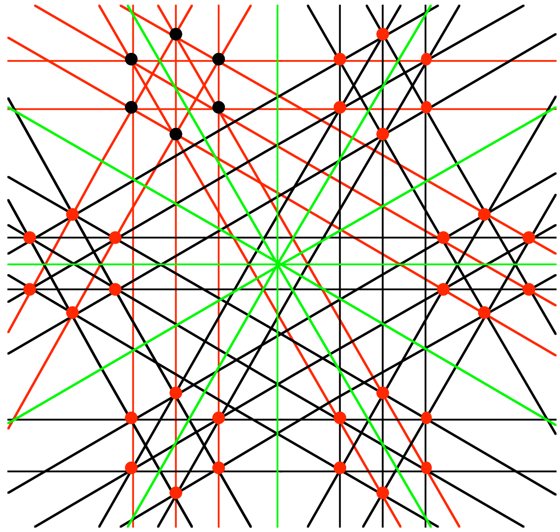
Construction ($\mathcal{M}_4$) is illustrated in Figures 4.5.8 and 4.5.9, by floral configurations (36_4) and (128_4). It should be noted that in contrast to the other constructions, ($\mathcal{M}_4$) leads to configurations in which some lines are incident with four florets. It is also worth mentioning that, as in construction ($\mathcal{M}_3$), instead of "adjacent" florets in some cases florets lying farther away may be used.

In order to illustrate the esthetic appeal of floral configurations, and their great variety, we shall now present a number of examples. Most deal with the more versatile ($\mathcal{M}_1$) construction.

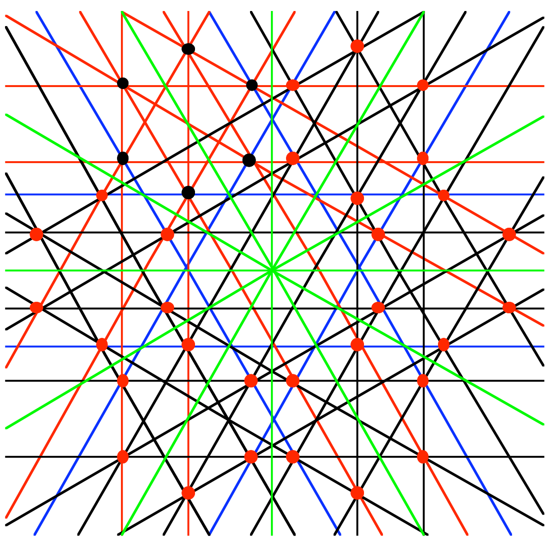
Figure 4.5.10 shows a floral [5,4]-configuration (120_5, 150_4), while Figure 4.5.11 has a floral [3,4]-configuration (72_3, 54_4), both obtained using construction ($\mathcal{M}_1$). A floral configuration (72_4) obtained by the same method is shown in Figure 4.5.12. Figure 4.5.13 shows a (98_4) floral configuration in which the points of the protofloret are at vertices of a regular heptagon, and the lines are diagonals of that heptagon. In contrast, the protofloret in Figure 4.5.14 is a (6_4, 12_2) configuration without any symmetry, used to construct by method ($\mathcal{M}_1$) a floral (108_4) configuration with d_9 symmetry.

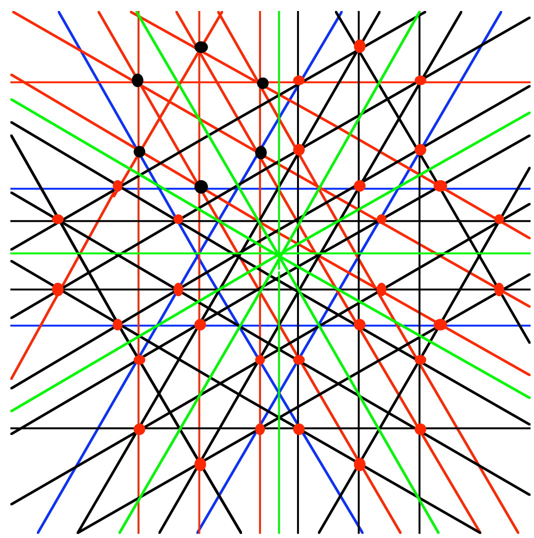
Construction method ($\mathcal{M}_2$) is illustrated by Figure 4.5.15 that shows a floral (150_3, 75_6) configuration with d_5 symmetry, in which the protofloret is a (15_3) configuration with d_5 symmetry; analogs of this configurations are the only example found so far of floral configurations in which each line is incident with six points. Construction method ($\mathcal{M}_2$) also yields the (72_4) configuration in Figure 4.5.16, in which the points of the protofloret are at the vertices of an isogonal dodecagon; the shape of the protofloret is variable.

Another method of constructing floral configurations starts with a floral [q,k]-configuration F_2, with protofloret F_1. Using F_1 as protofloret we can construct a **3-strata** floral configuration F_3. With this terminology, F_2 would be a 2-strata configuration, and F_1 a 1-stratum configuration. This construction method is illustrated in Figure 4.5.17. It shows a 3-strata floral (4,8)-configuration (250_4,125_8), in which the protofloret F_1 has points at the vertices of a regular pentagon. The second stratum is a (25_4) floral configuration obtained by the ($\mathcal{M}_3$) method; with it as protofloret the complete configuration is obtained by the ($\mathcal{M}_1$) construction.



(a)





(b) (c)

Figure 4.5.8. Floral configurations (36_4). The protofloret in (a) has points at the vertices of a regular hexagon, and symmetry group d_2; the configuration is obtained by the ($\mathcal{M}_2$) method and has symmetry group d_6. The other two configurations are obtained from it by the ($\mathcal{M}_4$) construction, which involves changing the size of the protofloret, and deleting different lines from the original.

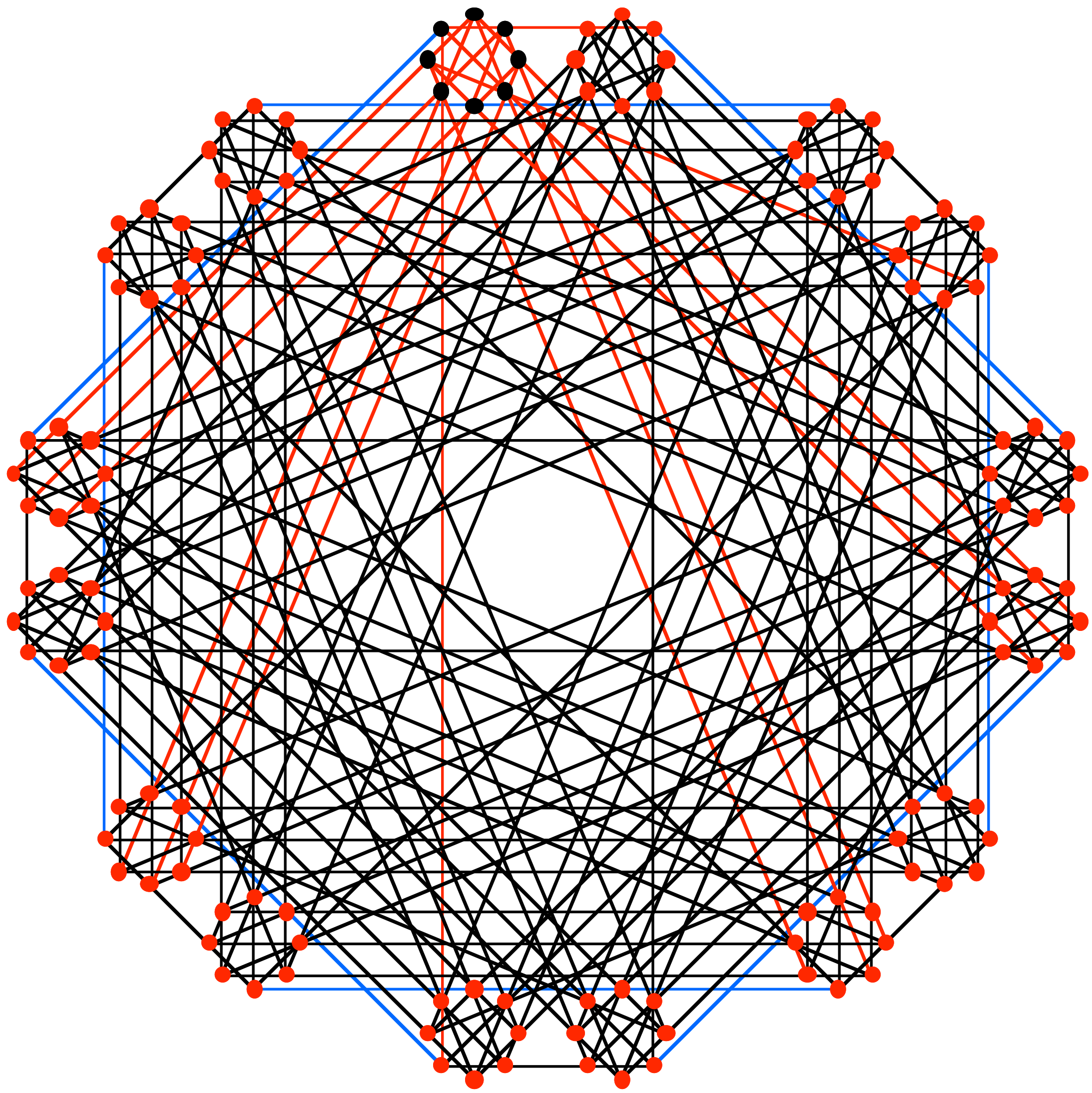


Figure 4.5.9. A floral configuration (128_4) with 16 florets and symmetry group d_8, obtained by construction ($\mathcal{M}_4$) from a (128_4) configuration reached through the ($\mathcal{M}_1$) method. The special lines are again shown in blue, but the mirrors are not shown.

Clearly, this procedure could be repeated to 4-strataconfigurations, and so on; however, the diagrams become for to crowded for visual comprehension.

Here is a brief comparison of our floral configurations with the presentation in [B12]. The main and fundamental difference is that in [B12] the protoflorets are considered as only sets of points that are restricted to coincide with either the vertices of a

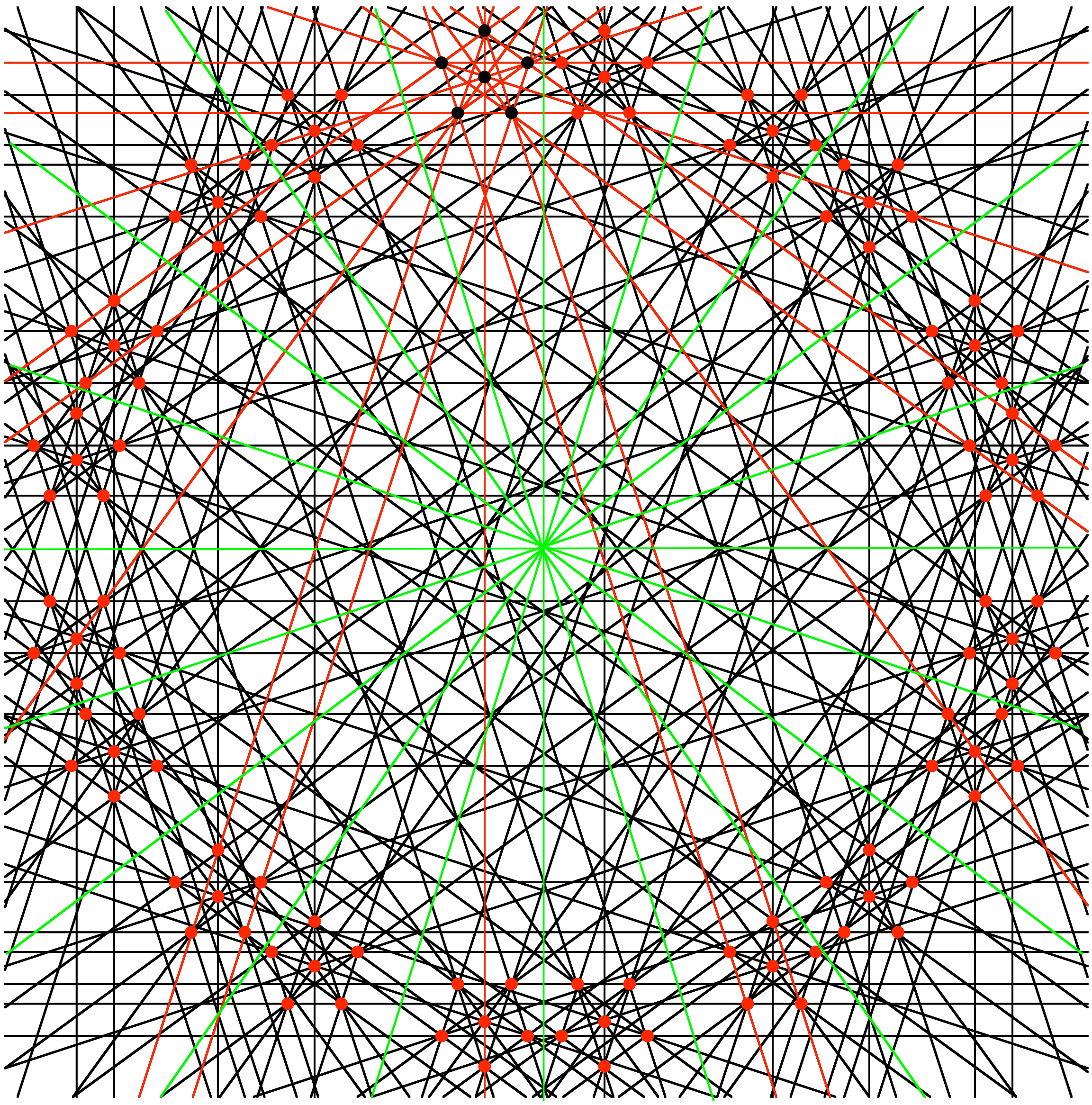


Figure 4.5.10. A floral [5,4]-configuration (120_5, 150_4), constructed by method ($\mathcal{M}_1$). The protofloret is a (6_5, 15_2) configuration consisting of the vertices of a regular pentagon and its center, and all the lines determined by these six points. The protofloret has symmetry group d_5, the configuration has d_{10}.

regular polygon, or the vertices of an isogonal but not regular polygon; a further restriction related the symmetries of the protofloret to those of the configuration. The main exposition in [B12] is restricted to 4-configurations, with more general types mentioned

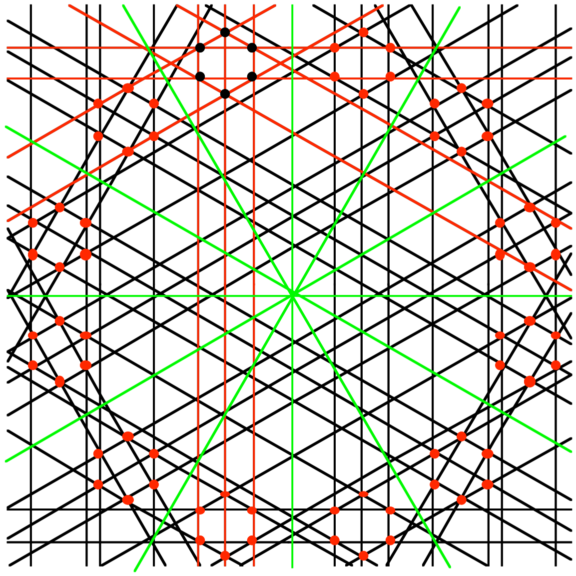


Figure 4.5.11. A floral [3,4]-configuration (72_3, 54_4) obtained using construction ($\mathcal{M}_1$), and with symmetry group d_6. The protofloret is a (6_3,9_2) configuration, with symmetry group d_2; there are 12 florets. The color conventions are the same as in Figure 4.5.3.

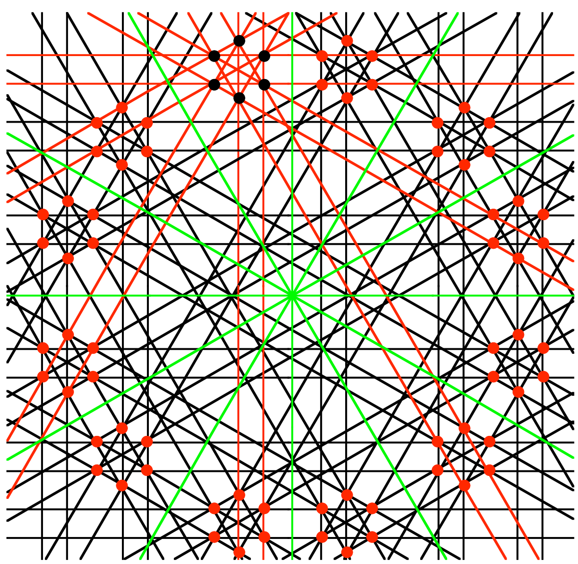


Figure 4.5.12. A floral configuration (72_4) obtained by method ($\mathcal{M}_1$). The protofloret has points at vertices of a regular hexagon, and symmetry group d_3. The configuration has twelve florets and symmetry group d_6.

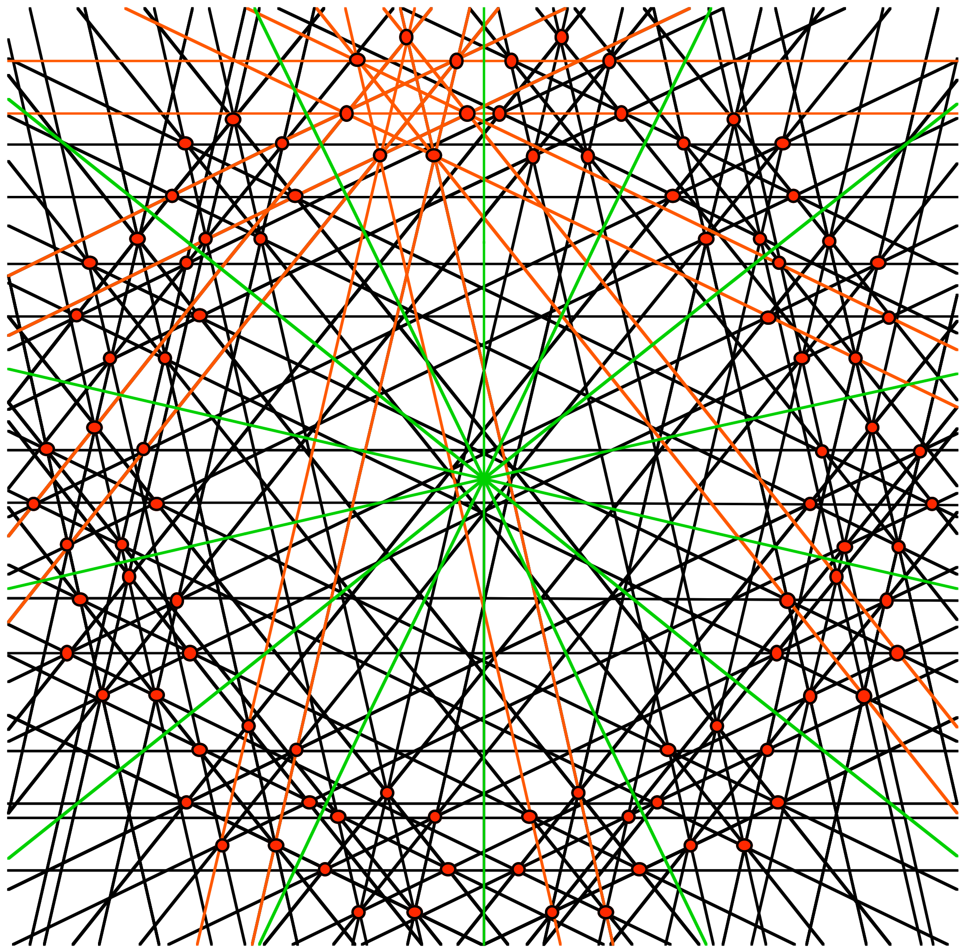


Figure 4.5.13. A floral configuration (98_4) obtained by method ($\mathcal{M}_1$). Both the protofloret and the configuration have symmetry group d_7.

only briefly as "generalized floral configurations". The lines of the protofloret are left to be examined in each case. With these differences in mind, the classification of floral configurations into five varieties can be explained in our terminology as follows. Varieties (A) and (C) of [B12] are obtained by the ($\mathcal{M}_1$) construction, (B) and (D) by the ($\mathcal{M}_2$) method. The protoflorets in (A) and (B) have points coinciding with vertices of isogonal, non-regular polygons, those in (C) and (D) at vertices of regular polygons. Variety (E) consists of configurations obtained by the ($\mathcal{M}_3$) method; they have no degrees of freedom beyond similarities.

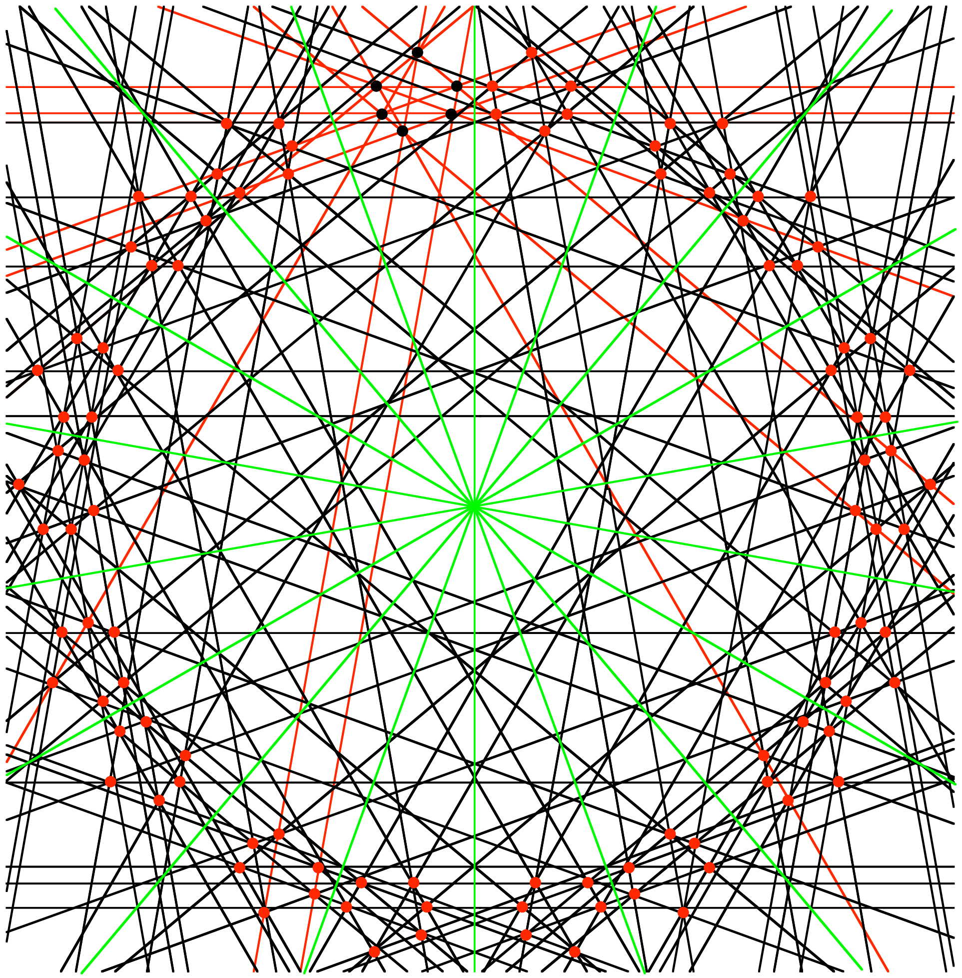


Figure 4.5.14. A floral (108_4) configuration with protofloret a $(6_4, 12_2)$ configuration devoid of any symmetry. The configuration was constructed using method $(\mathcal{M}_1)$ and has 18 florets and symmetry group d_9.

Many of the configurations illustrated in this section do not fit into the classification of [B12], since they do not satisfy the definition of floral configurations adopted there. Those that do are: Figure 4.5.1 is of variety (A), Figure 4.5.16 of variety (B), Figures 4.5.12 and 4.5.13 are of variety (C), Figures 4.5.2, 4.5.5(a) and 4.5.8(a) are of variety (D), and 4.5.5(b,c,d) of variety (E). Figure 4.5.17 is analogous to the generalized floral configuration $(512_4, 256_8)$ in [B12].

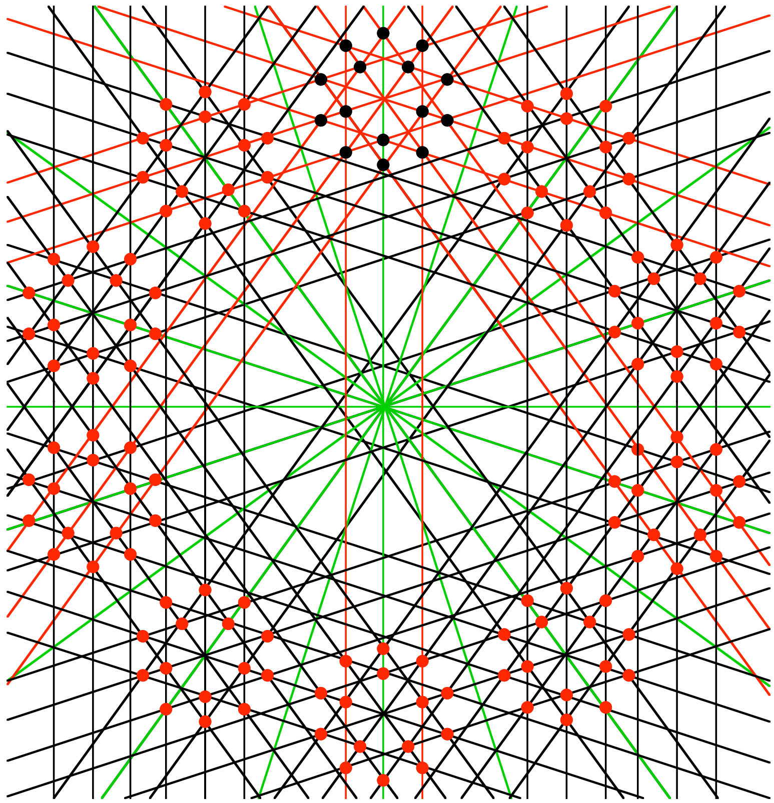


Figure 4.5.15. A (150_3, 75_6) floral configuration with d_{10} symmetry, resulting from the ($\mathcal{M}_2$) construction using as protofloret a (15_3) configuration with d_5 symmetry.

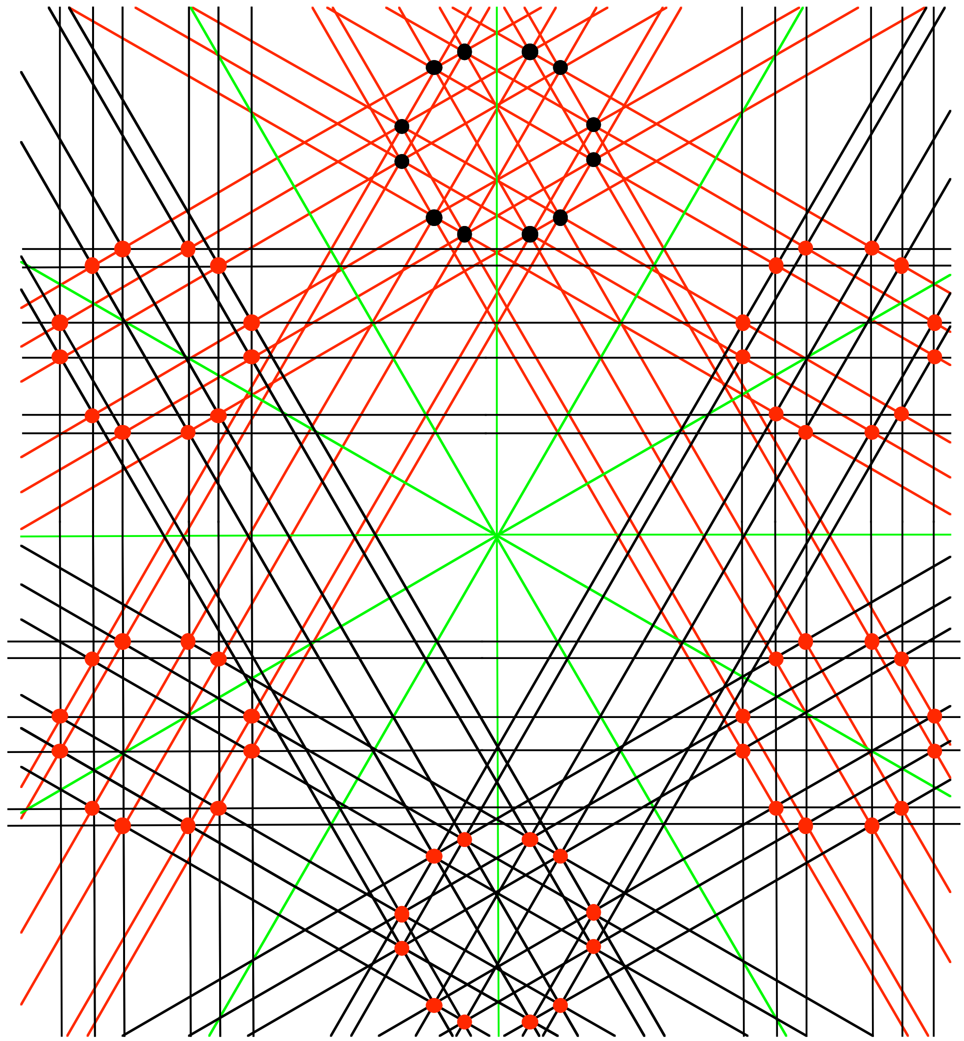


Figure 4.5.16. A floral configuration (72_4) obtained by method ($\mathcal{M}_2$). The points of the protofloret are at the vertices of an isogonal 12-gon, and the protofloret has symmetry group d_2.

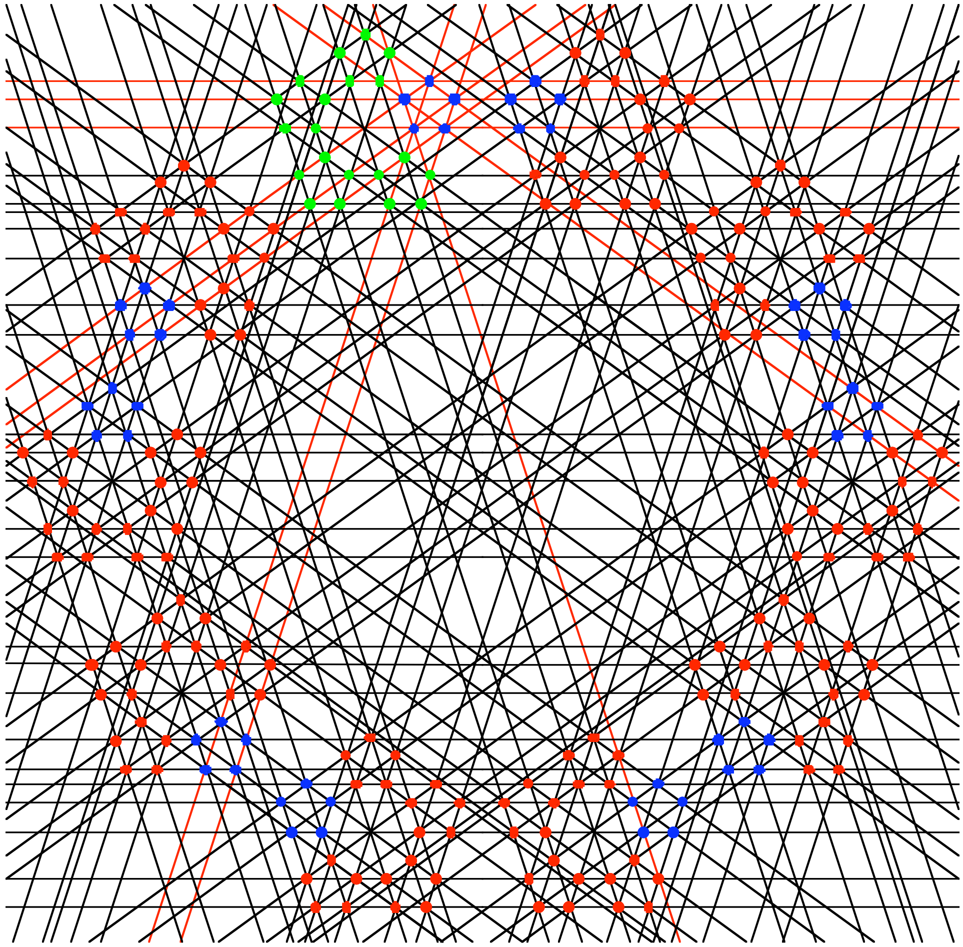


Figure 4.5.17. A floral [4,8]-configuration $(250_4,125_8)$ with three strata. The protofloret F_0 of the first stratum has points at the vertices of a regular pentagon. Using method $(\mathcal{M}_2)$, five copies of the protofloret form a second stratum F_1, which is a floral configuration (25_4). By method $(\mathcal{M}_1)$, ten copies of F_1 form the third stratum F_2. The protoflorets of the first two strata have symmetry group d_5, as does the complete configuration. The protofloret of the first stratum in each second stratum floret is shown in blue, and one second stratum protofloret is shown in green (and blue). The lines incident with one floret of the first stratum are shown in red.

We conclude with a brief description of chiral floral configurations. We have already encountered a wide class of these, when investigating the chiral astral 3-configurations in Section 2. In Figure 4.5.18 we show two examples of such configurations. Many additional examples can be found in Sections 2.7 (a protofloret consists of two points and two lines) and 2.9 among the multiastral 3-configurations.

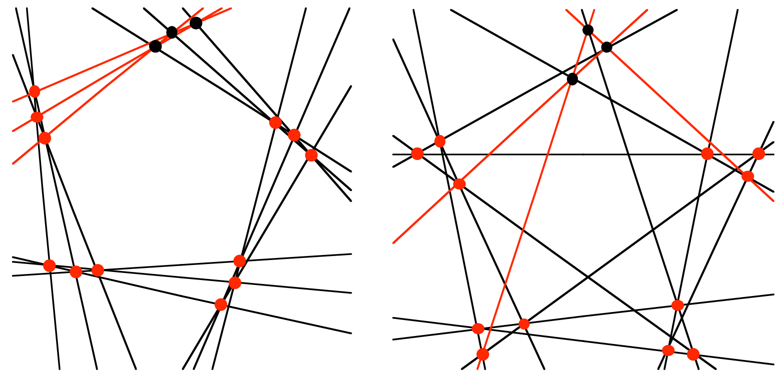


Figure 4.5.18. Two 3-astral configurations (15_3) that are also chiral floral configurations.

A different example of a chiral floral 4-configuration was first presented in [B12], it is reproduced in Figure 4.5.20b. A general method of generating such floral configurations is based on the $(\mathcal{M}_1)$ construction, and illustrated in a simple case in Figure 4.5.19. The crucial step is the replacing of one-half of the florets in the dihedral configuration by their mirror images.

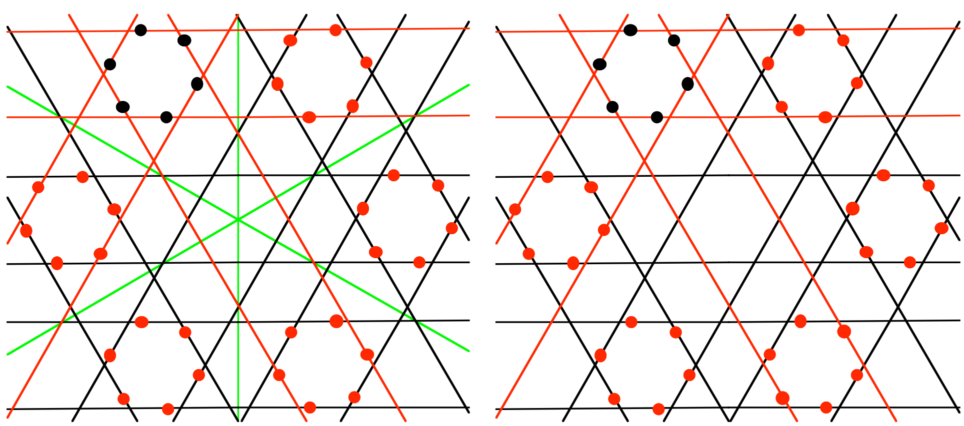


Figure 4.5.19. (a) An application of the $(\mathcal{M}_1)$ construction to a (6_1) configuration (that is, a protofloret consisting of six points and six lines). (b) Replacing one half of the florets by their mirror images yields a chiral floral configuration.

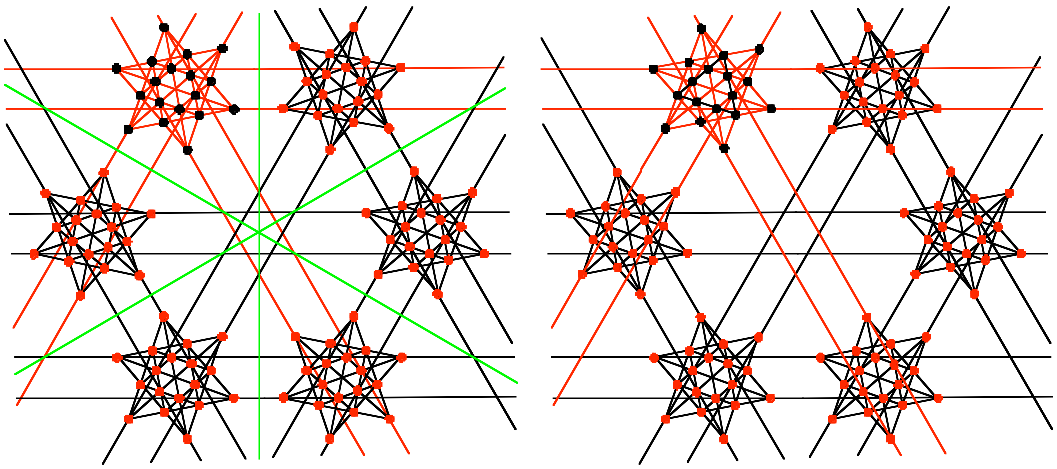


Figure 4.5.20. (a) An adaptation of the ($\mathcal{M}_1$) construction to the case in which not all lines in a protofloret are perpendicular to one of the mirrors. (b) Replacing half of the florets by their mirror images yield a chiral floral configuration.

Exercises and problems 4.5

1. Construct your own floral configurations, using each of the four methods ($\mathcal{M}_i$) .

2. Using the Martinetti "module" shown in Figure 2.4.2, show that a chiral floral (geometric) configuration (n_3) can be constructed for every $n = 10m$, $m \geq 3$.

4.6 TOPOLOGICAL CONFIGURATIONS

Studies of topological configurations have begun only in the very recent past. While in many ways analogous to geometric configurations, there are significant differences that deserve to be investigated in more detail. Here I will try to present the material that is available at this time.

The distinction between geometric and topological configurations became evident long ago, through Schroeter's proof [S6], [S8] that one of the ten combinatorial configurations (10_3) cannot be geometrically realized; see Section 2.1 for more details. The fact that it *almost* can be realized geometrically (as in Figure 1.2.2, with lines just a bit bent) means that it is topologically realizable. However, neither this, nor the fact that it is not known whether there exist geometrically non-realizable 3-connected (n_3) configurations with $n > 10$ that are topologically realizable, resulted in any consistent effort to find clarification. It took almost forty years after Schroeter's discovery for Levi [L3] to even define the appropriate concepts.

Another rather frustrating aspect of the situation concerning topological 3-configurations comes about through Steinitz's theorem (see Section 2.6). In the case of *topological* 3-configurations unintended incidences pose no problem, and one may formulate the resulting statement as follows:

Theorem 4.6.1. Every connected combinatorial 3-configuration with $n \geq 9$ can be realized by pseudolines if the incidence of an arbitrary point-line pair is disregarded.

Naturally, just as in the case of the Steinitz theorem itself, the unfulfilled incidence can always be restored by allowing a curve of degree at most 2. But there is no guarantee that this curve can be chosen in such a way that we obtain a topological configuration. As we have seen in Section 2.1, for $n = 7$ or 8 this is, in fact, impossible and there is no topological realization of these configurations.

A separate question is whether in certain families of 3-configurations (such as astral, or 3-astral, or others) there exist topological configurations that cannot be realized by geometric ones of the same character. An affirmative answer to one of these questions arises from the examples in Section 2.7 (in particular, see Figure 2.7.6). However, the full extent of such situations for connected astral 3-configurations has not been determined. More precisely, in Figure 4.6.1 we show four different astral 3-configurations of pseudolines which arise from unintended incidences in geometric astral configurations — all four resulting in the same astral 4-configuration (24_4).

A different situation happens with the astral 3-configuration 12#(5,1;3). Its drawing does not produce either the intended (24_3), nor a (24_4). Instead, the resulting family of points and lines has some points on three lines and some on four, while some line are incident with just three points and some with 4. This is illustrated in Figure 4.6.2(a). Again it is possible to avoid unintended incidences by replacing one orbit of lines by pseudolines, as indicated in Figure 4.6.2(b).

In all these cases it is not known whether actual geometric realizations of the 3-configurations can be obtained if one does not impose symmetry restrictions.

Concerning topological 4-configurations, we have already discussed in Section 3.2 the non-existence of topological (n_4) configurations for $n \leq 16$ and the fact that for every $n \geq 17$ there exist topological (n_4) configurations. Very recently, L. Berman [B4], determined the conditions for the existence of astral (that is, 2-astral) configurations of pseudolines with dihedral group of symmetries. The main result of [B9] is the following:

Theorem 4.6.2. Astral topological configurations (n_4) exist if and only if n is even and $n \geq 22$.

For the existence part of the proof it is sufficient to provide examples. An astral (22_4) configuration of pseudolines was first shown in [G50], and has been reproduced in several other publications; see Figure 4.6.3. Applying the notation we used in Sections 3.5 and 3.6 to topological configurations, this is 11#(5,4;1,4). It can be used as a template for all even $n = 2m \geq 22$: For each $m \geq 11$, the symbol m#(5,4;1,4) represents such a configuration. An example (with m = 17) is provided in Figure 4.6.4. To establish the inequality for n, it is necessary to first notice that due to the requirements for topological

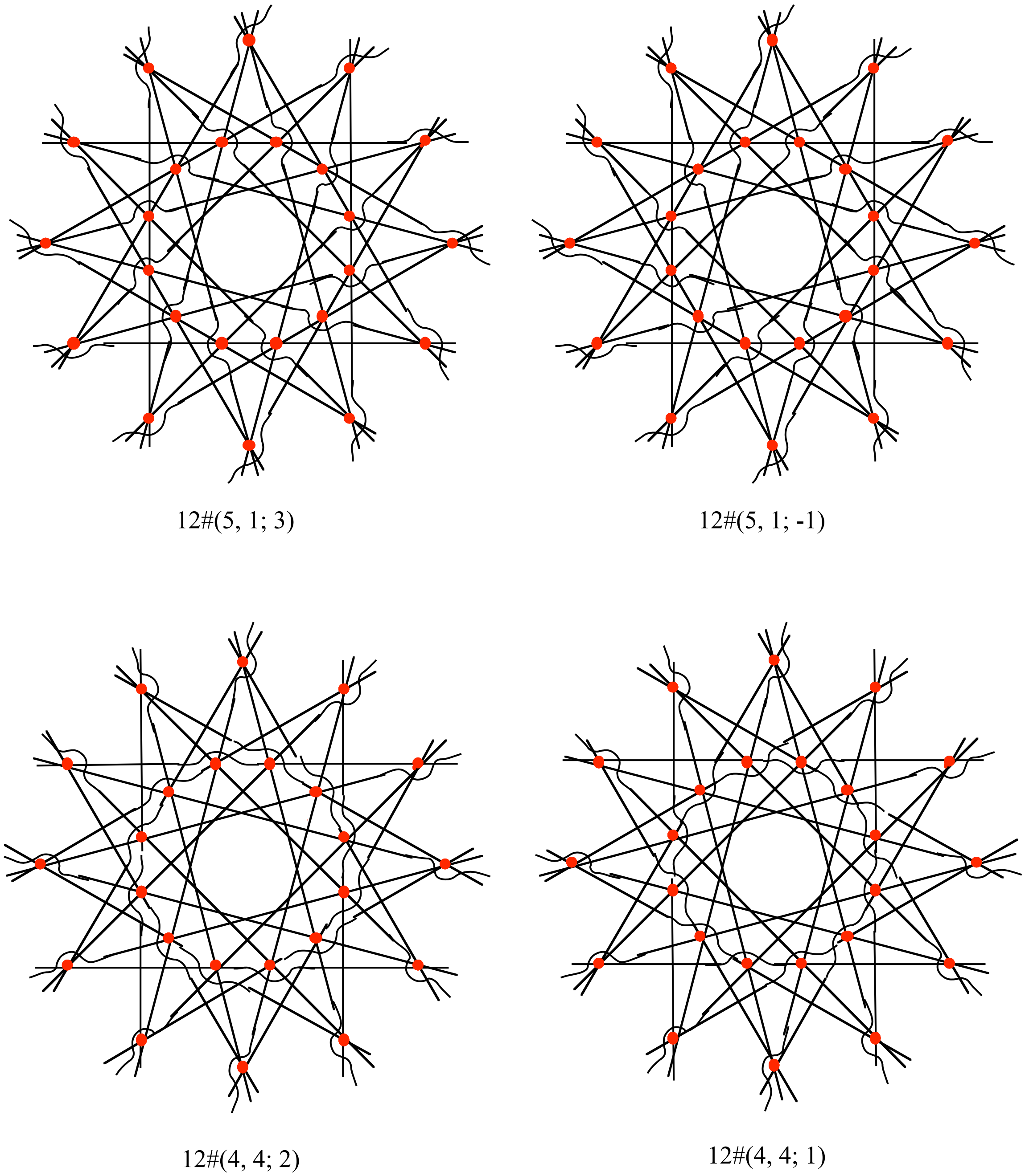
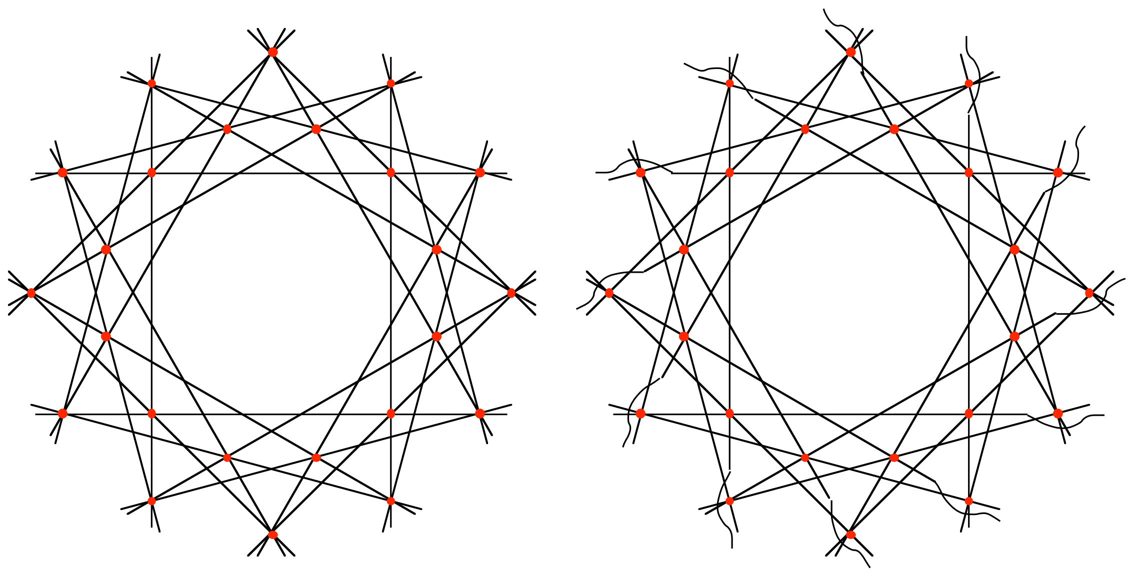


Figure 4.6.1. Four instances where an astral geometric 3-configuration (12_3) leads to the astral 4-configuration (24_4). The pseudolines can avoid the unintended incidences.

astral configurations, one can assume the configuration to be connected, have its points coincide with the vertices of two concentric regular m-gons, and have the concept of "span" of diagonals available — just as for geometric configurations. Then it is easy to verify that the shorter span must be at least 4, hence the larger span at least 5, and therefore m greater than twice 5. (This is an abbreviated version of the detailed arguments in [B3].) ♦



(a) (b)

Figure 4.6.2. A drawing (a) of the astral 3-configuration 12#(5,1;3) produces no geometric configuration, but can be modified to a topological configuration (b).

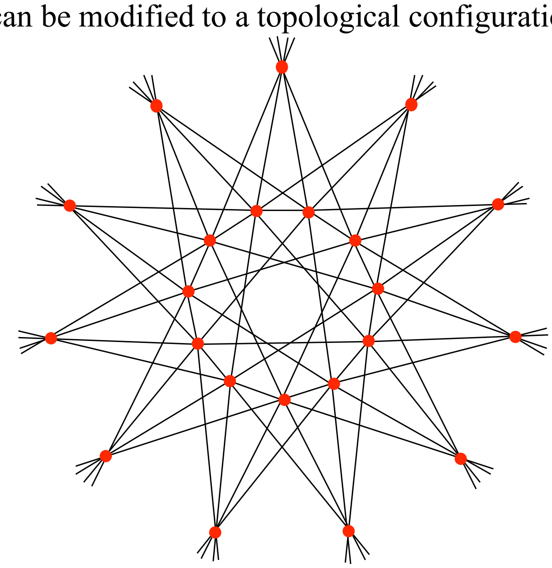


Figure 4.6.3. A topological astral configuration (22_4), that can be described as 11(5,4;1,4).

A more detailed description of astral topological 4-configurations is given in [B9] as well. It concentrates on those with dihedral symmetry. With a slight modification of the notation in [B9], we may summarize the results as follows. Using the symbol m#(b,c;d,e) in the same meaning as explained in Sections 3.5 and 3.6, we note that:

- The configuration points of the inner orbit can be situated on a circle of a radius that can vary between certain limits;
- The points of the inner orbit are either aligned with those of the other orbit (Type 1), or else situated at positions that enclose with them angles that are odd multiples of π/m (Type 2).
- m#(b,c;d,e) and m#(d,e;b,c) are equivalent; moreover $c \neq d$ and $b \neq e$; we conventionally assume that $b < e$;
- It follows that $c < b$ and $d < e$, and $b - c > e - d$;
- For Type 1 configurations we have $b - c \equiv e - d \equiv 0 \bmod 2$, and

  For Type 2 configurations we have $b - c \equiv e - d \equiv 1 \bmod 2$.

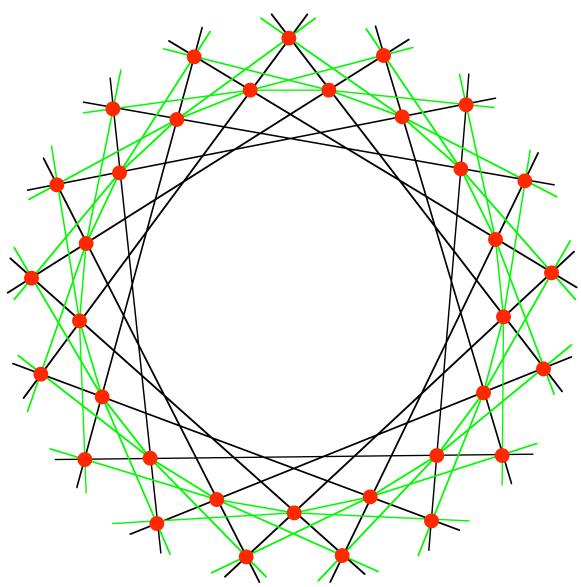
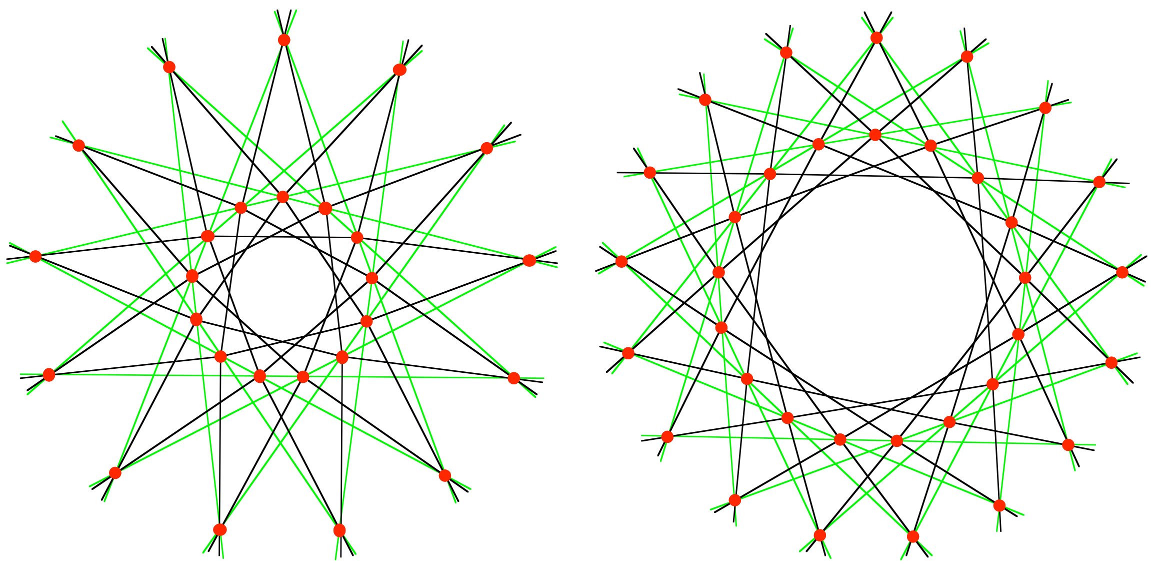


Figure 4.6.4. An astral topological configuration (34_4) of Type 2. It can be specified as 17#(5,4;1,4).

While these conditions impose many restraints on astral topological configurations, it is also clear that most of them cannot be "straightened" or "stretched" into geometric astral configurations. The reason in that the geometric m#(b,c;d,e) configurations exist only if m is a multiple of 6, while no such restriction holds in the topological case.

The smallest topological astral configurations is 11#(4,1;4,5) shown in Figure 4.6.3 above. It is the only astral configuration (22_4), and is of Type 2. The smallest astral topological configuration of Type 1 is 13#(5,1;4,6), shown together with 17#(5,1;4,6) in Figure 4.6.5.

Even when m is divisible by 6 there are topological astral configurations m#(b,c;d,e) that are not stretchable. The smallest such configuration is 18#(6,1;5,8), shown in Figure 4.6.6.



(a) (b)

Figure 4.6.5. Two astral topological configuration of Type 1. (a) The configuration 13#(5,1;4,6), the smallest such configuration. (b) Another (34_4) topological astral configuration, that can be specified as 17#(5,1;4,6).

Berman's paper [B9] contains a number of other results that we cannot get into here. It should only be mentioned that there are examples of essentially chiral configurations, that is, configurations that are astral under a cyclic symmetry group but are not even isomorphic to an astral configuration with mirror symmetries. Such configurations do not exist for geometric astral configurations. An example of an essentially chiral astral configuration is shown in Figure 4.6.7. A complete description of such configurations is still lacking, as is also any treatment of k-astral topological configurations for $k \geq 3$.

An interesting conjecture in [B9] can be formulated as follows:

Conjecture 4.6.1. If the outer orbit of points in a astral topological configuration m#(b,c;d,e) with dihedral symmetry is on a circle of radius 1, then the inner orbit is on a circle of radius r, where

$$0 < r < \cos((b-c-1)\pi/m)/\cos(\pi/m).$$

For a study of simplicial arrangements of pseudolines see [B8].

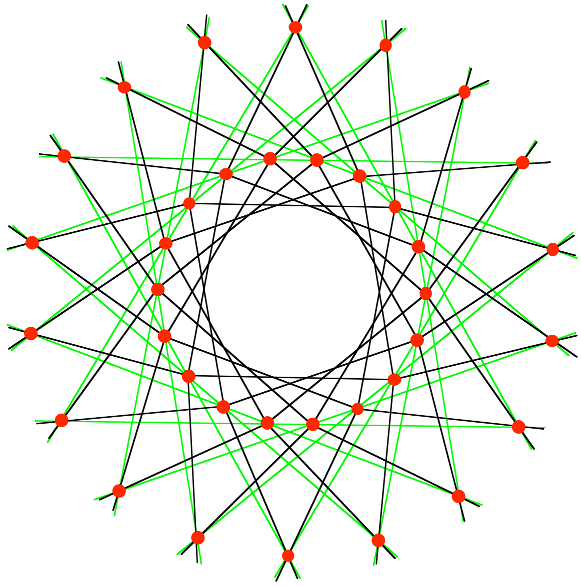


Figure 4.6.6. The astral topological configuration 18#(6,1;5,8), the smallest configuration with m divisible by 6 that is not a geometric astral configuration.

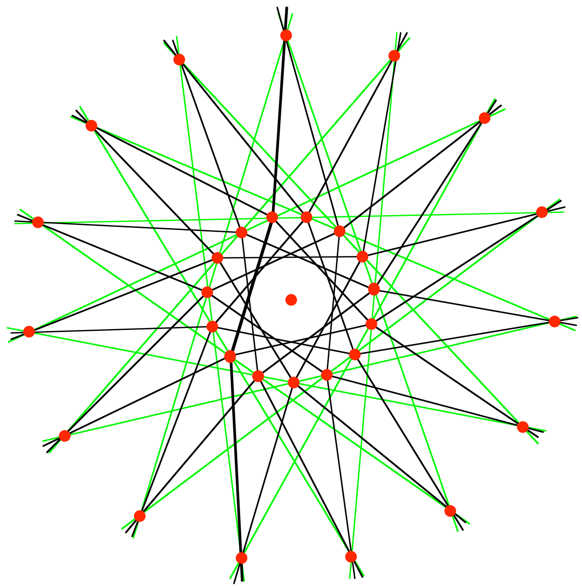


Figure 4.6.7. A chiral astral configuration (30_4) that is not isomorphic to any astral configuration that admits mirror symmetries. One of the pseudolines is drawn by heavy segments. It can be labeled 15#(6,1;5,7), and the fact that $6 - 1 \equiv 1 \bmod 2$ while $7 - 5 \equiv 0 \bmod 2$ shows that it cannot be dihedral of either Type 1 or Type 2.

Exercises and problems 4.6

1. Justify the claims that the configurations in Figures 4.6.5a and 4.6.6 are the smallest of their kind.

2. What is the smallest topological 5-configuration you can find?

3. How many distinct astral topological configurations (26_4) and (30_4) can you find?

4. What are the smallest topological 3-astral 4-configurations you can find?

5. Generalize the statement (in the proof of 4.6.2) that the symbol m#(5,4;1,4) describes a valid topological astral 4-configuration for each $m \geq 11$. What about analogous statements for 3-astral configurations?

4.7 UNCONVENTIONAL CONFIGURATIONS

In this section we shall consider several families of objects that we shall call "configurations" even though they do not fit the definition of that word accepted in all the other sections of this book.

The first of these families are "configurations of points and circles". Some examples are shown in Figures 4.7.1 and 4.7.2. In analogy to configurations of points and lines we may denote them by a symbol such as (p_q, n_k), where p, n are the numbers of points and of circles, and q, k are the number of circles incident with each point and the number of points incident with each circle; in case the numbers are equal, we use the notation (n_k). Hence the three configurations shown are (4_3), $(8_3,6_4)$, and (10_4).

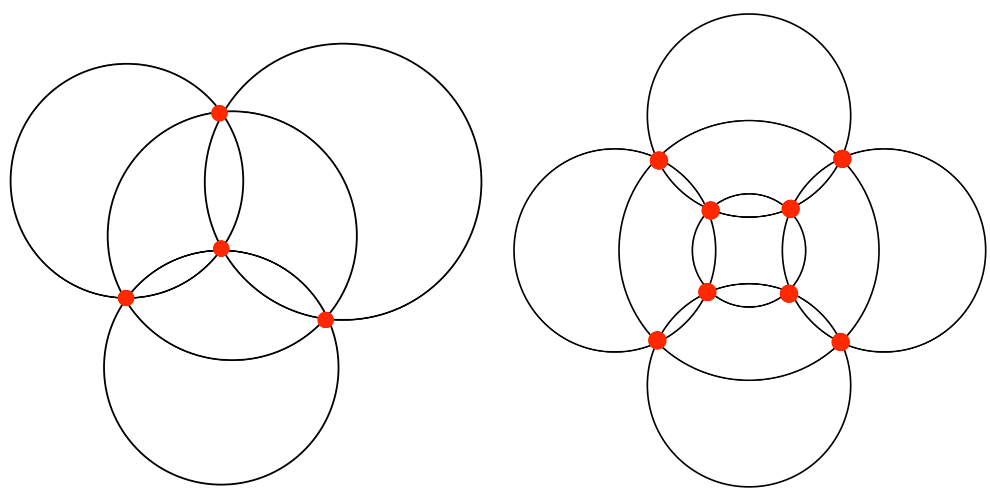
Several aspects of configurations of points and circles deserve notice.

First, such configurations are generalizations of configurations of points and lines in a very direct way: Every configurations of points and lines in the projective plane can be shown as a configuration of antipodal pairs of points and great circles in the model of the projective plane on the sphere; a stereographic projection then maps this into a configuration of points and circles in the plane. However, these are only very special cases of such configurations — none of those in Figures 4.7.1 and 4.7.2 is of this kind.

Second, in all but name, configurations of points and circles made their appearance before configurations of points and lines. For example, the configuration in Figure 4.7.1b is an illustration of a theorem of A. Miquel [M19a] dating to 1838, asserting that if four pairwise intersections of four circles are concyclic, the other four intersections of the same pairs are concyclic as well. This is one of several results of Miquel, some of which have been greatly generalized by many writers, starting with Clifford [C2**] in 1871 and de Longchamp [L4*] in 1877. One of the achievements are the so-called "chains of theorems" bearing the names of Clifford and de Longchamps. The former establishes the existence of configurations of points and circles $((2^{n-1})_n)$ for all $n \geq 1$. The cases n = 1 or 2 are trivial, and n = 3 is shown in Figure 4.7.1a. For more recent works on this topic and its relatives see, for example, Ziegenbein [Z8*], Rigby [R3*], Longuet-Higgins [L5*], Longuet-Higgins and Perry [L5**], and references given there to other works.

Third and last — why is there no greater activity regarding these configurations? I venture to guess that the preoccupation with just a few specific results (such as the "chains of theorems") tended to discourage more general inquiries. There are various subclasses of circle configurations that may well be worth investigating: Are pairs of circles required to intersect twice, are touching circles allowed, can disjoint circles appear, are straight lines admitted, does one wish to consider symmetries in the inversive plane, – – the choices and possibilities are very wide and almost entirely unexplored. (The inversive plane seems an appropriate setting for many of the considerations of symmetries of configurations of points and circles; see, for example Coxeter [C7], Eves [E2], Yaglom [Y1].)

The configuration (10_4) in Figure 4.7.2 is an example of configurations $((2n)_n)$ that exist for all $n \geq 5$ and exhibit remarkable symmetry in the inversive plane. The (10_4) configuration has a single orbit of points and a single orbit of circles under inversive transformations. I do not know what other configurations are as symmetric, but probably there are many additional ones.



(a) (b)

Figure 4.7.1. Configurations of points and circles. (a) A (4_3) configuration. (b) A $(8_3,6_4)$ configuration.

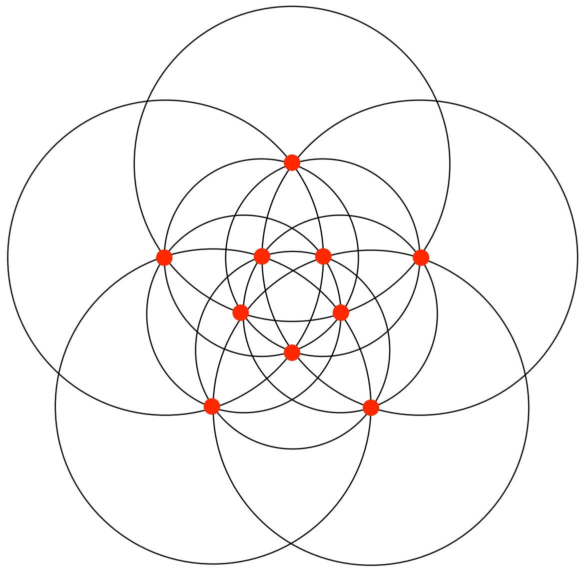


Figure 4.7.2. A (10_4) configuration of points and circles.

The second family of "unconventional configurations" is illustrated by the examples in Figures 4.7.3 and 4.7.4. The objects in this family are the traditional points and

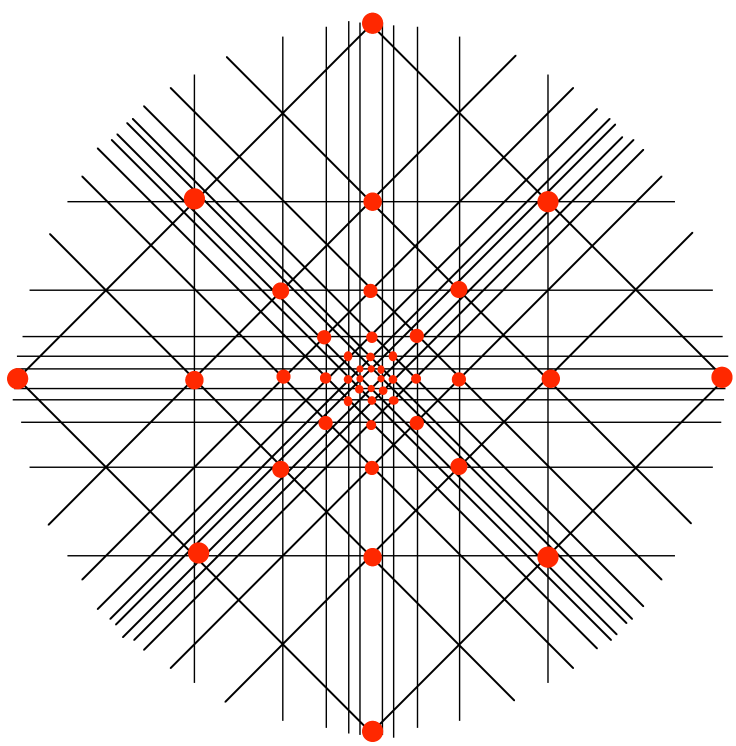


Figure 4.7.3. An infinite 3-configuration with 4-fold dihedral symmetry and single transitivity classes of points and of lines under the group of similarity transformations.

lines of the Euclidean plane, and the configurations satisfy all the conditions assumed throughout the book — except the requirement that there are only finite numbers of points and of lines. More precisely, we are now looking at infinite families of points and lines such, for some finite k, each point is incident with k lines, each line with k points, and the family is discrete in the sense that every point [line] has a neighborhood that contains no other point [line] of the family. We shall call a family of this kind an *infinite k-configuration.*

While many different kinds of infinite k-configurations (or of analogously defined infinite [q,k]-configurations) can be contemplated, the two examples we show have few orbits of points and of lines under similarity transformations.

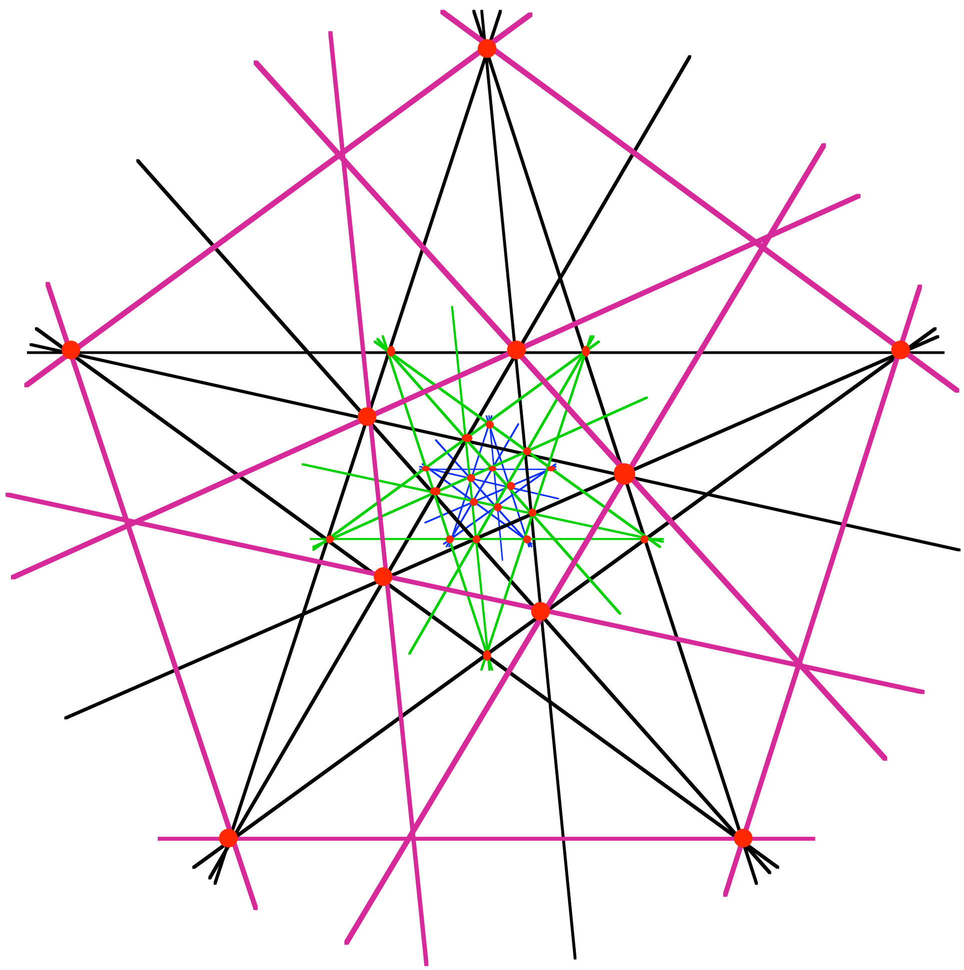


Figure 4.7.4. An infinite 5-configuration obtained by repeated inscribing/circumscribing of copies of the astral configuration 5#(2,2;1). The copies are distinguished by colors.

These configurations can be interpreted as an iterative analogue of the (4m) construction we considered in Section 3.3. The infinite 3-configuration in Figure 4.7.3 arises by repeatedly inscribing (4_2) configurations in each other. A construction of this type can be performed starting with any regular m-lateral, leading to an infinite 3-configuration with m-fold dihedral (or cyclic — with a suitable placement of the m-laterals) symmetry. Such configurations can therefore be considered as infinite analogues of the families of inscribed/circumscribed multilaterals we shall consider in Section 5.3.

The infinite 5-configuration in Figure 4.7.4 arises in the same way from repeated inscription/circumscription of copies of the astral configuration (10_3) shown in Figures 1.3.3 and 1.5.4. It is the only example of this kind that I found in the literature; it is explicitly mentioned in van de Craats' paper [V1]. It is clear that this type of construction can be carried out with other astral 3-configurations.

The third (and last) family of unconventional configurations is illustrated by the remaining figures of this section. In these configurations the role of points and lines are

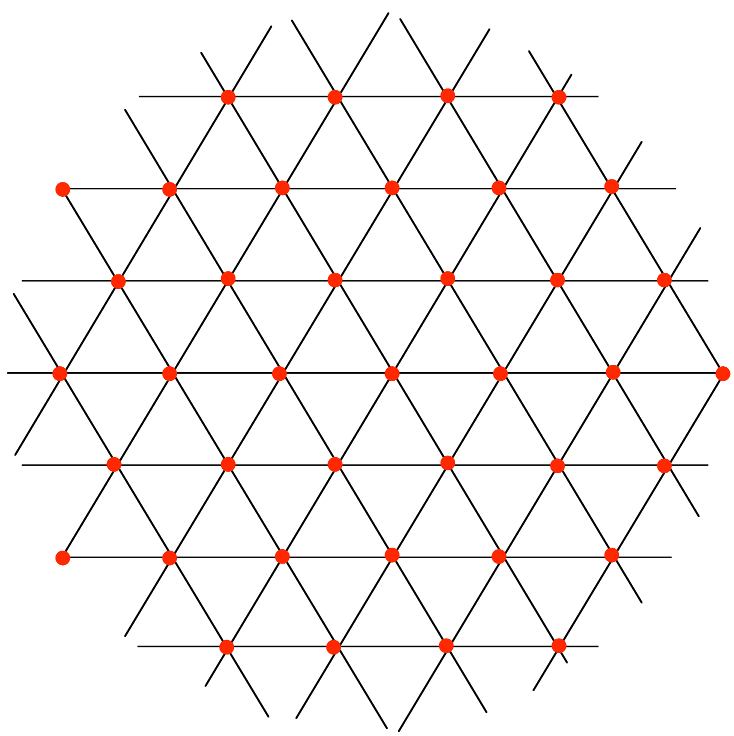


Figure 3.7.5. An infinite [3]-configuration with a single orbit of points and of lines under isometric symmetries of the plane.

different, although there are infinitely many of both: Each point is on precisely k lines for some finite k, but each line contains infinitely many points. We call such configurations *infinite [k]-configurations.* To avoid complications we also require that there be no accumulation points or lines. It is again convenient to consider configurations with a high degree of symmetry under the group of isometric maps of the plane. It is easy to verify that infinite [k]-configurations exist for all $k \geq 1$.

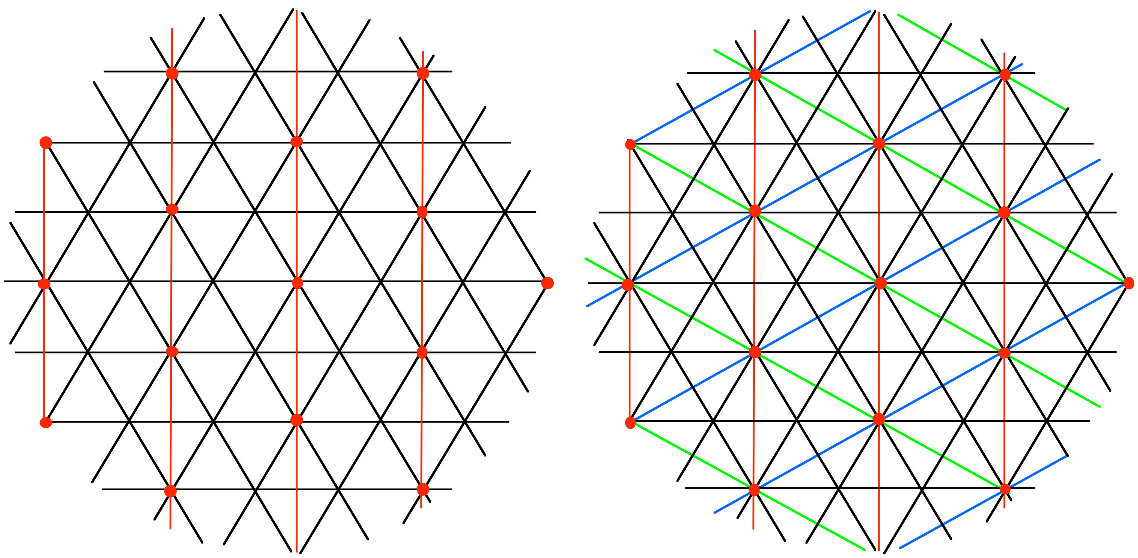


Figure 4.7.6. Examples of infinite [4]- and [6]-configurations.

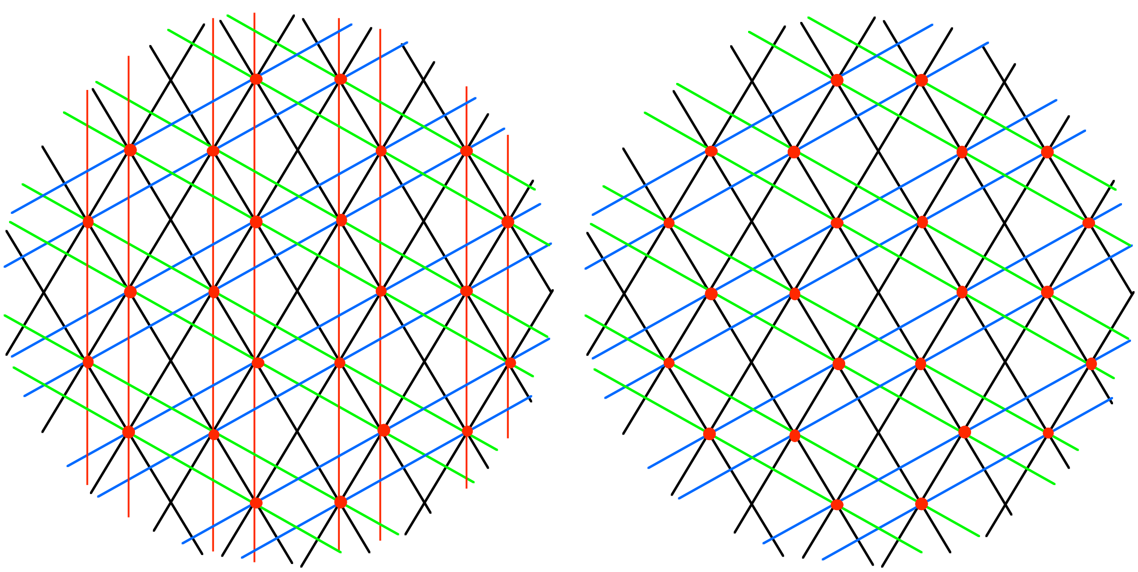


Figure 4.7.7. Examples of infinite [5]- and [4]-configurations.

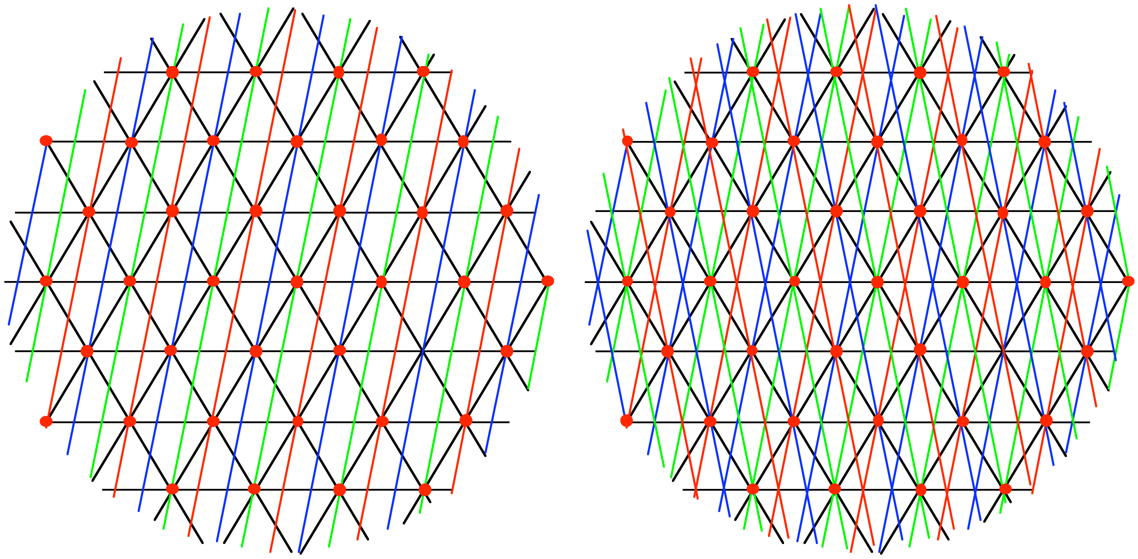


Figure 4.7.8. Additional example of infinite [4]- and [5]-configurations.

Exercises and problems.

1. Construct the (12_4) and (14_4) analogues of the configuration of points and circles in Figure 4.7.2.

2. Decide whether there are configurations (n_k) of points and circles for arbitrarily large k.

3. Modify the construction in Figure 4.7.3 to obtain a *chiral* infinite 3-configuration (that is, with cyclic symmetry group) and with a single orbit of points and one of lines.

4. Justify the claim that the van de Craats construction fan be carried out for other astral 3-configurations.

5. Is there any infinite k-configuration such that its points have no accumulation point?

6. Find infinite [k]-configurations that differ in some essential aspect from the ones shown here.

7. Construct infinite configurations of points and circles that share some features with the infinite configurations of points and lines described above.

4.8 OPEN PROBLEMS

There is so little known about the various kinds of configurations described in this section that it seems presumptuous to propose specific problems about any of them. But let us try to present a few that would seem capable of being solved within our lifetime.

1. Are any cyclic 5-configurations geometrically realizable? Any cyclic k-configurations for $k \geq 6$?

2. Develop a theory of k-astral 5-configurations for some $k \geq 4$.

3. Determine whether there exist k-configurations (n_k) for all sufficiently large n, that is for $n \geq N(k)$, where N(k) depends on k only. Similar question for unbalanced configurations, taking into account the divisibility properties resulting from the symmetry of the incidence relation.

4. Clarify the relation between the configurations $((4r)_3, (3r)_4)$ for $r \geq 5$ and cubic curves in the real plane. Can such curves contain all vertices of configurations of this kind for all r ? Are all such configurations realizable with all vertices on suitable cubic curves? If not, what are the smallest ones that are not realizable in that manner?

5. Consider geometric configurations of points and lines realized in 3-dimensional Euclidean or extended Euclidean space and spanning it. Find some that are astral in that setting, but have no astral realization in the plane.

6. There seems to be no information whatsoever available concerning k-astral 4-configurations for $k \geq 3$.

7. Develop some concept and some results on *configurations of curves* — that is, objects that can be described as "topological configurations of points and circles" in the same sense that configurations of pseudolines are "topological configurations of points and lines".

8. Is it possible to use astral 4-configurations to construct infinite k-configurations with an accumulation point, for some k ?