CHAPTER 3. 4-CONFIGURATIONS.

3.0 OVERVIEW

As explained in Section 3.1, the first publications dealing with 4-configurations appeared before the end of the nineteenth century, but not much development happened till relatively recently.

In Section 3.2 we present the general results concerning the existence of topological and geometric 4-configurations. The difference between the present case and that of 3-configurations is quite striking – there still are gaps in the numbers n for which we know that an (n_4) geometric configuration exists.

The various methods of construction of reasonably sized (n_4) geometric configurations — all less than twenty years old — are detailed in Section 3.3. These constructions are then applied in Section 3.4 to determine the values of n for which it is known that a geometric 4-configuration (n_4) exist. Although the development of new methods has made the construction of visually understandable 4-configurations possible in many cases, for some of the small numbers there still are only unattractive diagrams, or no known configurations at all.

Section 3.5 sets up the framework for the study of the k-astral 4-configurations; these are the configurations with a very high degree of geometric symmetry.

Based on that, in Section 3.6 we present one of the few complete results about 4-configurations — the complete enumeration of the 2-astral 4-configurations. These are configurations in which there are only two orbits of points, and two orbits of lines, under Euclidean symmetries of the configuration. This topic is related to (and depends upon) the investigation of the intersection-points of diagonals in regular polygons, in itself a subject with a classical flavor but with surprising twists in its unfolding.

Section 3.7 is devoted to 3-astral configurations. The presence of three orbits of points and three orbits of lines results in a family of configurations with properties very different from the ones considered in Section 3.6.

Section 3.8 is concerned with the k-astral 4-configurations with $k \geq 4$. There is again a sea-change in properties compared to 2-astral and 3-astral configurations, as well as in our knowledge of the possibilities.

A few problems not mentioned in the earlier sections are presented in Section 3.9.

3.1 COMBINATORIAL 4-CONFIGURATIONS

The history of configurations (n_4) is much shorter than that of configurations (n_3), and more easily told.

The first explicit mention of such configurations seems to be in a paper [K11, p. 440] by Felix Klein in 1879, which deals with quartic curves in the complex plane. He noted that there is a family of 21 points and 21 lines with incidences that make it into a (21_4) configuration, in our terminology — albeit in the complex plane. Although this particular configuration continued to interest mathematicians (various references can be found in Coxeter [C10], and Burnside [B32] discovered it independently), it did not have any noticeable direct influence on the study of (n_4) configurations in general. However, it did play later a significant role in the theory of geometric configurations, which we shall discuss in Section 3.2.

The first slightly more general treatment of such configurations was by Georges Brunel (1856 – 1900) in [B31], a paper that seems to have escaped the attention of all writers on the topic of configurations (n_4) prior to [G46][1]. In an earlier paper [B30] Brunel followed an idea quite popular at that time: a polygon inscribed and circumscribed to itself (with sides understood as lines). Clearly, these are a special class of (combinatorial or geometric) 3-configurations, which we will discuss in Section 5.2. Aware of the need to distinguish between combinatorial and geometric configurations, in [B31] Brunel pursued this idea farther, by considering a "polygon *doubly* inscribed and circumscribed" to itself. In the current terminology we call such polygons "Hamiltonian circuits (or multilaterals)" of the configuration, and we will consider them in more detail in Sections 5.2 to 5.4. Each line of such a doubly self-inscribed and self-circumscribed "polygon" is incident, besides the two points (vertices of the polygon) that define it as a side of the poly-

[1] Biographical data on Brunel, and comments on his work, may be found in [B2] and, in great detail, in [D11]; see also [G29].

gon, with precisely two additional vertices of the polygon. Brunel determines that any combinatorial configuration (n_4) must satisfy $n \geq 13$, and gives two constructions.

In the first, Brunel presents a configuration table (that is actually an orderly configuration table, in the terminology of Section 2.5), and states that while the verification that this indeed determines a combinatorial configuration (35_4) is easy, the graphical representation requires some effort. (Unfortunately, the remarks in [D11, p. LXVIII] concerning the geometric realization of this configurations are, at best, misleading.) From Brunel's statement (especially in view of his later comments concerning the other construction) one may conclude that he had found a geometric realization of this configuration. In fact, this configuration turns out to be isomorphic to the geometric configuration (35_4) mentioned in [G50], communicated to the authors by Ludwig Danzer. (See also [G49].) Although no reasonable diagram of this configuration seems to be available, the configuration can be described easily enough by a construction of the kind used by Cayley and others in similar contexts a century and a half ago. In the case under discussion, start with seven points in general position in real 4-space; consider the 35 2-planes and 35 3-spaces they generate, and intersect this family by a 2-dimensional plane in general position to obtains the required geometric configuration (35_4). The absence of any reasonable geometric symmetry makes this configuration visually unattractive.

Brunel's second construction yields combinatorial configurations (n_4) on which a cyclic group operates transitively. This includes explicitly specified configurations for $13 \leq n \leq 16$. Unfortunately, the results Brunel presents are marred possibly by typos, but also by outright errors. Among the latter, in several cases Brunel lists isomorphic doubly selfinscribed and selfcircumscribed polygons as distinct. For example, in case $n = 13$ Brunel lists cyclic translates of {0,1,4,6} and {0,1,3,9} as the two polygons, although the permutation (0)(1)(2)(3,4)(5)(6,9,8,10,12,7)(11) maps the first polygon onto the second. Moreover, it is rather easy to prove that up to isomorphism, there can be only one such combinatorial configuration; this is completely analogous to the proof (in Section 2.2) that the configuration (7_3) is unique. But even allowing for these shortcomings, we see that Brunel anticipated the corresponding results of Merlin [M8], and even went a bit beyond them. A corrected list would show one cyclic configuration (or polygon) for n = 13 and 14, three for n = 15, and two for n = 16. This coincides with the recent list of cyclic configurations given by Betten and Betten [B13], to which we shall return soon. Brunel also noted that translates of $\{0,1,4,6\}$ yield a configuration (n_4) for all $n \geq 13$; this anticipated by nearly a century a result of Gropp [G8].

Merlin mentions in [M8] that configurations (n_4) have not been investigated systematically, although some isolated ones were discovered by F. Klein [K11], W. Burnside [B32], and others. Like Brunel, he constructs a combinatorial configuration (13_4); moreover, he proves its uniqueness and minimality. He also constructs a configuration (14_4) and proves it is unique. Merlin states that there are exactly **three** distinct configurations (15_4) which, however, are not presented. In fact, he is mistaken. As shown by Betten and Betten [B13], there are **four** different configurations (15_4), three of which are cyclic and coincide with the three doubly selfinscribed and selfcircumscribed polygons of Brunel (who did not comment on the possibility of noncyclic configurations (15_4), or (n_4) in general). In the same context, Merlin makes two additional errors:

(i) He claims that his three configurations (15_4) can be distinguished by the number of vertex-disjoint triangles present in them, which he claims to be 5, 1 and 0, respectively. In fact, all four configurations (15_4) have five such triangles, the maximal possible number.

(ii) He states that his configurations (13_4), (14_4) and (15_4) have orderly configuration tables; this is correct — see Section 2.5 — and has been proved by Steinitz in [S17] for all configurations (n_k). However, Merlin then claims that it follows that there is no Hamiltonian circuit for any of them — which is wrong. Steinitz's orderliness result has no such implications, and cyclic 4-configurations such as Brunel's explicit constructions in [B31] (of which Merlin is unaware) provide counterexamples to Merlin's claim.

By a construction analogous to the one devised by Martinetti (in [M2], see Section 2.3) for configurations (n_3), Merlin shows that for every $n \geq 30$ there are combinatorial configurations (n_4). In fact, it is easy to show that there are such configurations for all $n \geq 13$; for example, as noted by Brunel and mentioned above, for all $n \geq 13$ it is enough to consider cyclic translates of the "line" {0,1,4,6}.

Concerning the number N(n) of distinct combinatorial configurations (n_4), the only known values are those given by Betten and Betten [B13]: the old N(13) = N(14) = 1, and their new results N(15) = 4, N(16) = 19, N(17) = 1972, and N(18) = 971171. These new numbers seem not to have been independently verified, except for the value N(17) = 1972 (see [B29]).

The configurations (13_4) and (14_4) can be obtained as cyclic configurations with generating "line" {0,1,4,6}. The four configurations (15_4) can be characterized as follows: The three cyclic ones are generated by the "lines" {0,1,4,6}, {0,1,5,7} and {0,1,3,7}, given already by Brunel. The other three configurations given by Brunel yield isomorphic configurations (two to the first, and one to the second). Betten and Betten [B13] give other generators for the three cyclic configurations: {0,2,8,12}, {0,1,9,11}, and {0,1,9,13}, respectively; these are shown in [B13] by Levi incidence matrices (see Section 1.4), – but matrices that do not exhibit the *cyclic* character of the configurations. Their fourth configuration (n_4) is clearly illustrated in [B13] by a Levi incidence matrix shown in Figure 3.1.1(a). As it is the only non-cyclic configuration (15_4), it is necessarily selfdual. An incidence matrix exhibiting one of the selfdualities is shown in Figure 3.1.1(b); it is obtained by suitable permutations of the rows and columns of the matrix in Figure 3.1.1(a).

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1				X	X	X	X								
2								X	X	X	X				
3												X	X	X	X
4	X			X				X				X			
5	X				X				X				X		
6	X					X				X				X	
7	X						X				X				X
8		X		X					X					X	
9		X			X			X							X
10		X				X					X		X		
11		X					X			X		X			
12			X	X						X			X		
13			X		X						X	X			
14			X			X			X						X
15			X				X	X						X	

(a)

	1	2	3	5	11	15	4	10	14	6	9	13	7	8	12
1				X			X			X			X		
2					X			X			X			X	
3						X			X			X			X
4	X						X							X	X
11		X						X					X		X
15			X						X				X	X	
5	X			X							X	X			
10		X			X					X		X			
14			X			X				X	X				
6	X							X	X	X					
8		X					X		X		X				
12			X				X	X				X			
7	X				X	X							X		
9		X		X		X								X	
13			X	X	X										X

(b)

Figure 3.1.1. (a) A Levi incidence matrix of the non-cyclic (15_4) configuration constructed by Betten and Betten [B13]. (b) A selfdual incidence matrix of this configuration.

Brunel's generating "lines" of the three cyclic configurations (15_4) given in [B31] have an advantage over the ones given by Betten and Betten [B13], even though they are isomorphic for the (15_4) configurations: Brunel's can serve as generating lines for combinatorial configurations (n_4) for **all** $n \geq 15$.

Concerning the (16_4) combinatorial configurations, it should be noted that the three generating lines of the cyclic (15_4) configurations listed above do serve to generate cyclic (16_4) configurations — but the three resulting configurations are isomorphic. There is one other configuration (16_4), also cyclic, specified in Betten and Betten [B13] by its generating line $\{0, 1, 6, 13\}$; Brunel renders the same configuration, but with a typo; when corrected, its generating line is $\{0,1,3,12\}$, or equivalently, $\{0,1,3,-4\}$. The generating lines $\{0, 1, 6, 13\}$ or $\{0,1,3,12\}$ do not yield a cyclic configuration for all $n > 16$; however, if the generating line is taken in the form $\{0,1,6,-3\}$ or $\{0,1,5,-2\}$, which are equivalent for (16_4), then they works for all such n. Obviously, any generating line for a cyclic configuration is also a generating line for all sufficiently large n.

Besides the two cyclic configurations, Betten and Betten [B13] describe 17 non-cyclic combinatorial configurations (16_4); they state that these 19 are the complete list, but give no details of the determination of this claim. There seems to have been no independent confirmation of this list. As with all the listings in [B13], it seems that no attention was given to finding presentations of the configurations as symmetric as possible; in particular, there is no mention of duality or selfduality. Beyond the cyclic configurations already mentioned, and the (15_4) configuration in Figure 3.1.1(a), this is illustrated by one of the seventeen (16_4) configurations illustrated in Figure 8 of [B13]. This example is shown in Figure 3.1.2(a).

Betten and Betten [B13] state (or at least imply) that there are only two cyclic configurations (17_4); their generating lines given are equivalent to the ones mentioned above, {0,1,4,6} and {0,1,5,–2}.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	X										X	X	X			
2		X	X	X										X		
3					X	X	X								X	
4								X	X	X						X
5	X	X			X			X								
6	X		X			X			X							
7	X			X			X			X						
8		X				X					X					X
9		X								X		X			X	
10			X				X						X			X
11			X					X			X				X	
12				X	X							X				X
13				X					X				X		X	
14					X				X		X			X		
15						X				X			X	X		
16							X	X				X		X		

(a)

	1	14	15	16	11	12	13	8	6	4	9	2	7	5	3	10
1	X				X	X	X									
2		X								X		X			X	
3			X						X				X	X		
4				X				X			X					X
7	X									X			X			X
6	X								X		X				X	
5	X							X				X		X		
10				X			X						X		X	
9			X			X						X				X
14		X			X						X			X		
12				X		X				X				X		
15		X					X		X							X
11			X		X			X							X	
13			X				X			X	X					
16		X				X		X					X			
8				X	X				X			X				

(b)

Figure 3.1.2. (a) A (16_4) combinatorial configuration as illustrated in [B13] by its Levi incidence matrix. (b) A symmetric incidence matrix of the same configuration, illustrating its selfduality.

As we shall see in Section 3.2, except for one of the (17_4), none of the *combinatorial* configurations (n_4) with $n \leq 17$, is even *topologically* realizable (see Section 3.2). Merlin [M8] shows that the configurations (13_4), (14_4) and the three cyclic (15_4) are not geometrically realizable. But he also notes that *geometric* configurations (n_4) do exist for infinitely many values of n. His construction uses "stacks" of 3-configurations and vertical lines through their vertices to construct [4,3]-configurations, and then stacks of duals of the projections of these into the plane to construct 4-configurations. While this yields geometric configurations (n_4) for infinitely many values of n, there are infinitely many n that are not covered.

Much new information on the question of existence of topological and geometric 4-configurations has become available recently. We discuss it in the following sections.

Exercises and problems 3.1

1. Decide whether the (35_4) configuration of Brunel is cyclic or not.

2. Prove that the three cyclic configurations (15_4) generated by the "lines" $\{0,1,4,6\}$, $\{0,1,5,7\}$ and $\{0,1,3,7\}$, given by Brunel, are distinct (non-isomorphic).

3. Prove that the three cyclic configurations (15_4) generated by the "lines" $\{0,1,4,6\}$, $\{0,1,5,7\}$ and $\{0,1,3,7\}$ are isomorphic to the three generated by $\{0,2,8,12\}$, $\{0,1,9,11\}$, and $\{0,1,9,13\}$, respectively.

4. Investigate the duality properties of the three cyclic configurations (15_4).

5. Validate the claim that the three generating lines in Exercise 2 yield isomorphic configurations (16_4).

6. Show that the cyclic (16_4) configurations with starting lines $\{0,1,4,6\}$ and $\{0,1,6,13\}$ are not isomorphic.

3.2 EXISTENCE OF TOPOLOGICAL AND GEOMETRIC 4-CONFIGURATIONS

As mentioned in Section 3.1, both Brunel [B31] in 1897 and Merlin [M8] in 1913 discussed *geometric* 4-configurations in the real Euclidean plane, and were clear about the distinction between combinatorial and geometric configurations. However, neither did actually show a drawing of any geometric configuration.

The first published diagram of a geometric configuration (n_4) appeared only in [G50], published in 1990. It is reproduced here as Figure 3.2.1. As it happens, it is a realization of Klein's configuration (21_4), introduced in [K11] and mentioned in Section 3.1. The paper [G50] marked the beginning of research of *geometric* configurations (n_4); the results of these investigations form the topic of the remaining part of Chapter 3. The results are intimately connected to the study of topological configurations (n_4), and we shall first describe the known facts concerning these configurations.

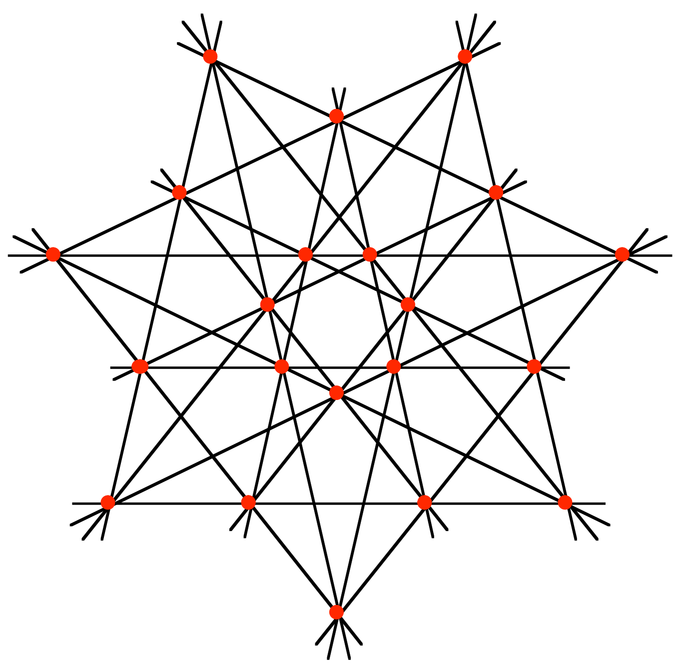


Figure 3.2.1. A geometric configuration (21_4).

The arguments given by Merlin [M8] to establish the non-existence of geometric configurations (n_4) for $n \leq 15$, do not carry over to topological configurations. However, we have:

Theorem 3.2.1. (Bokowski and Schewe [B23]) For $n \leq 16$ there are no topological configurations (n_4).

This is the best possible, since we also have

Theorem 3.2.2. (Bokowski, Grünbaum and Schewe [B22]) Topological configurations (n_4) exist for every $n \geq 17$.

In contrast to this situation, we have:

Theorem 3.2.3. (Bokowski and Schewe [B24]) For $n \leq 17$ there are no geometric configurations (n_4).

Theorem 3.2.4. (Bokowski and Schewe [B24]) There exist geometric configurations (n_4) for all $n \geq 18$ except possibly for $n = 19, 22, 23, 26, 37, 43$.

Theorems 3.2.3 and 3.2.4 demonstrate how the understanding of the (n_4) configurations has developed during the past twenty years. In [G50] it was conjectured that there are no geometric configurations (n_4) with $n \leq 21$ other than the configuration in Figure 3.2.1. Similar conjectures were repeated in various other publications, such as [G41], [G42], [G43]. However, the recent discovery (see [G47]) of a (20_4) configuration led to a modified conjecture, that geometric configurations (n_4) exist only for $n \geq 20$. But this was also short-lived, and was resolved in the negative by the discovery of a geometric (18_4) configuration by J. Bokowski and L. Schewe [B24]. Thus Theorems 3.2.3 and 3.2.4 settle the 20-years quest for the smallest geometric configuration (n_4).

The history of the Theorem 3.2.4 illustrates the rapid improvement in the understanding of configurations (n_4). The first version, in [G47], established that connected (n_4) configurations exist for every $n \geq 21$ except possibly if $n = 32$ or $n = p$ or $n = 2p$ or $n = p^2$ or $n = 2p^2$ or $n = p_1p_2$, where p, p_1, p_2 are odd primes and $p_1 < p_2 < 2p_1$. The number of exceptional cases was soon reduced (in [G41]) to a finite number: There are (n_4) configurations for all $n \geq 21$ except possibly if n has one of the following thirty two values: 22, 23, 25, 26, 29, 31, 32, 34, 37, 38, 41, 43, 46, 47, 49, 53, 58, 59, 61, 62, 67, 71, 77, 79, 89, 97, 98, 103, 113, 131, 178, 179. Newly found construction methods [G43] reduced the list of possible exceptions to the following ten values: 22, 23, 26, 29, 31, 32, 34, 37, 38, 43. All this was while the general belief was that 21 is the smallest number of points in an (n_4) configuration. After Theorem 3.2.3 was established, and additional constructions found, the result became that connected (n_4) configurations exist if and only if $n \geq 18$, except possibly if n has one of the eight values 18, 19, 22, 23, 26, 34, 37, 43. Finally, the discovery of a (18_4) configuration led to the result stated above [B24].

The proofs of Theorems 3.2.3 and 3.2.4 will be given in the next section; here we shall give outlines, and some details, of the proofs of Theorems 3.2.1 and 3.2.2.

The proof of Theorem 3.2.1 given in [B23] is easy for $n \leq 15$. The case $n = 16$ is much more complicated, and forms the bulk (six pages) of that paper. It follows a large number of *a priori* possible topological subconfigurations, and in each case leads to a contradiction. We have to refer the reader to the original paper. In contrast, the case $n \leq 15$ is easily explained, and for fixed k is applicable to all combinatorial configurations (n_k) with n sufficiently small. We present the proof from [B23] with only minor adaptations.

Assume that a combinatorial (n_k) configuration is realized by pseudolines in the projective plane. Due to the possibility of locally perturbing pseudolines at points that are not vertices of the configuration, we may assume that in the *arrangement* (see Appendix A2) generated by the perturbed pseudolines each vertex of the arrangement is incident with either k or 2 pseudolines. Since each of the former accounts for k(k-1)/2 pairwise intersections of pseudolines, the total number of vertices of the modified arrangement is $f_0 = n + n(n-1)/2 - nk(k-1)/2 = n(n - k^2 + k + 1)/2$. Similarly, the number of edges of the modified arrangement is $f_1 = n(n - k^2 + 2k - 1)$. From Euler's theorem for the projective plane it follows that the number of cells (faces) of the arrangement is $f_2 = f_1 - f_0 + 1 = n(n - k^2 + 3k - 5)/2$. On the other hand, arrangements of pseudolines have no digons, hence counting incidences of edges and cells yields $3f_2 \leq 2f_1$. Therefore we have $f(n) = -n^2 + nk^2 + nk - 5n + 6 \leq 0$ as a necessary condition for the existence of a topological realization of a combinatorial (n_k). For fixed k, this function f(n) of n has its only maximum for $n = (k^2 + k - 5)/2$ and is decreasing for all larger n. Simple checking shows that for k = 4 we have $(4^2 + 4 - 5)/2 < 8$ and $f(15) = 6 > 0$, hence (n_4) is not topologically realizable for $n \leq 15$. Since $f(16) = -10 < 0$, this criterion is not applicable for n = 16. On the other hand, this result shows that there are no topologically realizable configurations (n_5) for $n \leq 24$, nor are there any topological (n_6) for $n \leq 36$.

Turning now to Theorem 3.2.2, the first thing to observe is that geometric configurations are, obviously, examples of topological configurations. Hence, assuming that Theorem 3.2.4 can be proved without reliance on Theorem 3.2.2 (as is in fact the case), we need only provide examples of topological configurations for those values of $n \geq 17$ for which there are no known geometric configurations. These values are n = 17, 19, 22, 23, 26, 37, 43. We shall now show such examples, together with a few others that we find appropriate for various reasons. Most of these examples are modified from [B22].

In Figure 3.2.2 we show a topological configuration (17_4) that is a realization of the configuration given by Table 3.2.1. This is taken from [B22], where a proof is outlined according to which this combinatorial configuration (17_4) is the only one admitting a topological realization. It should be noted that this realization has 4-fold rotational symmetry in the extended Euclidean plane. It is not known whether there are realizations with any symmetry in the Euclidean plane proper, or whether there are additional combinatorial automorphisms. Since the configuration is the only topologically realizable (17_4)

configuration, it is necessarily self-dual. (Although it seems not well-known, topological configurations in the projective plane do have dual configurations. This can be inferred from results in [G6].)

1	1	1	1	2	2	2	3	3	3	4	4	4	8	9	10	10
2	5	8	11	5	6	7	5	6	7	5	6	7	13	13	11	12
3	6	9	12	8	9	11	12	8	9	11	10	12	15	14	14	16
4	7	10	13	14	15	16	15	16	17	17	13	14	17	16	15	17

Table 3.2.1. A configuration table of the only (17_4) configuration that admits a topological realization.

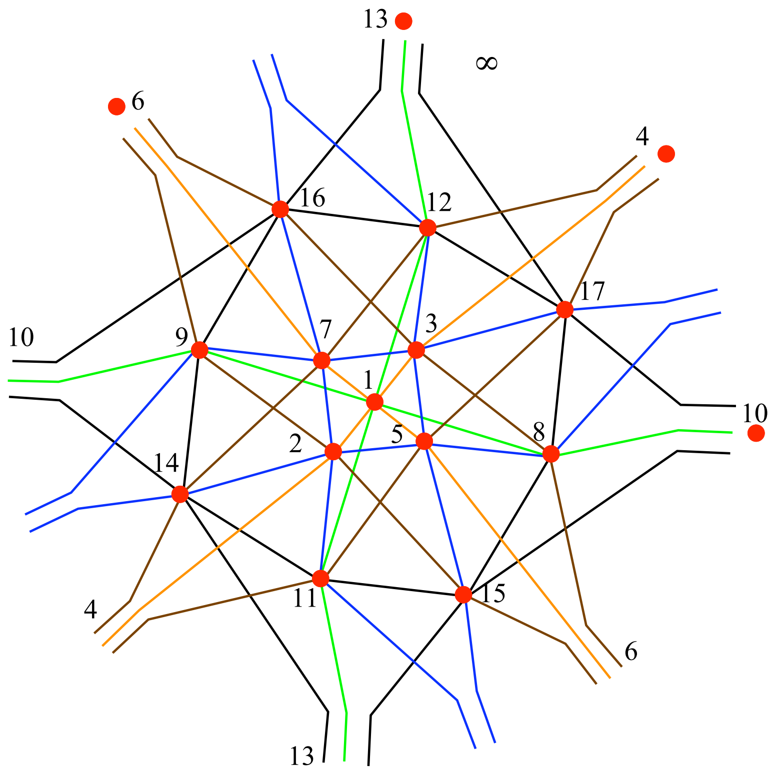


Figure 3.2.2. A topological configuration (17_4). It is a realization of the unique combinatorial configuration (17_4), specified in Table 3.2.1, that has a topological realization.

A topological configuration (18_4) is shown in Figure 3.2.3. This configuration is not isomorphic to the geometric configuration (18_4) we shall see in the next section, and it is not known whether is can be realized geometrically. On the other hand, it has a six-fold rotational symmetry.

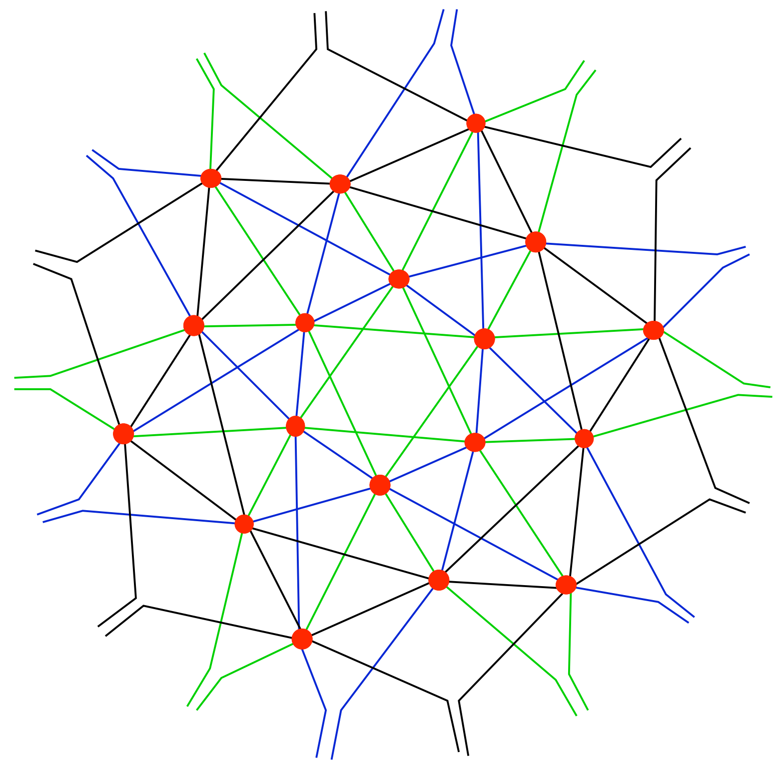
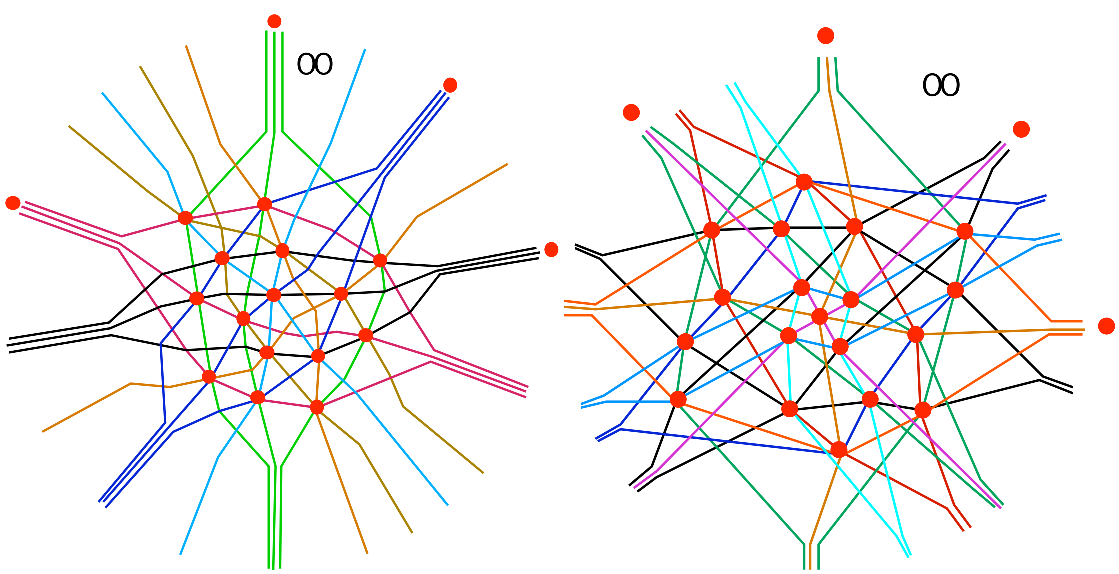


Figure 3.2.3. An example of a topological configuration (18_4) with six-fold rotational symmetry in the Euclidean plane. Adapted from [B22].



(a) (b)

Figure 3.2.4. Topological configurations (a) (19_4) and (b) (23_4). Adapted from [B22].

In contrast, all known topological configurations (19_4) and (23_4) have only trivial symmetry groups. Examples of these configurations are shown in Figure 3.2.4.

The examples of topological configurations presented so far have been *ad hoc*, obtained essentially through (lots of) trial and error. Their rather ungainly appearance is a reminder of their genesis. In contrast, the examples of configurations (22_4) and (26_4) shown in Figure 3.2.5, are members of a systematic family: they are *topological* examples of *astral* configurations; the *geometric* members of the family will be studied in detail in several sections, starting with Section 3.5. The two examples in Figure 3.2.5 are representatives of configurations (n_4) possible for all even $n \geq 22$. In the terms of astral configurations we shall discuss in Sections 3.5 and 3.6, these configurations have *spans* 4 and 5; other possibilities exist, increasing in number with increasing n. Additional information will be given in the discussion of geometric astral configurations, and in Section 5.8.

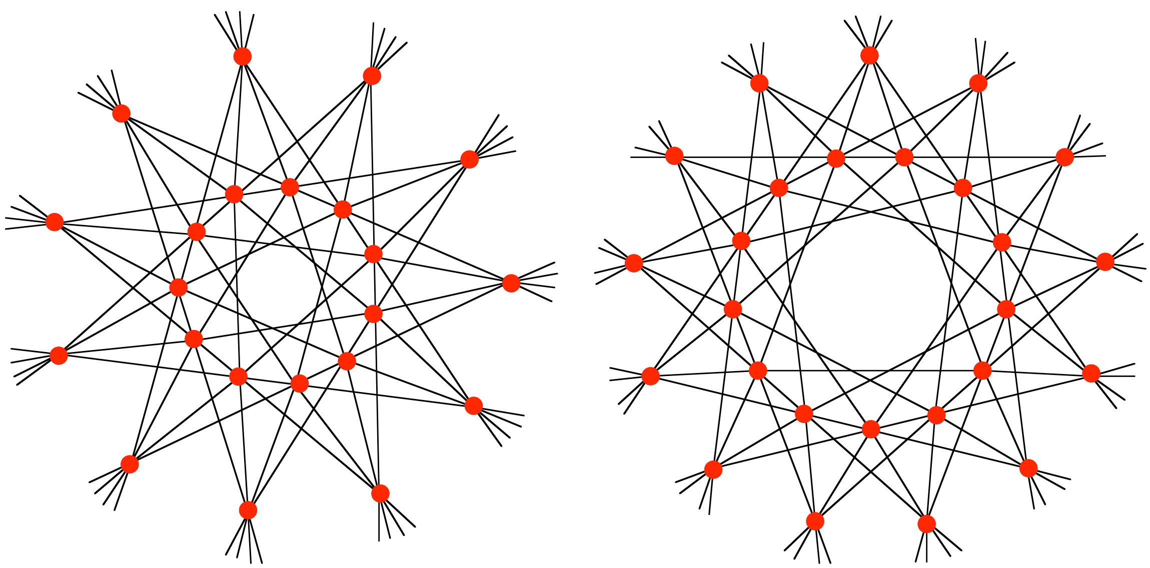


Figure 3.2.5. Topological configurations (22_4) and (26_4), with 11-fold resp. 13-fold dihedral symmetry. They are typical of topological astral configurations (n_4) possible for all even $n \geq 22$.

The examples we provide for (37_4) and (43_4) are special cases of a much more general construction, that is actually very simple. Assuming we have a (p_k) topological configuration, and a (q_k) topological configuration, for some $k \geq 2$, we can construct an (n_k) topological configuration, where $n = p + q - 1$, in the following way: Delete one pseudoline from the former configuration, and a point from the latter, and make the k pseudolines that are now incident with only $k-1$ points, pass through the k points that are incident with only $k-1$ pseudolines. In Figure 3.2.6 is shown the case of (24_4) and (20_4) geometric configurations, leading to a (43_4) topological configuration; the significant points and lines are shown in red. Another (43_4) configuration could be obtained by pairing in the same way an (18_4) configuration with a (26_4). The same kind of construction with (20_4) and (18_4) configurations (either geometric or topological) yields the last of the required topological configurations, (37_4); alternatively, the topological (17_4) could be paired with the geometric (21_4). A different topological configuration (37_4) is shown in [B22]. We shall revisit the same idea for construction of geometric configurations in the next section.

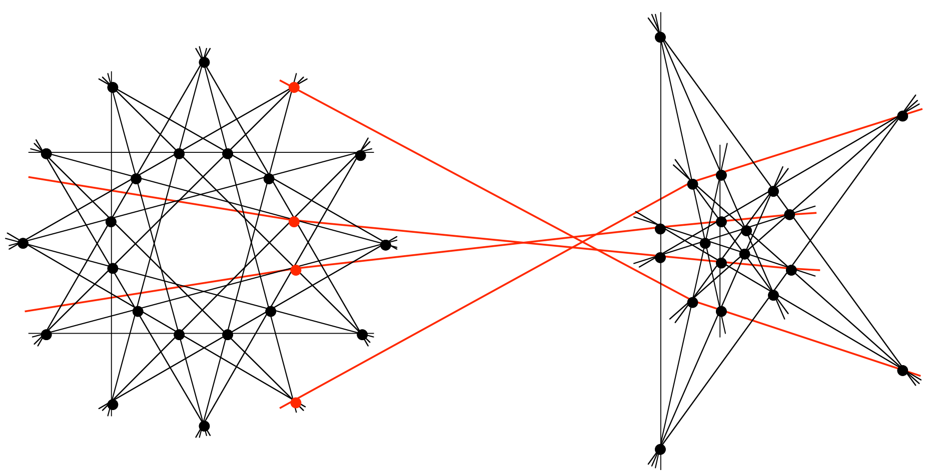


Figure 3.2.6. A topological configuration (43_4). The red points are collinear on the deleted line, the red (pseudo)lines were concurrent at a deleted point of the (20_4) configuration.

Since the combinatorial (n_4) configurations, for n = 13, 14, 15, 16, as well as a large majority of such configurations for $n \geq 17$, cannot be realized by topological configurations, two kinds of questions arise naturally.

First, what relaxation of incidence requirements would be sufficient to enable the construction of topological near-configurations realizing these combinatorial ones?

Second, what are the obstructions preventing topological realizations of some of the combinatorial configurations? For the smallest combinatorial configurations (such as (7_3), (13_4), (21_5), (31_6), (49_8), ...) the existence of *ordinary points* in any family of pseudolines not all of which pass through the same point (see Lemma 2.1.1) can be interpreted as such an obstruction. Indeed, it implies that these configurations cannot have topological realizations since all intersections of pairs of pseudolines would have to be "used up" in points incident with multiple pseudolines, leading to an absence of ordinary points.

The inequality $-n^2 + nk^2 + nk - 5n + 6 \leq 0$ mentioned above as a necessary condition for the existence of an (n_k) topological configuration is another kind of obstruction. It shows that combinatorial configurations (n_k) with $n \leq k^2 + k - 5$ cannot be topologically realized. Since $n \geq k^2 - k + 1$ in all cases, that shows that for each k certain values of n lead to topologically non-realizable configurations (n_k). However, it must be noted that for quite a few of the relevant pairs n, k there exist no combinatorial configurations either – and there is no necessary and sufficient criterion for their existence.

Exercises and problems 3.2.

1. Construct the configuration table dual to the one in Table 3.2.1, and show that it is realized by the configuration in Figure 3.2.2.

2. Prove that the topological (18_4) configuration in Figure 3.2.3 is not isomorphic to the geometric (18_4) configuration shown in Figure 3.3.4 and 3.3.5.

3. Find a topological (18_4) configuration that is dual to the one in Figure 3.2.3.

4. Construct a topological (37_4) configuration.

5. There seems to be no *a priori* reason that would preclude the existence of topological (19_4) or (23_4) configurations with halfturn symmetry. Do any exist?

6. Determine how many topological configurations (26_4) with dihedral symmetry d_{13} exist.

7. Which multiastral combinatorial configurations (n_4) have topological realizations?

3.3 CONSTRUCTIONS OF GEOMETRIC 4-CONFIGURATIONS

The fact that the first graphic realization of *any* (n_4) configuration (see Figure 3.2.1) is less than twenty years old attests to the difficulties that have to be overcome in realizations of such configurations in any intelligible manner. One reason for this situation is that an (n_4) geometric configuration implies the (non-trivial) satisfaction of 2n collinearity conditions, while on the other hand, any finite set of n points (not all collinear) has an affine image that depends on 2n – 6 parameters. Hence there must be some dependences – obvious or hidden – between the collinearity conditions in every geometric configuration (n_4). For a relevant discussion of this topic see Michalucci and Schreck [M18].

In contrast to the situation concerning (n_3) configurations we have presented in Section 2.4, there is no reasonable method or algorithm to go from a combinatorial configuration (n_4) to a topological or geometric one — even if any of these does exist. Nor are any criteria known to distinguish topological configurations which admit geometric realizations from those that do not. Hence, if we wish to find geometric 4-configurations we are, by necessity, forced to resort to more or less *ad hoc* arguments. This does not preclude constructing by the same method large (even infinite) families of examples; however, *finding* such methods or isolated examples is more of an art than a deductive science.

In this section we shall describe several kinds of such constructions. The various families or constructions will be designated in the form (sm), where s is a suitable integer (or another short symbol); the reason for such a name is that for appropriate values of m, the construction leads to a configuration (n_4) with n = s m (or some other value that depends on m).

Following this preamble, let's turn to some concrete cases. In most instances, the construction starts from some given configuration and yields a 4-configuration.

The first construction, which we call **(5m),** starts with an arbitrary (m_3) configuration C; in the example in Figure 3.3.1 this is the (9_3) configuration shown with blue points and lines. We select in the plane a line L (heavy black line in Figure 3.3.1) which misses all the points of C and is neither parallel nor perpendicular to any line determined by any two points of C. We construct three additional copies of C by stretching C through three different ratios in the direction perpendicular to L; only one such copy is shown (red points and lines) in Figure 3.3.1 in order to avoid crowding. The resulting configuration C* consists of the four replicas of C, together with the m intersection points of C with L (shown as hollow dots, which are also intersection points with L of the copies of C), and of the m lines perpendicular to L (shown dashed green) which pass through the points of C (and the other copies). Hence this construction yields a configuration C* of type (n_4), with $n = 5m$. Since — by Theorem 2.1.3 — (m_3) configurations are well-known to exist if and only if $m \geq 9$, this establishes the existence of configurations (n_4) for all $n \geq 45$ which are divisible by 5. Very important for the sequel is the observation that, as follows from the construction, each such configuration C* contains a set of m parallel lines. Moreover, this construction yields "movable" configurations in the sense explained in Section 5.7.

It should be noted that this construction —as well as the ones discussed below — leads in some cases to unwanted incidences, that is, to *prefigurations*. However, this can in all cases be avoided by selecting appropriate parameters for the construction.

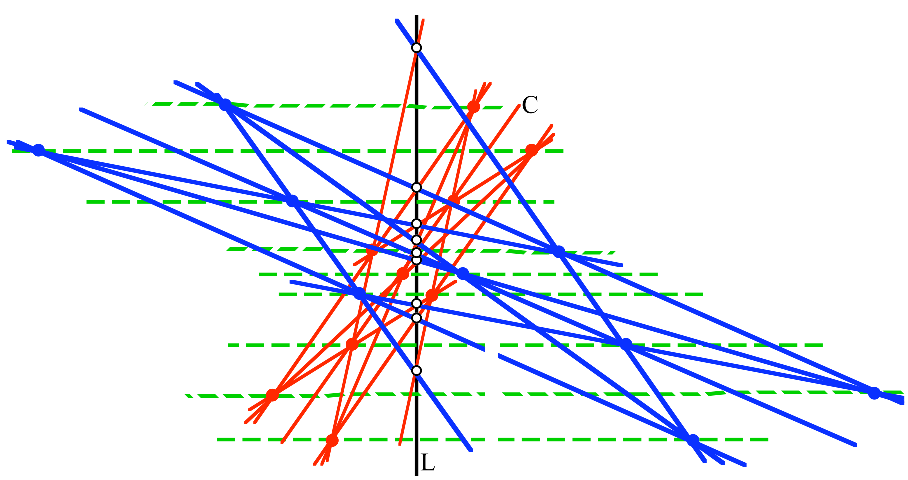


Figure 3.3.1. An illustration of the (5m) construction.

Our second construction is called **(5/2m).** It starts with a $(2m_3)$ configuration C that has a line L of mirror symmetry with the following properties: No point of C lies on the mirror L, no point on L belongs to more than two lines of C, and no line of C is perpendicular to L. It follows from the mirror property of L that there are m points of L at which pairs of lines of C meet. From C another copy is obtained by shrinking C towards L by a certain factor f (say f = ½), and then adding the m intersection points of the lines of the two copies with L, and the m lines perpendicular to L that pass through the points of the two configurations. This is illustrated for (10_4) and (12_4) in Figure 3.3.2, yielding configurations (25_4) and (30_4), respectively. We note that this construction also yields configurations $(5m_4)$ with m parallel lines. Moreover, this construction is **movable**, that is, nontrivial parts of it can be changed in a continuous manner without changing other nontrivial parts. (As already mentioned, we shall discuss movable

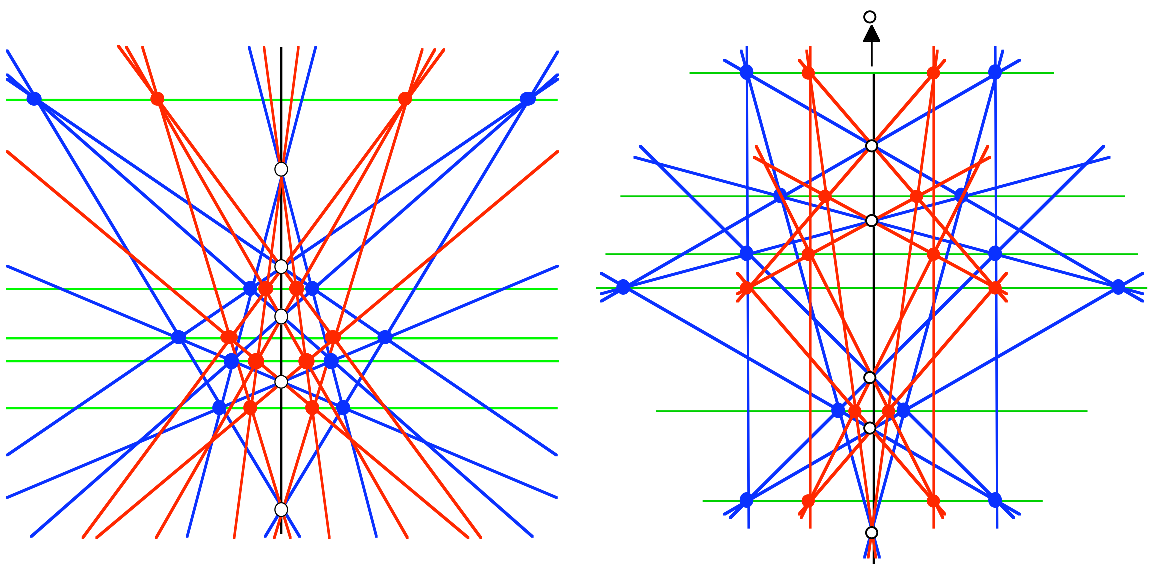


Figure 3.3.2. A (25_4) configuration with five parallel lines, and a (30_4) configuration with six parallel lines. The one at left starts with a (10_3) configuration, the other one with a dihedral astral (12_3) configuration (blue points and lines); copies of these are obtained by shrinking in ratio f = ½ towards the vertical line of symmetry (black line). Adding the five or six intersection points on the line of symmetry (hollow points, at right one at infinity) and five or six horizontal lines (green), completes these typical (5/2m) constructions.

configurations in Section 5.7.) This implies that the cross-ratio of the four points on each of the new (horizontal) lines (which is the same for all m of these lines) can be made equal to any predetermined value by an appropriate choice of f. In Figure 3.3.3. is shown an example of a (14_3) configuration to which the (5/2m) construction is applicable.

A construction of the only known (18_4) configuration was discovered very recently by J. Bokowski and L. Schewe; it is illustrated in Figure 3.3.4, and two different realizations of the same configuration are shown in Figure 3.3.5. This configuration can be considered the smallest member of an infinite family; we shall call this the **(6m) construction** or **family**. The idea to look for such a family came from noticing that the original rendering of the configuration (in Figure 3.3.4) contains a well-known subconfiguration (9_3), which we encountered in Figure 1.1.6, see Figure 3.3.6. This observation led to the construction of a whole family of analogous configurations. The (6m) construction is explained on hand of the typical case illustrated in Figure 3.3.7. The precise membership in the (6m) family has not been determined so far, but the family includes members (n_4) for every $n = 6m$ with odd $m \geq 3$. An additional example is shown in Figure 3.3.8.

The next case to consider is (20_4), first described in [G47], shown in Figure 3.3.9. It too was discovered as a single configuration, and the family to which it belongs was

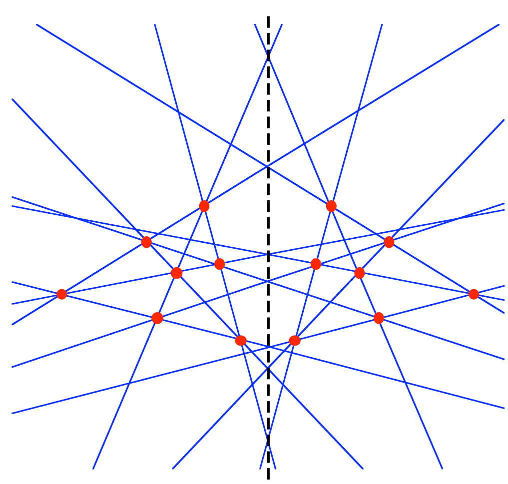


Figure 3.3.3. A (14_3) configuration that can be used to construct a configuration (35_4) by the method in Figure 3.3.2; this (35_4) configuration will have seven parallel lines.

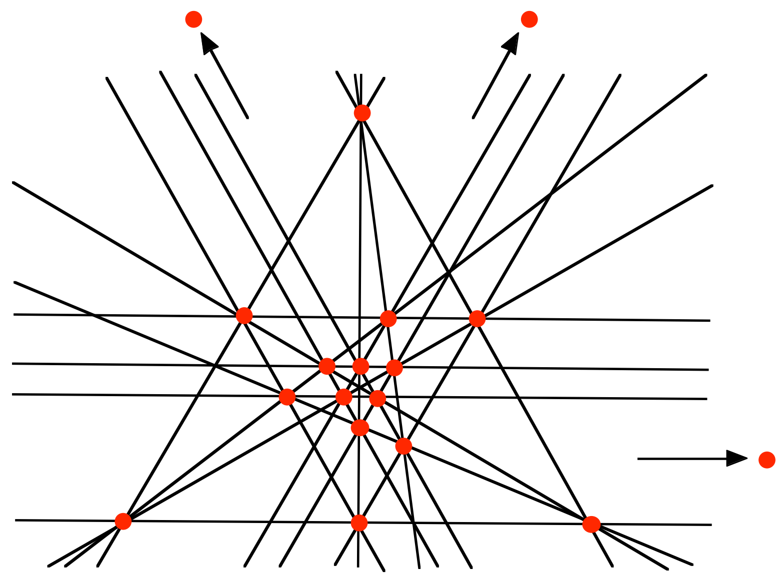


Figure 3.3.4. The only known geometric configuration (18_4) (after [B24]).

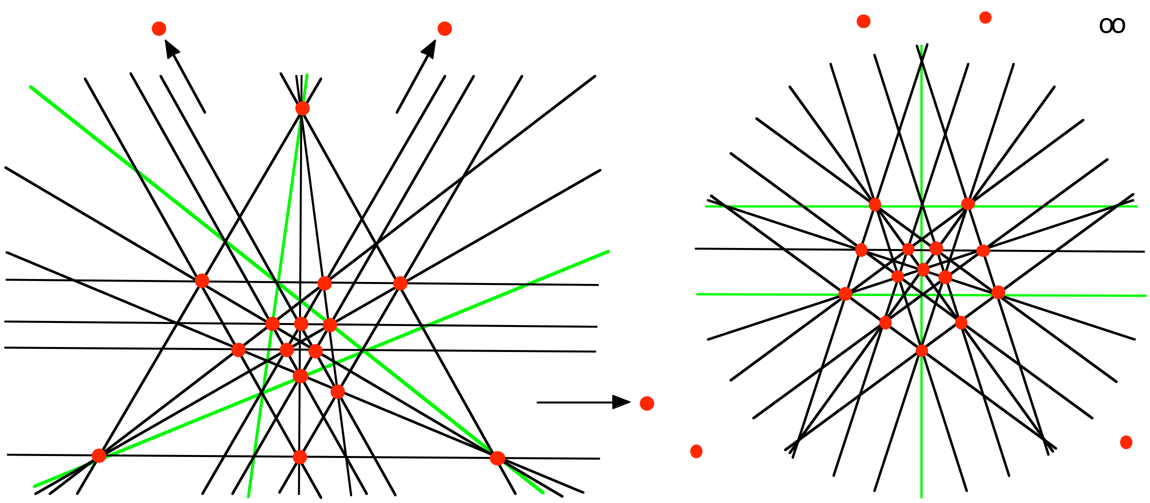


Figure 3.3.5. Two versions of the configuration (18_4) (red points and black lines) from Figure 3.3.4. In each version, adding the three green lines yields a simplicial arrangements of 21 lines (denoted A(21,2) in the catalog [G48]).

found only later; for obvious reasons we call this the **(4m) family** or **construction**. At the time of its discovery the construction seemed quite strange; particularly surprising is the use of two chiral configurations of the same handedness in order to obtain a mirror symmetric configuration. By now we have a much better understanding of the process, although a general proof of the validity of the construction is still not available.

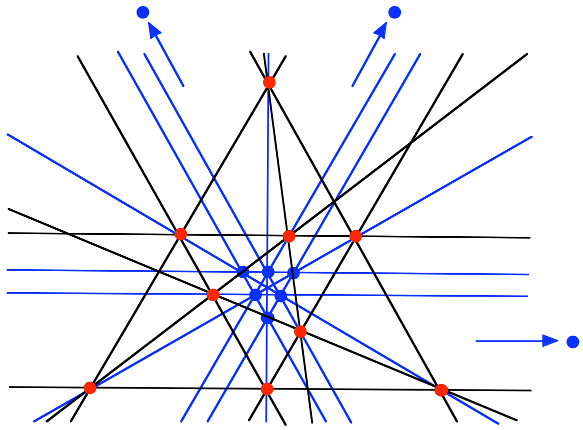


Figure 3.3.6. The configuration (18_4) from Figure 3.3.4 arises from a copy of the configuration $(9_3)_2$ taken from Figure 1.1.6, shown here in red points and black lines, by the addition of nine additional points and lines (shown in blue).

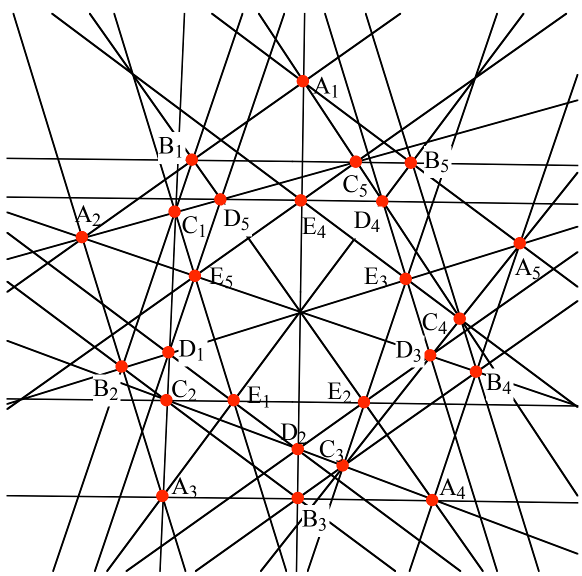


Figure 3.3.7. A (30_4) configuration in the (6m) family, the family that includes the configuration (18_4) in Figure 3.3.4. More generally, the construction of a $((6m)_4)$ configuration starts with a regular m-gon A_1, ... A_m, where $m \geq 3$ is odd. The point B_i is the midpoint of A_i and A_{i+1}, and C_i is selected on B_i,B_{i+1} so that the line C_i,C_{i+1} passes through A_{i+2}. Then D_i is determined on C_i,C_{i+1} so that $D_iC_i/C_{i+1}C_i = C_iB_{i+1}/B_iB_{i+1}$, and E_i is the midpoint of D_i and D_{i+1}. Finally, m points at infinity (not shown) are added, in the directions A_iA_{i+1}. Lines are: A_iA_{i+1}, B_iB_{i+1}, C_iC_{i+1}, D_iD_{i+1}, E_iE_{i+1} and A_iB_{i+2}. All subscripts are understood mod m.

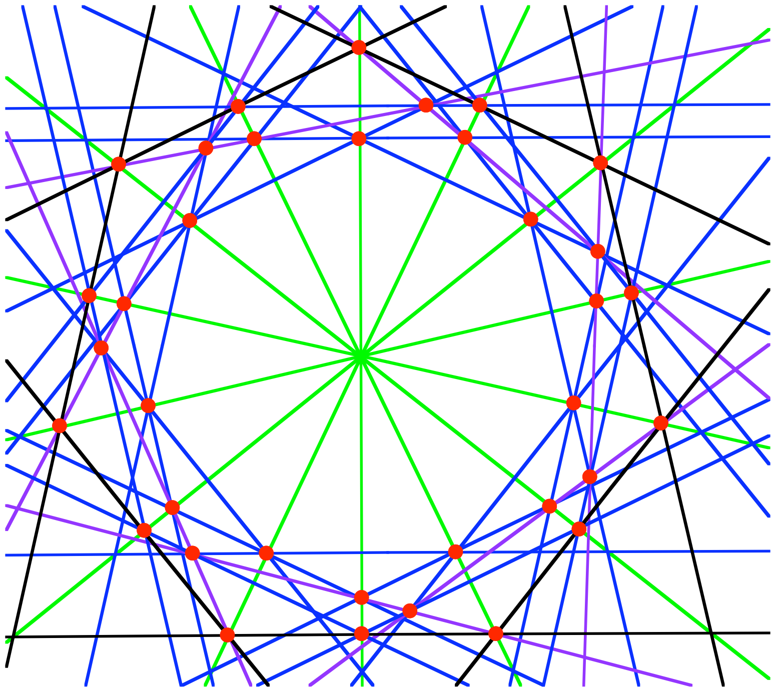


Figure 3.3.8. Shown here is the (6m) construction in the case of the regular 7-gon (black lines), leading to a (42_4) configuration. The seven points at infinity are again not shown; they are in the directions of the quadruplets of parallel black and blue lines.

Extensive experimental evidence led to the general understanding explained below. It leads to the conclusion that geometric configurations (n_4) exist for all $n = 4m$, with $m \geq 5$.

The construction can be described as follows; the explanation is illustrated in Figures 3.3.10 and 3.3.11. We start (see parts (a) in these illustrations) with an astral configuration m#(b,c;d), which we denote C, where $b \geq c > d > 0$ in the notation detailed in Section 2.6. We call this the "outer part" of the construction, and we note that the outermost points of the configuration C determine diagonals of span c. The other m points of C determine diagonals of span b; through each of the outermost points of C passes one of these diagonals. The lines of symmetry of the two diagonals of span c at each outermost point of C (one of these is shown by the green line in (a)) can be used as

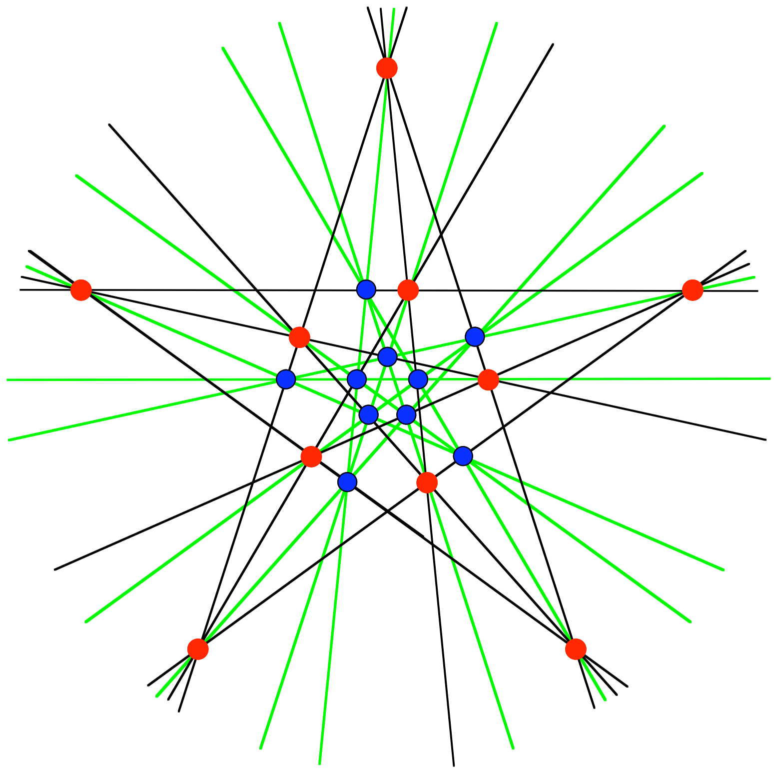


Figure 3.3.9. A (20_4) configuration belonging to the (4m) family. Here m = 5. The construction uses two astral (10_3) configurations; one is shown with red points and black lines, the other with blue points and green lines.

mirrors to reflect the m inner points of C as well as the diagonals of span b (see parts (b) and (c)). The m new points become the outermost points of the "inner part" of the configuration we are constructing. To find the last m (inner) lines, we connect each of the new "outermost" points with one of the original inner points – specifically, we connect it to the $(b+1)^{st}$ of these points, counting in the same orientation as used in calculating the symbol m#(b,c;d). This is indicated by the purple segments in parts (c). The new lines (see parts (d)) pass through previous intersections of two lines, creating the last m points of the $((4m)_4)$ configuration.

It is worth stressing that if the starting outer configuration is the selfpolar m#(b,b;d) as in Figures 3.3.9 and 3.3.10, then the inner configuration is another copy

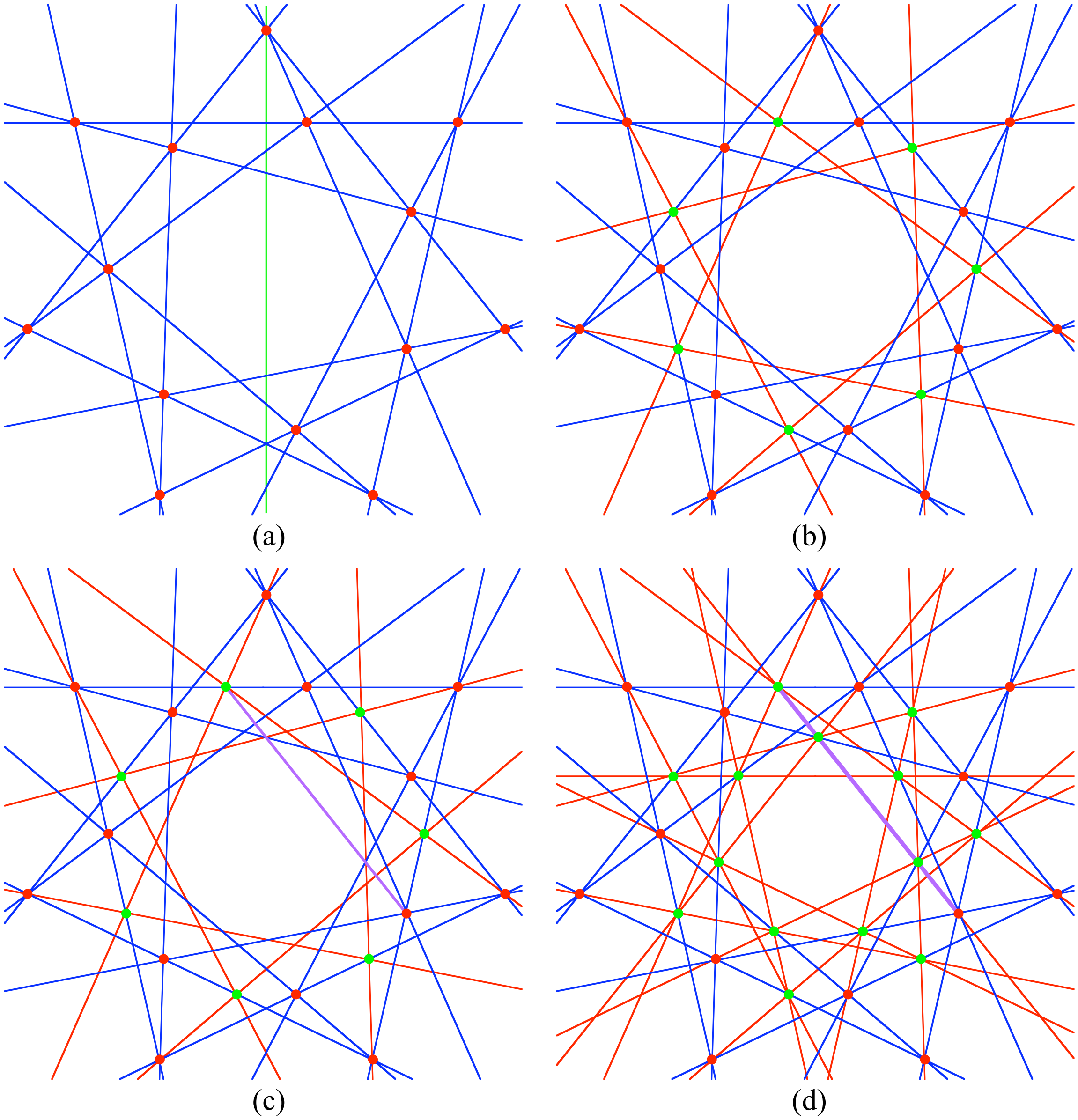


Figure 3.3.10. The steps in the (4m) construction of a (28_4) configuration from the (14_3) configuration 7#(2,2;1), as explained in the text.

(similar to the outer one) of m#(b,b;d). On the other hand, if $b > c$ as in the illustration in Figure 3.3.11, then the outer and inner parts are the two isomorphic and mutually polar configurations with symbol m#(b,c;d).

It is also worth mentioning that if $d > c$ then this construction (or any analogous one I could think of) does not seem to work. This includes the case of selfpolar configurations m#(b,c;d) with $d = (b + c)/2$.

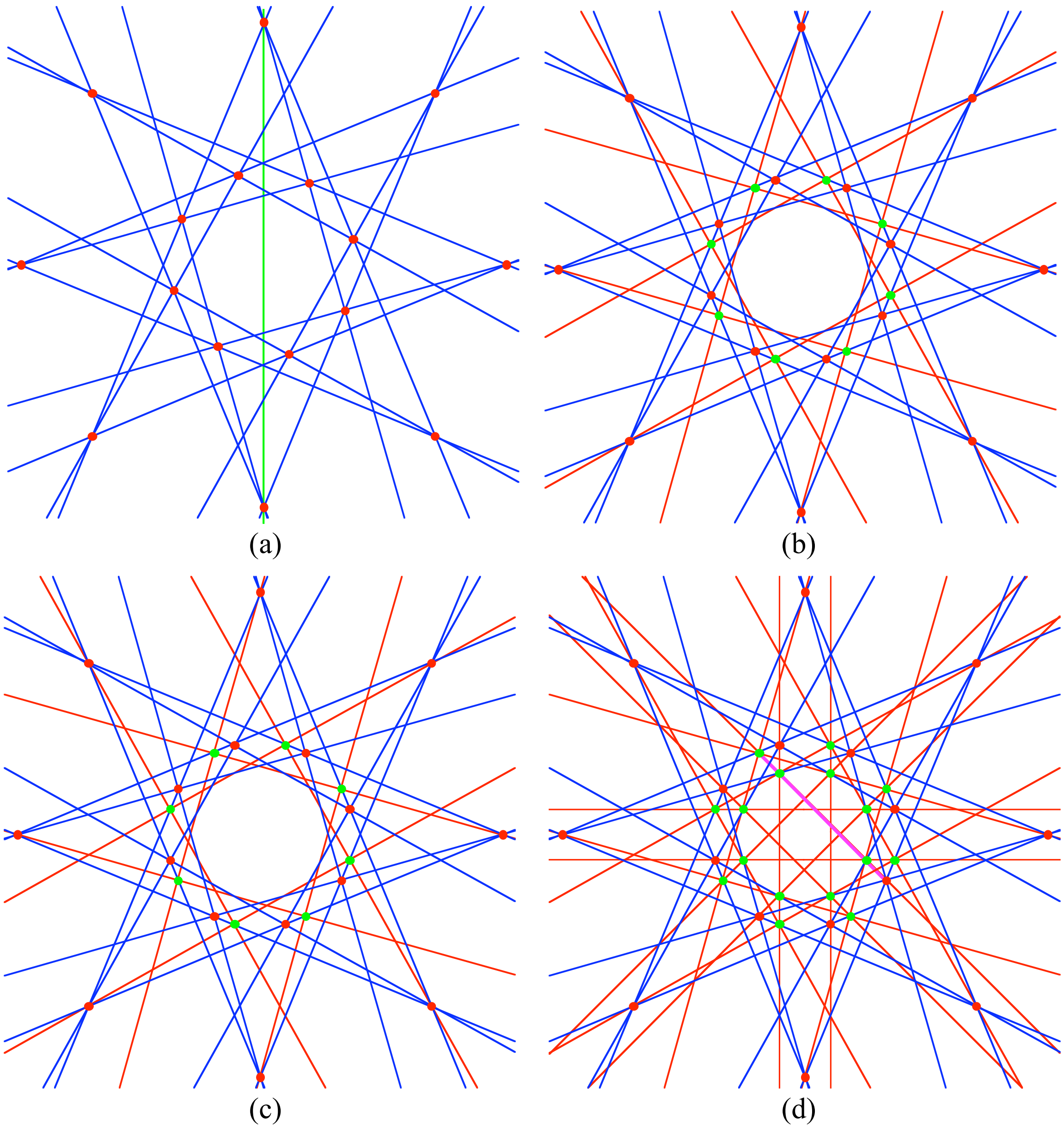


Figure 3.3.11. Another illustration of the construction. We start with a (16_3) configuration $8\#(3,2;1)$ and obtain a (32_4) configuration. Note that the outer and inner parts are not similar, but are polar to each other.

Another infinite family, which we designate as the **(5/6m) family**, is constructed as follows, starting from a 3-astral configuration (n_4) with $n = 6m$, where $m \geq 5$. Let us assume this configuration satisfies the following conditions:

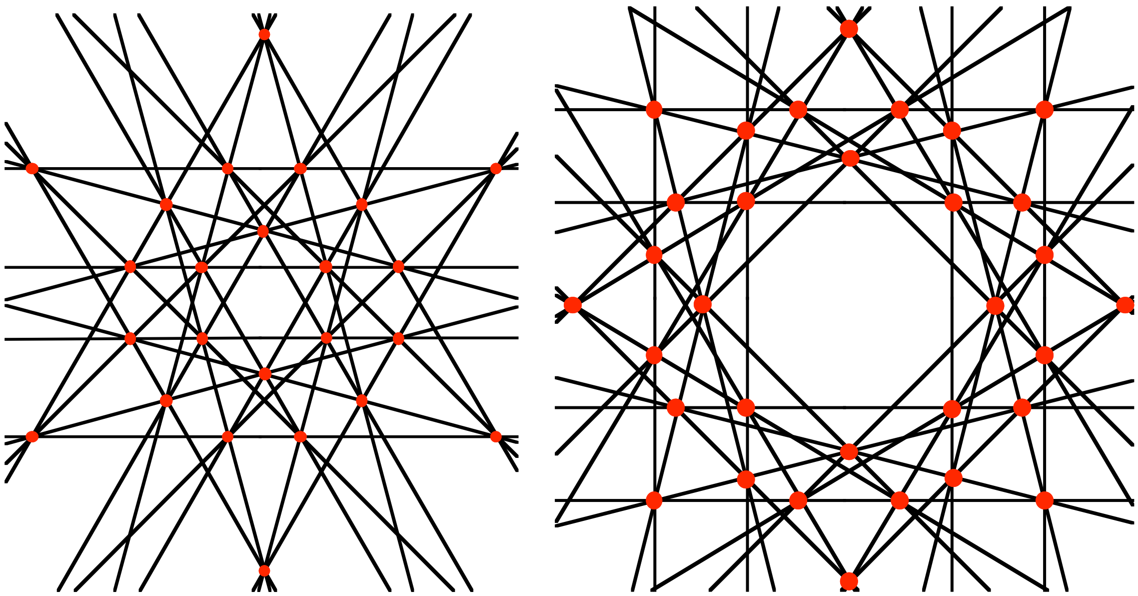


Figure 3.3.12. Configurations (24_4) and (32_4) from the (4m) family; the latter is different from the one in Figure 3.3.11.

(i) It has 2m-gonal dihedral symmetry.

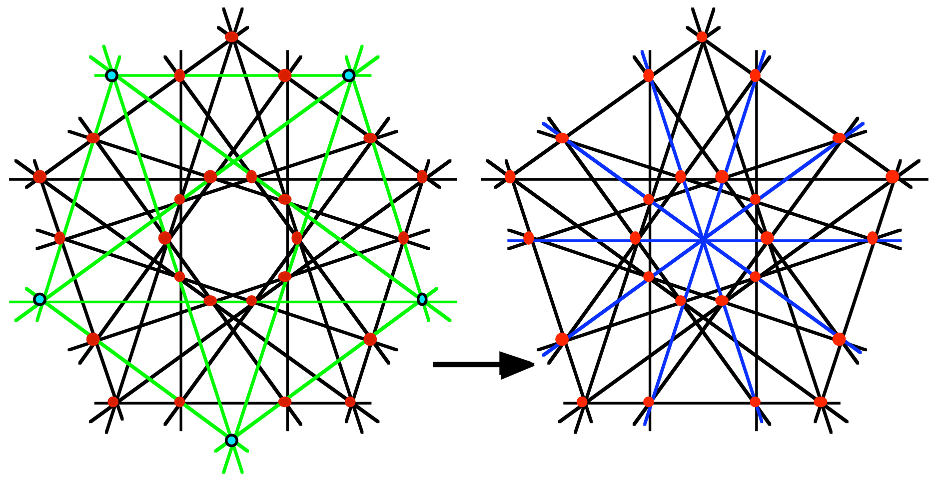
(ii) The configuration is encoded by the symbol $(2m)\#(s_1,t_1;s_2,t_2;s_3,t_3)$, where s_i is the span of the i-th family of diagonals of the i^{th} level polygon P_i, and t_i is the order of the intersection point, counting from the midpoint of the diagonal s_i, and considering only diagonals of span s_i of the polygon P_i. For more details see Section 3.6.

(iii) s_1 and t_3 are distinct, and both are even; this implies $m \geq 5$.

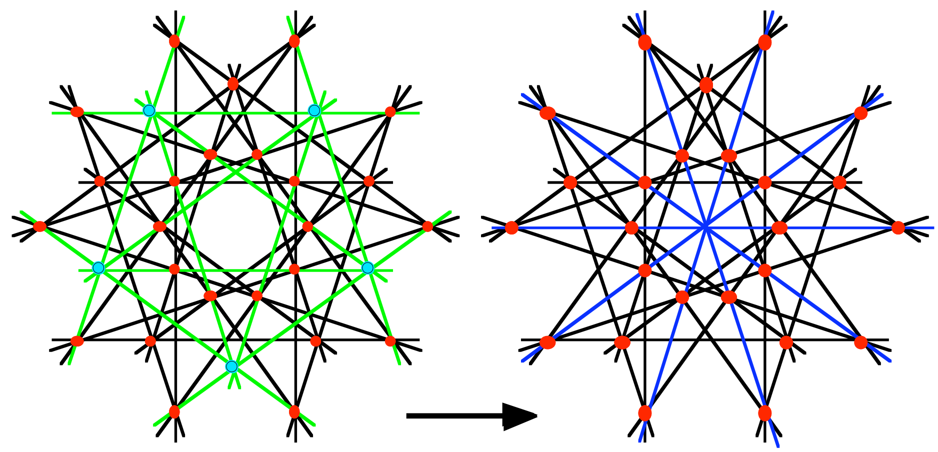
(iv) t_1 and s_3 are odd.

(v) s_2 and t_2 have same parity.

Condition (iii) implies that both kinds of diagonals ending at points of P_1 have even lengths. Therefore, omitting every other point of P_1 and all the lines incident with these points leads to a loss of m points and 2m lines. (Note that, as shown in Figures 3.3.13 and 3.3.14, "level 1" does not mean that P_1 is the "outermost level".) The claim is that the above conditions imply that one can add to the remaining lines and points m suitable lines through the center to obtain a $((5m)_4)$ configuration. The examples in Figures 3.3.13 and 3.3.14 illustrate the construction.



10#(4,1,2,4,1,2)



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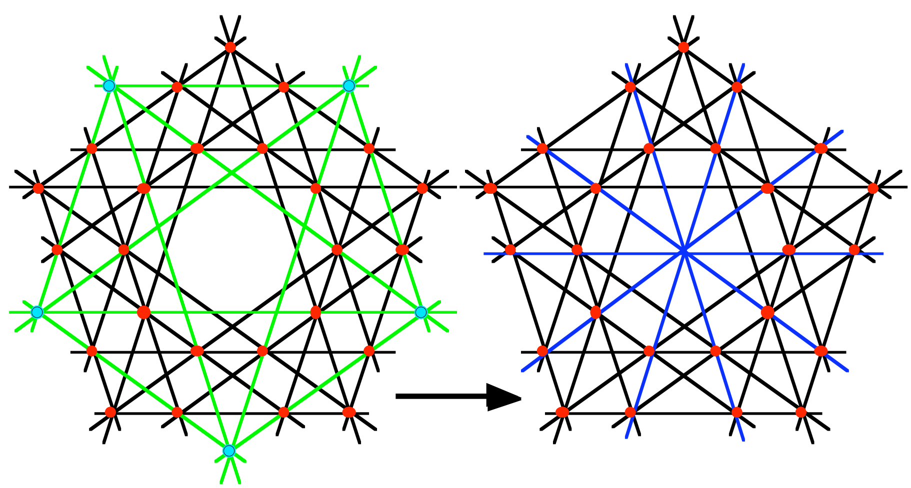


10#(4,3,1,3,1,2)

Figure 3.3.13. The three (25_4) configurations obtainable by the (5/6m) construction.

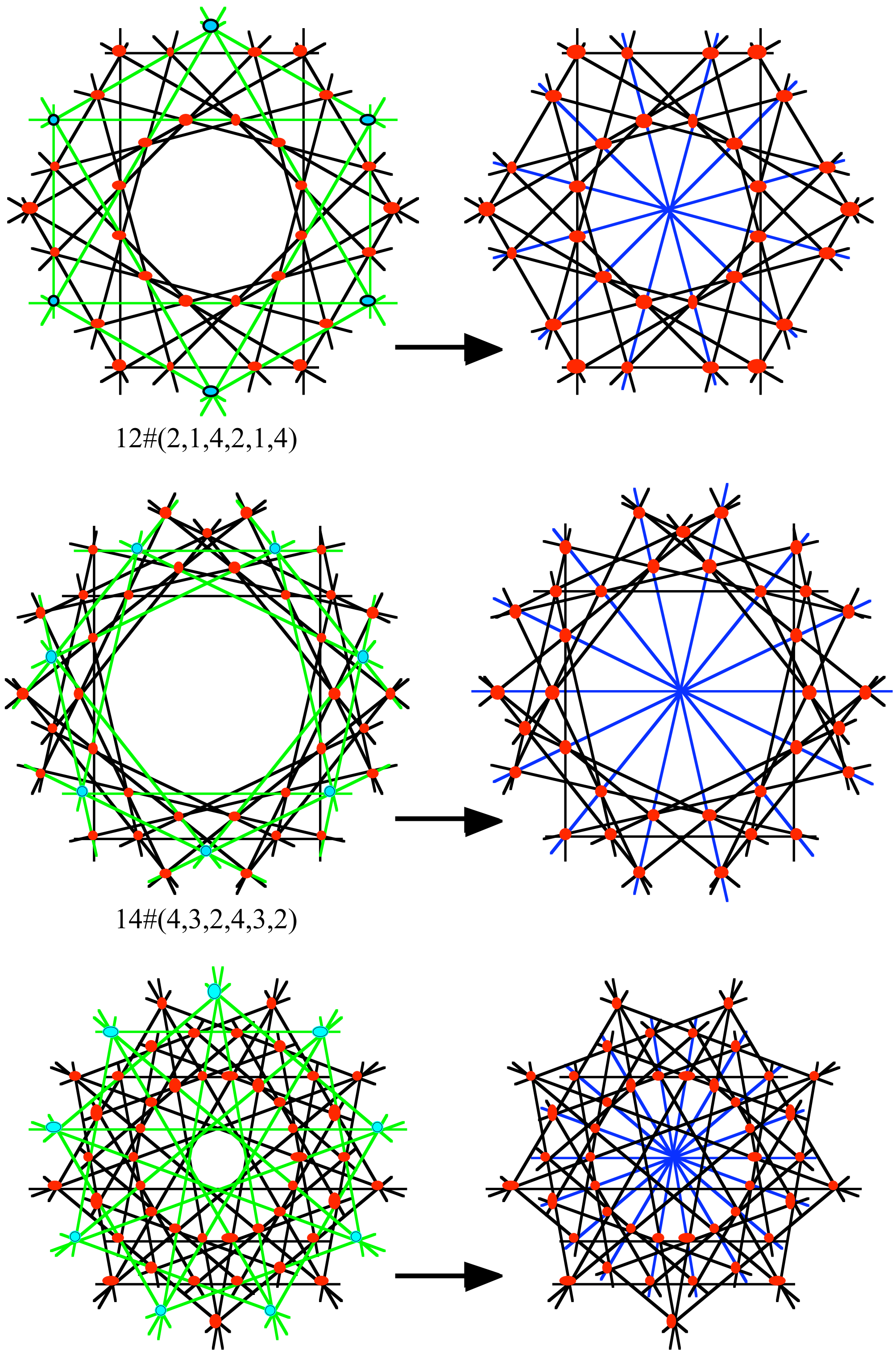


Figure 3.3.14. Configurations (30_4), (35_4) and (45_4) belonging to the (5/6m) family.

The reason the construction works is the following. All points of a 3-astral configuration with (2m)-gonal dihedral symmetry (that is, based on a regular (2m)-gon G) are on lines through the center that are mirrors for the symmetries of the configuration. The points of G are on mirrors that enclose angles that are multiples of p/m. More specifically, the two types of points on an s_i diagonal of P_1 are spaced an even multiple of p/m if s_i and t_i have the same parity, and an odd multiple of p/m if these parities are different.

In view of the above, conditions (iii), (iv) and (v) imply that viewed from the center, the points of level 1 are not aligned with the points of the other two levels. Hence these latter points are aligned, and provide the m lines required for the formation of a $((5m)_4)$ configuration.

Since configurations (2m)#(2,1,4,2,1,4) and (2m)#(2,3,4,2,3,4) exist for all $m \geq 5$, our existence claim is justified. In fact, for every $m \geq 5$ there exist additional possibilities. This is illustrated in Figure 3.3.13, for m = 5. This case was the starting points of this construction. Some of the configurations in Figure 3.3.13 were first constructed, independently and by *ad hoc* methods, by T. Pisanski and J. Bokowski.

A few other configurations in the (5/6m) family are illustrated in Figure 3.3.14.

The constructions we have seen so far started from given configurations that had to satisfy certain conditions. The resulting (n_4) configurations always had as n a composite number — more specifically, a multiple of 4, or 5, or 6. Now we shall describe constructions that are applicable quite generally, but are apt to give (n_4) configurations with other values of n.

The general construction, which we call the **(3m+) construction**, has the interesting feature that it is more easily visualized and explained in 3-space; the resulting configuration is then readily projected into the plane. We start with an (m_4) configuration C in the plane. We assume that this is the (x,y)-plane in a Cartesian (x,y,z)-system of coordinates, and that C has $p \geq 1$ lines parallel to the x axis, and $q \geq 1$ lines parallel to the y axis, such that no two of them have a point of the configuration in common. (Note that by an affine transformation — which does not change incidences — any two sets of parallel lines can be made orthogonal. The orthogonality is assumed only in order to simplify the description.) We select a real number $h > 1$ and keep it constant throughout the discussion; it is convenient (but not necessary) to think of $h = 10$. We construct two copies of C. One is C', obtained from C by stretching C in ratio $(h-1)/h$ (that is, in fact, shrinking it) towards the y-axis, stretching it in ratio $(h+1)h$ towards the x-axis, and then translating it to the level $z = 1$. A schematic representation of a section parallel to the x-axis is shown in Figure 3.3.15.The other is C", obtained similarly but by using the ratio $(h+1)/h$ for stretching towards the y-axis, $(h-1)h$ for the ratio towards the x-axis, and translation to the plane $z = -1$. Thus, C' is obtained from C by the map $f(x, y, 0) = (x(h-1)/h, y(h+1)/h, 1)$, and C" by $g(x, y, 0) = (x(h+1)/h, y(h-1)/h, -1)$. It is easy to check that for each point $A = (x, y, 0)$ the points A, f(A) and g(A) are collinear, and that the points $h(A) = (0, 2y, h)$ and $h^*(A) = (2x, 0, -h)$ are collinear with them. Now, for any four points A_j $(j = 1, 2, 3, 4)$ of C that are on a line L parallel to the x-axis — that is, have the same y-coordinate — the point $h(A_j)$ will be the same since it does not depend on the x-coordinate. Therefore we can conclude that by deleting the line L from the configuration C and its parallels in C' and C", while adding the

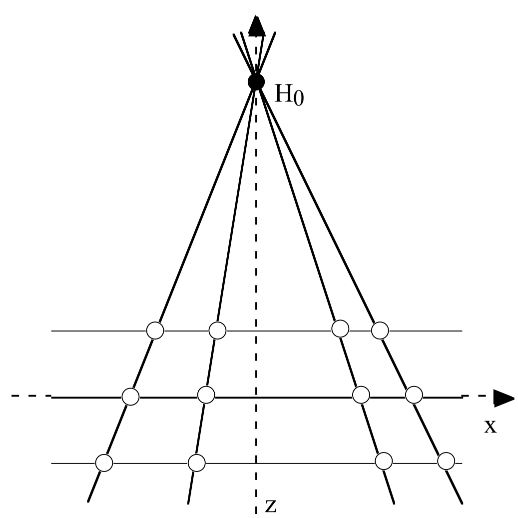


Figure 3.3.15. A schematic illustration of the (3m+) construction.

lines from A_j to $h(A_j)$, the points A_j and the corresponding points in C' and C" will remain incident with four lines, and the new point $h(A_j)$ will also be incident with four lines. We deleted three lines and added four, and also added one point. Thus, from the starting (m_4) configuration we obtained a configuration (n_4) where $n = 3m+1$. Analogously, any four points of C collinear on a line parallel with the y-axis may lead to an additional increase in the number of points and lines; the assumed disjointedness of the two families of parallels is needed here to assure that no 5-point lines arises. Proceeding similarly with some or all lines parallel to either the x-axis or the y-axis, we see that from (m_4) we can obtain configurations (n_4) for each n such that $3m + 1 \leq n \leq 3m + p + q$.

Next, we have the **deleted unions constructions** (DU-1) and (DU-2). Consider any configurations $C_1 = ((n')_4)$ and $C_2 = ((n'')_4)$, such that the cross-ratio of points of C_1 on a certain line coincides with the cross-ratio of lines through a certain point of C_2. Then omitting the line and the point in question, and adjusting the positions and sizes of the deleted configurations appropriately, we obtain a configuration with $n' + n'' - 1$ points. In every case one can use for C_2 a polar of C_1, to go from (n_4) to $((2n-1)_4)$. This is construction (DU-1); illustrations are provided in Figures 3.3.16 and 3.3.17. For (DU-2) we need to delete two disjoint lines and two unconnected points, respectively. An illustration is given in Figure 3.3.18. Again, the only requirement is that the cross-ratios of the appropriate quadruplets of points and of lines be equal.

With this we have completed the description of the various constructions that will enable us to find geometric configurations (n_4) for almost all values of $n \geq 18$. The proof of this assertion, which we have already formulated as Theorem 3.2.4, will be given in the next section. In it we shall utilize various configurations with very high symmetry — astral, multiastral, and other. Since their construction and properties are both interesting and complicated, we are not describing them here; instead, we shall devote to them several later sections.

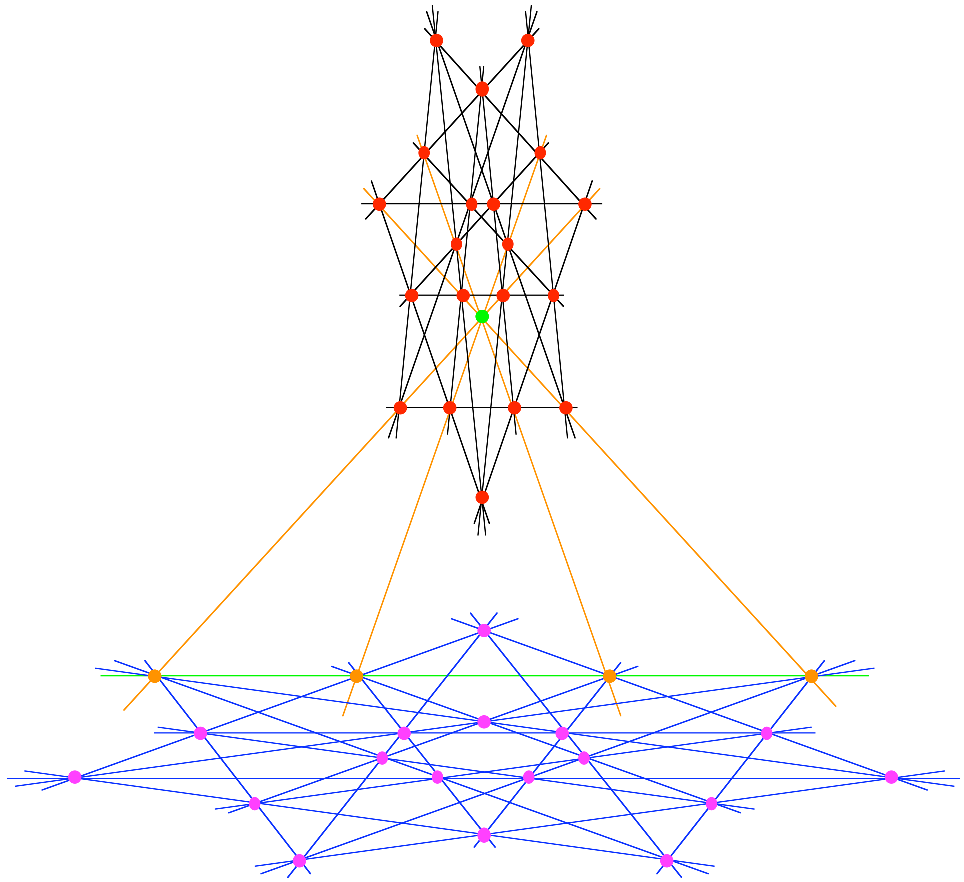


Figure 3.3.16. (41_4) from two copies of (21_4) using (DU-1).

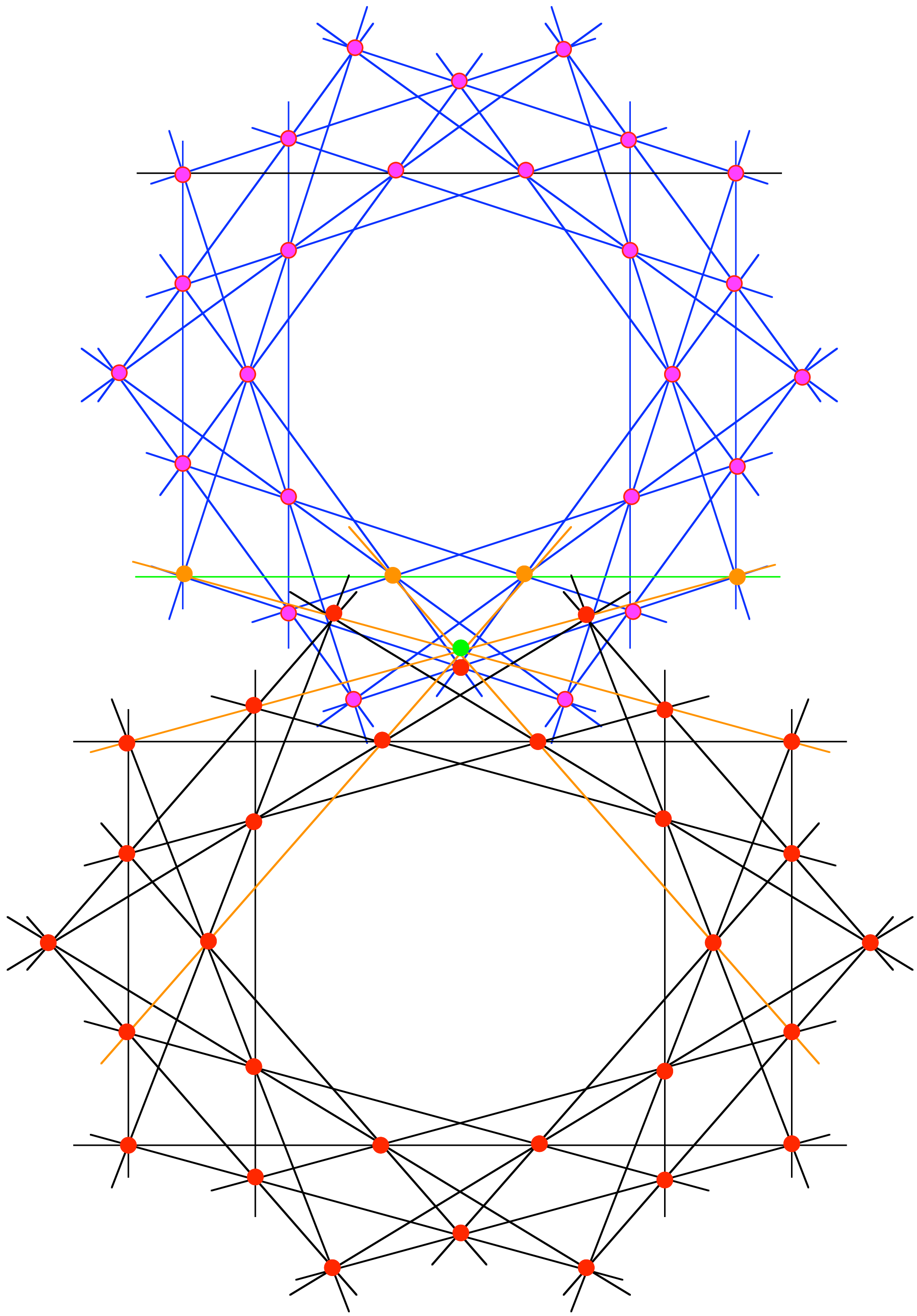


Figure 3.3.17. (59_4) from two copies of (30_4) using (DU-1). p+q = 8

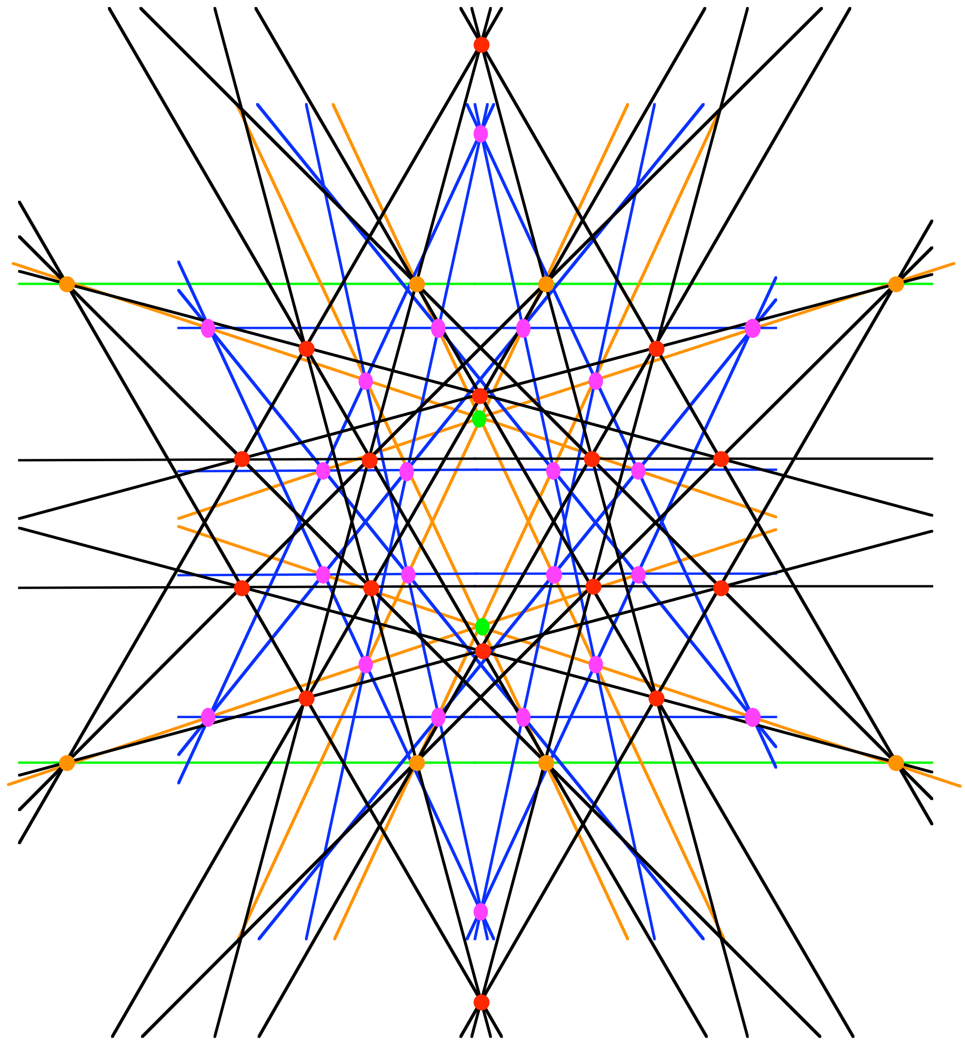


Figure 3.3.18. (46_4) from two copies of (24_4), using (DU-2).

Exercises and problems 3.3

1. Carry our the construction of the (35_4) configuration described in Figure 3.3.3.

2. Determine whether any of the three configurations (25_4) in Figure 3.3.13 are isomorphic.

3. Devise a general proof for the validity of the (4m) construction, as detailed in the text.

4. Formulate the analog of the (3m+) construction that leads from 3-configurations to 3-configurations. Illustrate by a simple example.

5. The (6m) construction is applicable to regular star-polygons as well. Explore the case of a pentagram, and of one of the regular star-heptagons.

6. Explain why the (DU-1) construction cannot be applied to get a (43_4) from (20_4) and (24_4).

3.4 EXISTENCE OF GEOMETRIC 4-CONFIGURATIONS

We start with a quick summary description of the construction methods detailed in Section 3.3.

The (5m) construction is illustrated in Figure 3.3.1. It starts with an arbitrary (m_3) configuration and yields an $((5m)_4)$ configuration.

The (5/2m) construction is illustrated in Figure 3.3.2. It starts with appropriate configurations $((2m)_3)$ and yields a $((5m)_4)$ configuration; the criteria for usable (m_3) configurations are given on page 3.3.3.

The (4m) construction starts with an astral configuration $((2m)_3)$ and yields a 4-orbit dihedral configuration $((4m)_4)$. As explained on page 3.3.7, it works for most (but not all) such configurations with $m \geq 5$.

The (6m) construction starts with a 3-orbit configuration $((3m)_3)$ and yields a 6-orbit configuration $((6m)_4)$. It assumes that $m \geq 3$ is odd. Some details are given on page 3.3.6.

(3m+) denotes the construction described in detail on page 3.3.18. It starts with an (m_4) configuration and yields an $((3m+p+q)_4)$

Deleted Unions constructions (DU-1) and (DU-2). Using (DU-1), from suitable configurations $C_1 = ((n_1)_4)$ and $C_2 = ((n_2)_4)$ we obtain a configuration with $n_1 + n_2 - 1$ points and as many lines. In particular, we can go from any (n_4) to $((2n-1)_4)$. For (DU-2) we delete two disjoint lines and two unconnected points, and obtain $((2n-2)_4)$ from (n_4).

In addition to these, we use the notation (t-A.m) for the multiastral configuration with t orbits and with symmetry group d_m. This implies that each orbit has m points. Details of these configurations and the notation used for them appear in Section 3.5. (2-A.m) denotes astral configurations. If no other indication is given, the references are to the "trivial" choices of parameters such as (1,2,3,1,2,3) or (1,2,3,4,2,1,4,3).

It is relatively simple to show that (n_4) configurations exist for all $n \geq 210$. Indeed, by the (5m) construction there is for each $m \geq 9$ a $((5m)_4)$ configuration with $p = m$ parallel lines. It follows by the (3m+) construction that for all $m \geq 9$ and $1 \leq p \leq m$ there exists a $((15m+p)_4)$ configuration. Since $15m = 5(3m)$, by the (5m) construction we can add $p = 0$ to the range of p. Thus (n_4) configurations exist all values of n such that $15m \leq n \leq 16m$; for $m \geq 14$ these ranges are contiguous or overlapping, and so the claim is established.

For smaller values of n we have to rely on the various constructions described above and in Section 3.3. We found it simplest to arrange the necessary data in a table (Table 3.4.1) in which we list examples of configurations (n_4) for each n. In most cases there are other configurations we could have listed — the present choice is largely accidental.

n	Reference or explanation
18	(6m) for m = 3; Figure 3.3.4
19	Not known
20	(4m) for m = 5; Figure 3.3.9;
21	(3-A.m) , 7#(3,2,1,3,2,1); Figure 3.2.1
22	Not known
23	Not known
24	(2-A.m), 12#(5,4,1,4); Figure XXX; (3-A.m)
25	(5/2m) for m = 10; starting with $(10_3)_{10}$. Figure 3.3.2.
26	Not known
27	(3-A.m)
28	(4m) for m = 7; Figure 3.3.10
29	Bokowski (unpublished)
30	(3-A.m)
31	Bokowski (unpublished)

32	(4m) for m = 8. Figures 3.3.11 and 3.3.12
33	(3-A.m)
34	(DU-2) from two (18_4)
35	(5/2.m) for m = 7; starting from a (14_3) shown Figure 3.3.3
36	(2-A.m) several possibilities with m = 18; (3-A.m), (4-A.m)
37	Not known
38	(DU-2) from two (20_4)
39	(3-A.m)
40	(4-A.m)
41	(DU-1) from (21_4) Figure 3.3.16
42	(3-A.m)
43	Not known
44	(4-A.m)
45	(5-A.m) for m = 9; e.g. 9#(1,2,3,4,2,1,2,3,4,2); (3-A.m)
46	(DU-2) from (24_4) Figure 3.3.18
47	(DU-1) from (24_4)
48	(2-A.m); (3-A.m); (4-A.m); (6-A.m)
49	(7-A.m); Figure 3.4.1.
50	(5-A.m), 10#(1,4,3,2,3,1,4,3,2,3)
51	(3-A.m)
52	(4-A.m)
53	(DU-1) from (27_4)
54	(3-A.m)
55	(5-A.m)
56	(4-A.m)
57	(3-A.m)
58	(DU-2) from (30_4)
59	(DU-1) from (30_4)
60	(2-A.m); (3-A.m); (4-A.m); (5-A.m); (6-A.m) 10#(1,3,2,4,2,3,4,1,3,2,3,2)
61	(DU-1) applied to (21_4) and (41_4)

62	(DU-1) applied to (21_4) and (42_4)
63	(3-A.m)
61 – 63	(3m+) from (20_4)
64 – 66	(3m+) from (21_4)
67	(DU-1) from (33_4) and (35_4), obtained by (5/2m) for m = 14.
68	(4-A.m)
69	(3-A.m)
70	(5-A.m)
71	(DU-1) from (36_4)
72	(2-A.m); (3-A.m); (4.-A.m); (6-A.m)
73 – 76	(3m+) from (24_4), p+q = 4
75	(5/2m), m = 30; (5-A.m)
76 – 80	(3m+) from (25_4) = (5/2m), m = 10, p+q = 5
81	(3-A.m), m = 27; (9-A.m), m = 9, 9#{3,4,2,1,4,1,4,3,2,3,4,2,1,4,1,4,3,2}
82 – 87	(3m+) from (27_4), p + q = 6
88 – 95	(3m+) from (29_4), p + q = 6
91 – 98	(3m+) from (30_4), 30#(4,6,9,4,6,9), p + q = 8
99	(3-A.m), m = 33;
100 – 105	(3m+) from (33_4), p + q = 6
106 – 112	(3m+) from (35_4) = (5/2m), m = 14, p + q = 7
109 – 114	(3m+) from (36_4), 12#(1,2,3,1,2,3), p + q = 6
115	(5/2.m), m = 46; (5-A.m)
116	(4-A.m), m = 29
117	(3-A.m), m = 39
118 – 123	(3m+) from (39_4), 13#(1,5,3,1,5,3), p + q = 6
121 – 128	(3m+) from (40_4) = (5/2m), m = 16, p + q = 8
127 – 132	(3m+) from (42_4), 14#(1,3,5,1,3,5), p+q = 6
133 _ 139	(3m+) from (44_4), 11#(1,2,5,4,2,1,4,5), p+q = 7
136 – 144	(3m+) from (5/2m) = (45_4), m = 18, p+q =9
145 – 152	(3m+) from (48_4), 12#(1,2,5,4,2,1,4,5), p+q = 8

151 – 160	(3m+) from (50_4) = (5/2m), m = 20, p+q = 10
157 – 164	(3m+) from (52_4), 13#(1,2,5,4,2,1,4,5), p+q = 8
165	(5m) (33_3)
166 – 173	(3m+) from (55_4), 11#(1, 2, 3, 4, 5, 1, 2, 3, 4, 5), p+q = 8
172 – 177	(3m+) from (57_4), 19#(1,4,7,1,4,7), p+q = 6
178 – 185	(3m+) from (59_4), p+q = 8, Figure 3.3.17
181 – 192	(3m+) from (60_4) = (5/2m), m = 24, p+q = 12
193 – 200	(3m+) from (64_4), 16#(1, 3, 7, 5, 3, 1, 5, 7), p+q = 8
199 – 207	(3m+) from (66_4), 11#(1, 2, 3, 4, 5, 4, 2, 1, 4, 3, 4, 5), p+q = 9
208 – 213	(3m+) from (69_4), 23#(1,3,5,1,3,5), p+q = 6
211 – 224	(3m+) from (5m), m = 14 = p+q
225	(5m) from (45_3)
226 – 240	(3m+) from (5m), m = 15 = p+q
241 – 256	(3m+) from (5m), m = 16 = p+q
256 – 272	(3m+) from (5m), m = 17 = p+q
271 – 288	(3m+) from (5m), m = 18 = p+q
286 – 304	(3m+) from (5m), m = 19 = p+q.

Table 3.4.1. Desriptions of the construction of (n_4) configurations for $n \leq 304$.

The arguments presented above, together with the data in Table 3.4.1, constitute a proof of Theorem 3.2.4.

The known constructions explained above for the configurations (n_4) with small n (such as 18, 20, 21, 24, 25) all rely on r-fold rotational symmetry with $r \geq 3$. As a consequence, none of these constructions can be carried out in the *rational* projective plane. While there is no proof available showing that some or all these configurations are not realizable in the rational projective plane, it is a challenging problem to decide for which n is such a realization possible. An easy argument shows that if we start with rational configurations then (5m) constructions can be performed so as to yield rational configurations. Similarly for (5/2m) and (3m+) constructions.

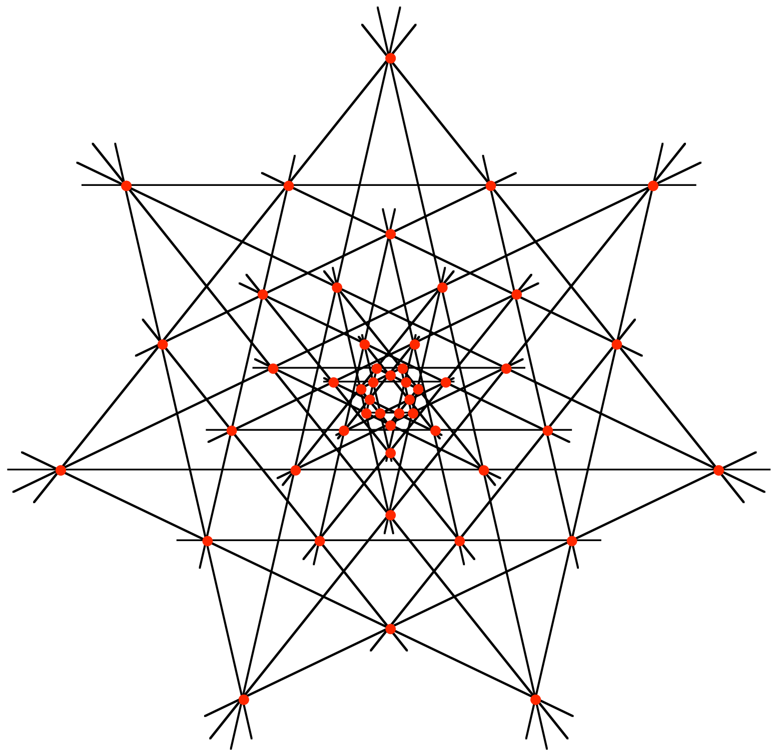


Figure 3.4.1. A (7-A.m) configuration (49_4), with symbol 7#(2,1,2,1,3,2,3,2,1,2,1,3,2,3).

Exercises and problems 3.4

1. Decide whether a suitable affine (or projective) image of the (18_4) configuration shown in Figure 3.3.4 can be put in the rational plane.

2. Determine for which n can one find a configuration (n_4) in the plane over a quadratic extension of the rationals.

3.5 ASTRAL 4-CONFIGURATIONS

In this section we start our investigation of a special class of 4-configurations which we call **k-astral** for some $k \geq 2$. They are of interest for several reasons. To begin with, such configurations were the first 4-configurations for which geometric realizations were found (see, for example, [G50], [G40], [B20], and the other publications that will be mentioned later). Next, they have a clear-cut definition that leads to a natural notation, as well to construction of the configuration given its symbol. Finally, they exhibit a variety of phenomena that add interest to their study, such as the relation of a configuration to its dual (actually, its polar) configuration, and questions of realizability versus representation.

k-astral configurations have appeared under several different names, and with several different definitions – not all of which coincide in all cases. In several publications configurations we call k-astral have been termed *celestial*. The intention in the present account of these configurations is to have an easily implementable decision algorithm for checking the membership of either a given configuration to the class, or of a symbol for correspondence to a geometric configuration.

Definition 3.5.1. A (n_4) configuration C is **k-astral** provided all the following conditions are satisfied:

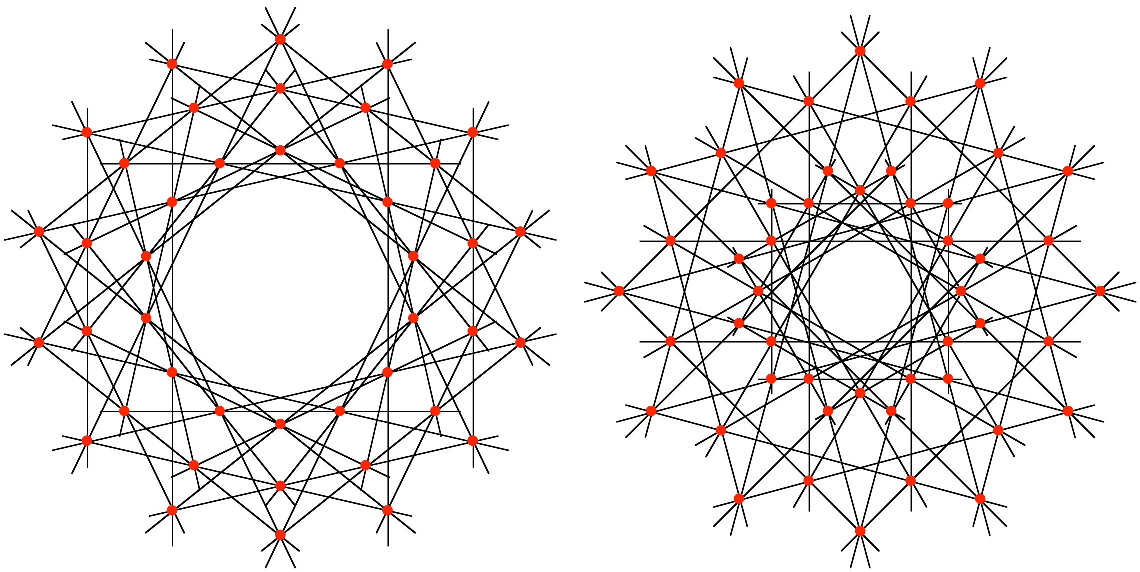
(A1) $k \geq 2$ and $n = k \cdot m$, for some $m \geq 7$.

(A2) The points of C are at the vertices of k regular convex m-gons, with common centers and such that all angles subtended from this center by the various points of C are multiples of π/m.

(A3) C has symmetry group d_m; the vertices of each k-gon form an orbit.

(A4) Each line of C contains two points from each of two m-gons (point orbits); each point is incident with two lines each from two line orbits.

We have already encountered various configurations that are k-astral, for example the ones in Figures 1.1.2 and 1.5.1(a). Two additional examples are shown in Figure 3.5.1.



(a) (b)

Figure 3.5.1. (a) A 3-astral configuration (42_4) with symbol 14#(5,3;2,4;1,3). (b) A 4-astral configuration (45_4) with symbol 12#(5,4;3,2;4,5;2,3).

Some comments deserve to be made regarding k-astral configurations.

(i) In most cases the k regular m-gons have different sizes; however, in some cases with $k \geq 3$ there may be pairs of polygons with the same size. We shall give examples later.

(ii) The conditions in Definition 3.5.1 could be weakened at the expense of complicating the verification.

(iii) It will turn out to be convenient to consider the case of connected k-astral configurations separately from the case of disconnected ones.

Theorem 3.5.1. Each k-astral configuration C can be assigned a symbol $m\#(s_1,t_1;s_2,t_2; \ldots ,s_k,t_k)$ in such a way that C is the only configuration from which that symbol arises. At most 2k distinct symbols correspond to each configuration; such symbols are said to be equivalent. The family of equivalent symbols can be obtained from any one of them by cyclic permutations of an even number of steps of the 2k entries in parentheses, or by reversal of these.

Proof. The proof consists of a description of the steps leading from the configuration to one of the symbols, and observing the stages at which distinct symbols may arise. The main tools in the derivation are a notation for the intersection points of the diagonals determined by each of the regular m-gons, and the "characteristic paths" along lines of the configuration.

For a regular convex m-gon M, the **span** s of a diagonal S is the number of edges of M between the endpoints of S, taken as the smaller of the two possible numbers; hence $s \leq m/2$. Despite talking about "endpoints", by "diagonal" we understand both the elementary-geometric meaning of the term as a segment, as well as the whole line determined by this segment. In Figure 3.5.1, the outer polygon has diagonals of spans 3 and 5 for both configurations (a) and (b).

The intersections of a diagonal S of span s with the other diagonals of span s of the same polygon M are denoted by the symbol (s//t), where t is the position of the intersection points on S, counting from the midpoint of S. (Instead of (s//t), the notation [[s,t]] has also been used.) Thus, for example, each endpoint of S has symbol (s//s). The intersection points are not limited to the diagonal considered as a segment, but continue "outside" and exist for all $t < m/2$. If m is even, one may consider the point-at-infinity on S as having $t = m/2$. An illustration of the notation (s//t) is given in Figure 3.5.2.

The use of polar coordinates is particularly convenient for the intersection points (s//t), since it is easily seen that in the setting of Figure 3.5.2, such a point has coordinates $(\cos s\cdot\pi/m \,/\, \cos t\cdot\pi/m,\ t\cdot\pi/m)$. If the endpoints of the diagonal are not on the unit circle but at distance r, then the first polar coordinate needs to be multiplied by r.

A **characteristic path** P of a (connected) k-astral configuration C consists of k *segments* of lines of the configuration, determined as follows. The procedure we describe here is illustrated by the example in Figure 3.5.3.

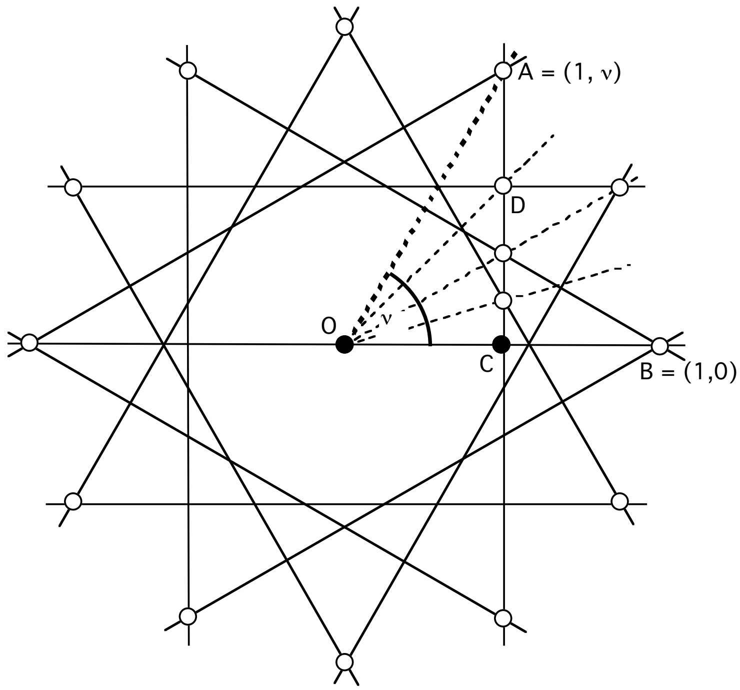


Figure 3.5.2. The determination of the symbol (s//t) of an intersection point of diagonals of a regular m-gon. Here m = 12, the diagonals are of span s = 4, the vertices of the m-gon with unit radius have polar coordinates (1, ν), where ν is a multiple of π/m. In the diagram ν = 4π/m, angle DOB is 3π/m, so that s = 4, t = 3. This gives for D the symbol (s//t) = (4//3). The right triangles OCA and OCD imply that OD = cos s·π/m / cos t·π/m, hence D has polar coordinates (cos s·π/m / cos t·π/m, t·π/m).

As the first step we orient all lines of C in the same sense, generally taken to be counterclockwise as seen from the center. Next, we choose an arbitrary point P_0 of C and through it an arbitrary line L_1 for which P_0 is the earlier of the two points in the same orbit; this involves the choice of one line from the two orbits of points through P_0. On L_1 we take the first point (in the orientation we adopted) of the *other* orbit of points incident with L_1, and denote it P_1. We choose as line L_2 a line through P_1 that is in the orbit different from L_1, and for which P_1 is the earlier point in its orbit. On L_2 we choose the earlier point in the orbit different from the one of P_1, and denote it P_2. Continuing in the same way, we select the line L_{j+1} through the already selected point P_j that is in the orbit different from L_j, and for which P_j is the earlier point among the points on L_{j+1} belonging

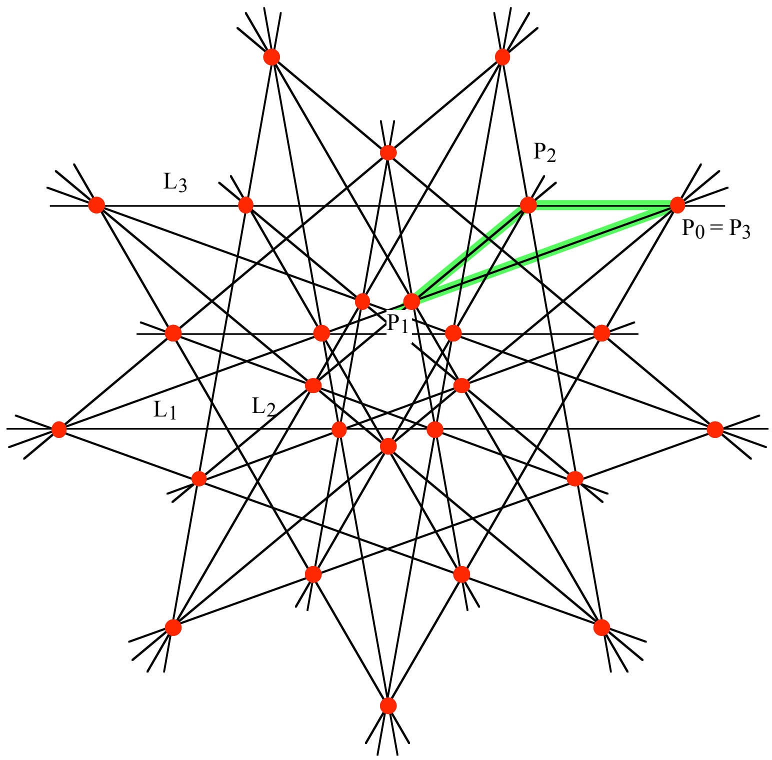


Figure 3.5.3. An illustration of the construction of a characteristic path (shown green) and the corresponding symbol of the configuration, according to the description given in the text. Since $P_1 = (4//2)$, $P_2 = (3//4)$, $P_3 = (2//3)$, and k = 3 while m = 9, the resulting symbol of this (27_4) configuration is 9#(4,2;3,4;2,3).

to the orbit different from the orbit of the earlier point P_j. This continues until we reach the line L_k and the point P_k. (in Figure 3.5.3 we have k = 3.) This point P_k necessarily belongs to the same orbit as the starting point P_0; in the illustration P_k coincides with P_0, but this is not necessarily the case. Figure 3.5.4 illustrates the possibility of P_k being different from P_0. By using the notation $(s_j//t_j)$ for the point P_j, the characteristic path $P_0,L_1,P_1,L_2,P_2, \ldots L_k,P_k$ determines a symbol $m\#(s_1,t_1;s_2,t_2; \ldots ;s_k,t_k)$ for the configuration.

What are the possible alternative symbols for a configuration? We arbitrarily chose the orientation of the lines, the starting point of the characteristic path, and the starting line through that point. The choice of orientation does not lead to any new symbols since a k-astral configuration has *dihedral* symmetry group d_k. However, the other

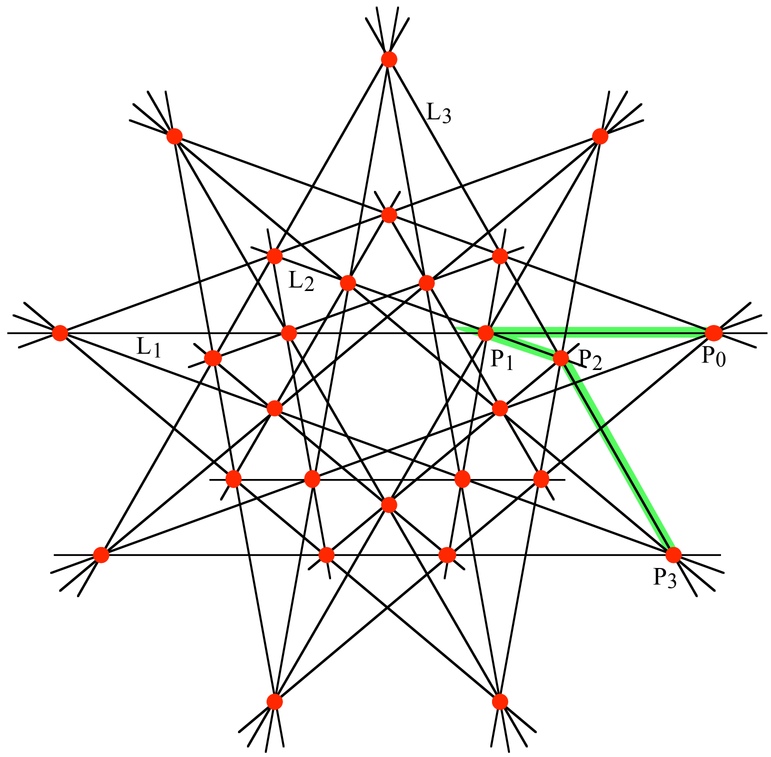


Figure 3.5.4. Another illustration of the construction of a characteristic path and the symbol of the configuration. Since $P_1 = (4//3)$, $P_2 = (2//3)$, $P_3 = (1//3)$, and k = 3 while m = 9, the resulting symbol of this (27_4) configuration is 9#(4,3;2,3;1,3), and $P_3 \neq P_0$.

two choices obviously matter, and in general lead to 2k distinct symbols — k choices of the orbit of the starting point of the path, and two choices for the starting line through that point. As an illustration we show in Figure 3.5.5 the four characteristic paths and the resulting four equivalent symbols for the 2-astral configuration (48_4).

The various equivalent symbols for a given k-astral configuration arise in one of the following two ways. For a given characteristic path, selecting on this path a different point as the starting point clearly permutes the symbols $(s_j//t_j)$ cyclically, that is, by an even number of steps in the symbol $m\#(s_1,t_1;s_2,t_2; \ldots ;s_k,t_k)$ of the configuration. This yields up to k distinct symbols. On the other hand, if we consider a diagonal of span s, the symbol (s//t) for the t^{th} intersection point (counting from the midpoint) can be interpreted as saying that on the orbit of all points (s//t) the same diagonal line has span t, and the original endpoints (that gave span s to the diagonal) now have symbol (t//s).

This means the following: Starting with a given characteristic path but traversing it in the opposite direction, will reverse the roles of s_j and t_j in all the diagonals, as well as the order of the points. Hence this leads to the reversal of all the entries in the original symbol, thus accounting for (up to) an additional k symbols.

The construction of the symbols for a k-astral configuration (n_4) leads to several notable properties of the symbols and the configurations. For ease of reference we list them as a continuation of the entries in Theorem 3.5.1.

(A5) Since the symbol (s//s) denotes the endpoints of a diagonal of span s, (hence would not constitute a step in the characteristic path) any two adjacent entries in the symbol $m\#(s_1,t_1;s_2,t_2; \ldots ;s_k,t_k)$ must be different; this includes the requirement that s_1 and t_k are distinct.

Next, as obvious from the reasoning concerning the symbol (s//t) and visible in Figure 3.5.2, the polar angle of the point (s/t) differs from 0 by a multiple of π/m. The parity of that multiple is the same as the parity of $s+t$. Since the endpoint of a characteristic path leads to a point in the orbit of the starting point, and the polar angles of any two such points differ by a multiple of $2\pi/m$, it follows that the sum of all entries in the parentheses of a symbol $m\#(s_1,t_1;s_2,t_2; \ldots ;s_k,t_k)$ must be even, or equivalently, that

(A6) $$\delta = \tfrac{1}{2} \sum_{1\le j\le k} (s_j - t_j) \quad \text{is an integer.}$$

If condition (A6) is not satisfied in a symbol that fulfils all other requirements, then the last point of the characteristic path ends midway between points of the orbit of the starting point — and consequently has only two lines incident with it, just as the starting point is incident with only two lines. We shall discuss this in more detail later, but already here we can supply in Figure 3.5.6 an example of such a situation.

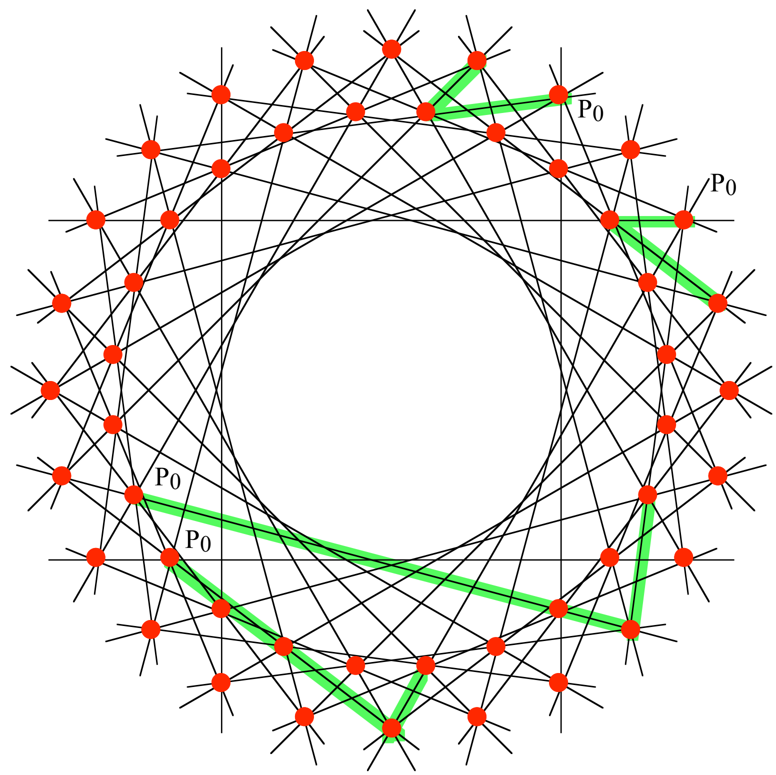


Figure 3.5.5. Four characteristic paths (green) for a 2-astral (48_4) configuration; all proceed counterclockwise. In order to avoid excessive clutter, in each path only the starting point is labeled. The path on top starts in the outer ring of points; it leads to the symbol 24#(5, 2; 7, 8), since the first point of the inner ring that is encountered by the path has symbol (5//2), and the first point met after that in the outer ring has symbol (7//8). The other characteristic paths lead to the symbols 24#(7, 8; 5, 2), 24#(2, 5; 8, 7), and 24#(8, 7; 2, 5), respectively, in counterclockwise order of the starting points.

One additional — and very important and useful — property of all k-astral 4-configurations follows from the comments we made after the introduction of the (s//t) notation. Since the radius of a point (s//t) of a regular convex m-gon with circumradius r is $r \cdot \cos(s \cdot \pi/m) / \cos(t \cdot \pi/m)$, the distance of the point P_j from the center is (assuming the starting point of the characteristic path is at unit distance from the center):

$$\prod_{1 \le i \le j} (\cos(s_i \cdot \pi/m) / \cos(t_i \cdot \pi/m)).$$

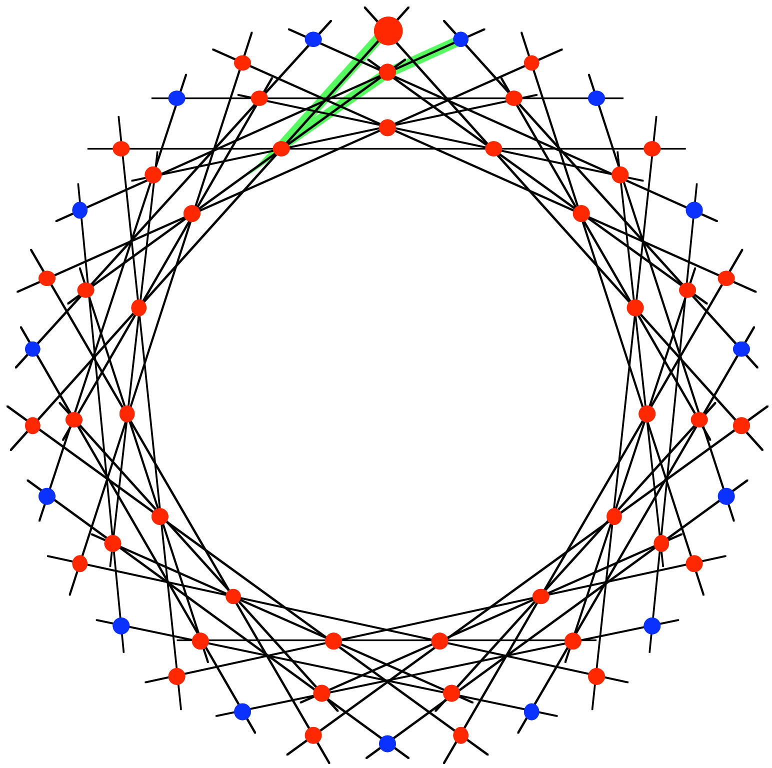


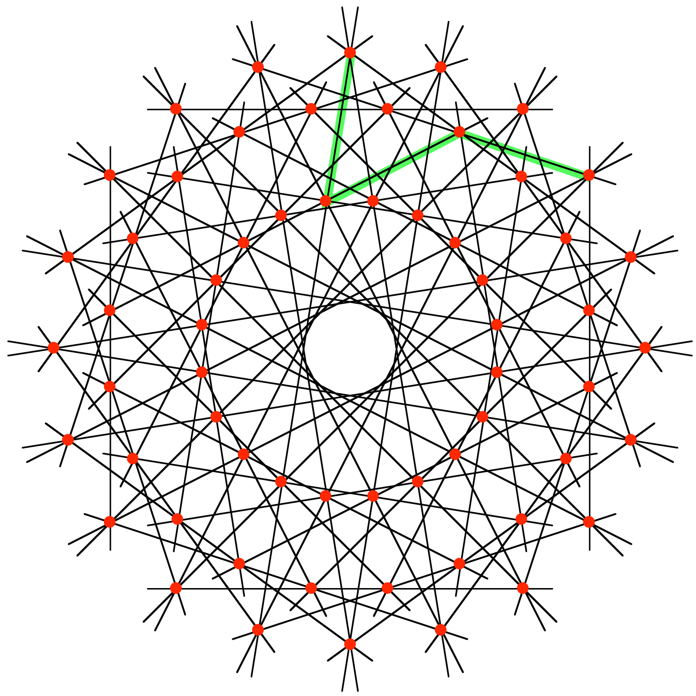
Figure 3.5.6. The symbol 15#(4,2;1,3;2,3) satisfies all conditions for a valid symbol of a 3-astral 4-configuration (45_4), except (A6). The characteristic path (green) that starts at the top point (large red) leads to a point (blue) at an in-between position. Both the starting point and the end point of the path are incident with just two lines each — hence the symbol does not correspond to any 4-configuration.

Since the endpoint of any characteristic path is in the same orbit as the starting point, this yields

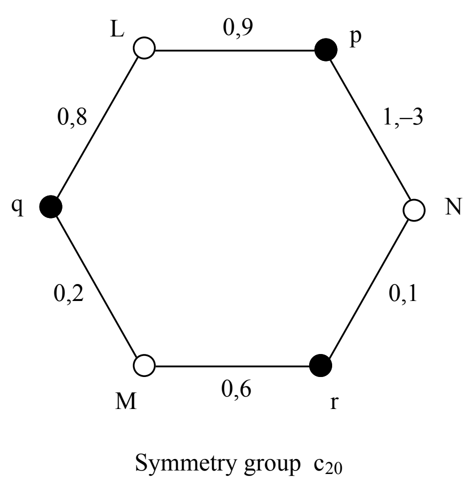
$$\textbf{(A7)} \qquad \prod_{1\le j\le k} \cos(s_j\cdot\pi/m) = \prod_{1\le j\le k} \cos(t_j\cdot\pi/m)$$

It is easy to verify that in all examples of k-astral configurations presented in this section the condition (A7) is fulfilled.

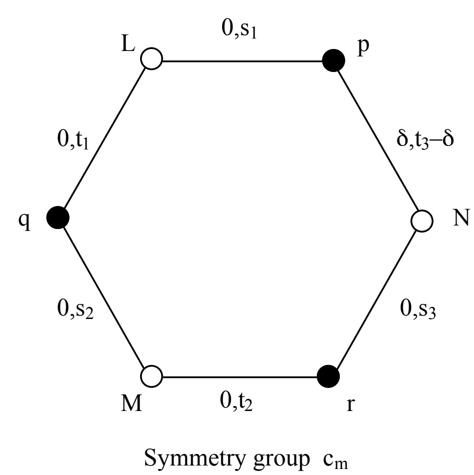
The appropriateness of the characteristic path approach to the notation for k-astral 4-configurations can be seen in the straightforward translation of the characteristic path into the reduced Levi diagram of the configuration. Without entering in lengthy descriptions of the procedure (which is essentially taken from Boben and Pisanski [B20]), we show in Figure 3.5.7 a typical example. The configuration 20#(9,8;2,6;1,4) and a characteristic path leading to this symbol are shown in part (a), while part (b) presents a reduced Levi diagram of this configuration. In part (c) we show the reduced Levi diagram of the



(a)



(b)



(c)

Figure 3.5.7. (a) The 3-astral configuration 20#(9,8;2,6;1,4) and a characteristic path . (b) The corresponding reduced Levi graph of 20#(9,8;2,6;1,4) . (c) The reduced Levi graph of the 3-astral configuration $m\#(s_1,t_1;s_2,t_2;s_3,t_3)$.

general case of a 3-astral configuration $m\#(s_1,t_1;s_2,t_2;s_3,t_3)$. The corresponding situation for a k-astral configuration differs only in the length of the circuit, so that it contains k white and k black points. The value of δ is determined by (A6).

Next, we explore what happens if the 2k entries between parentheses of a symbol $m\#(s_1,t_1;s_2,t_2; \ldots ;s_k,t_k)$ of a k-astral configuration C are changed by a cyclic permutation that moves them an *odd* number of steps. What — if anything — can we say about a configuration C* that would correspond to $m\#(t_1,s_2;t_2, \ldots ,s_k;t_k,s_1)$?

Considering the well-known relations between points and lines polar to them with respect to a circle of a given radius and center (illustrated in Figure 3.5.8, see also, for example, [<C12 Chapter 6]), we see that for a configuration corresponding to the symbol $m\#(t_1,s_2;t_2, \ldots ,s_k;t_k,s_1)$, the distance of the line L_j of C* that is polar to the point P_j of C with respect to a circle of unit radius should satisfy

$$\text{distance}(O, L_j) = OP_j^* = \prod_{1\le i\le j} \left(\cos(t_i\cdot\pi/m) / \cos(s_i\cdot\pi/m)\right) = 1/OP_j =$$

$$1/\prod_{1\le i\le j} \left(\cos(s_i\cdot\pi/m) / \cos(t_i\cdot\pi/m)\right) = \prod_{1\le i\le j} \left(\cos(t_i\cdot\pi/m) / \cos(s_i\cdot\pi/m)\right).$$

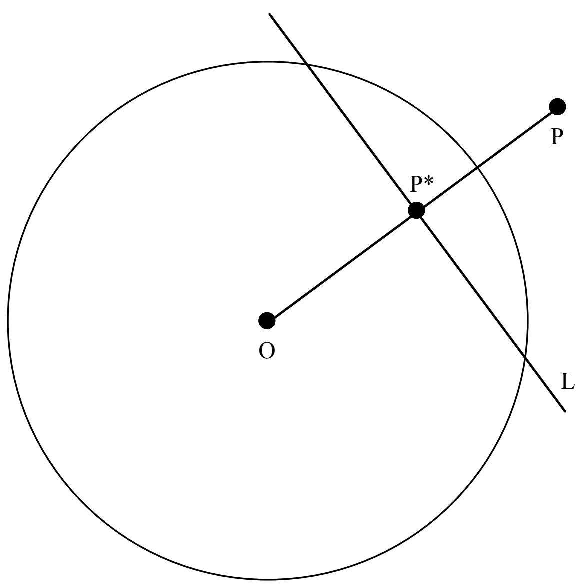


Figure 3.5.8. If the point P and the line L are polars of each other with respect to the circle of radius r and center O, then the distance between O and L is the same as the distance OP*, and the relation between the distances is $OP\cdot OP^* = r^2$.

Hence distances from the center O of all lines of the putative configuration C* are correct for them being the polars of the points of C, and since incidences and symmetry are all preserved under polarity, we can conclude:

Theorem 3.5.2. If the symbol $m\#(s_1,t_1;s_2,t_2; \dots ;s_k,t_k)$ corresponds to a k-astral 4-configuration then the symbol $m\#(t_1,s_2;t_2, \dots ,s_k;t_k,s_1)$ corresponds to a k-astral 4-configuration that is polar to the former with respect to the unit circle with center at the common center of both configurations.

A notational remark. Unless there is a definite reason to do otherwise, we shall always strive to use the lexicographically highest symbol for each k-astral 4-configuration.

Several concepts simplify the listing and classification of possible k-astral configurations. One is based on the observation that if we switch the positions of two entries separated by an odd number of other entries in the symbol of an astral configuration, the modified symbol automatically satisfies all the conditions listed above, except possibly (A5). By repeated application of this observation while avoiding a violation of (A5), we arrive to the conclusion that it is sensible to introduce the **cohort concept** and **notation**. For a k-astral configuration with symbol $m\#(s_1,t_1;s_2,t_2; \dots ;s_k,t_k)$ the **cohort symbol** is $m\#\{s, t\} = m\#\{\{s_1,s_2, \dots ,s_k\},\{t_1,t_2, \dots ,t_k\}\}$; it stands for all the valid assignments of suitable permutations of the s_i's and permutations of the t_i's into a symbol for a k-astral configuration. For example, for the configuration in Figure 3.5.1(a) the symbol is 14#(5,3;2,4;1,3), and the cohort symbol is 14#{{5,2,1},{4,3,3}}; This cohort symbol indicates, and is shared by, the six distinct 3-astral configurations 14#(5,4;2,3;1,3), 14#(5,3;2,4;1,3). 14#(5,3;2,3;1,4), 14#(5,4;1,3;2,3), 14#(5,3;1,4;2,3), and 14#(5,3;1,3;2,4).

The second comes from the observation that all the conditions, except possibly (A5), are satisfied if in the cohort symbol m#{s,t} the sets s and t are the same. As an example, the configuration we used in Figure 3.5.3 has symbol 9#(4,2;3,4;2,3), hence s = t ={4,3,2}. Since the condition (A7) is satisfied without the need to make any calculations, we shall say that the cohort 9#{{4,3,2},{4,3,2}} is **trivial**. On occasion we shall use "trivial" also for an individual configuration in a trivial cohort. For odd k, a typical representative of a trivial cohort is m#(a,b;c, ... ;u,v;w,a;b,c; ... ,u;v,w), while for even k we can use m#(a,b;c,d; ... ;v,w;b,a;d,c; ... ;w,v). We should mention here that there cannot be any 2-astral configurations of the trivial type.

We shall say that a cohort symbol m#{s,t} of k-astral configurations is **systematic** provided m and the elements of s and t depend on one or more parameters in such a way that the validity on (A7) can be ascertained using only trigonometric identities and without the need to calculate specific values of the parametrized s_i's and t_i's. As we shall illustrate in Section 3.6, the cohort with m = 6k, s = {3k-j,j}, t = {3k-2j,2k} is a systematic 2-astral cohort.

If a k-astral configuration belongs neither to a trivial cohort nor to a systematic one, we shall say that the configuration and its cohort are **sporadic.** For k = 2 all sporadic configurations are known, and we list them in Section 3.6. However, already for k = 3 we have only examples of such configurations (as discussed in Section 3.7), but no complete characterization.

If a cohort symbol m#{s,t} of a k-astral configurations contains the same integer in both s and t, a cohort symbol of a (k-1)-astral configurations may result if this integer is deleted from both s and t. As is easily verified, all the conditions for (k-1)-astral symbol are automatically verified, except possibly (A5). We call **clade** of m#{s,t} all the cohorts resulting from one or several applications of this procedure. This will be illustrated in Section 3.7.

* * * * * *

We end this section with an unsolved problem of methodology in the study of k-astral configurations. We required in the definition that the symmetry group of every astral configuration is d_k. In fact, the other conditions show that this happens automatically if we require that the cyclic group c_k is a symmetry group of the configuration. The characteristic path, the symbol, and the reduced Levi graph of each k-astral configuration are all based on the cyclic group. The reason is that (as far as I know) nobody has come up with a reasonable version of all these based on the dihedral symmetry group. The construction of a reduced Levi graph that is based on the dihedral symmetry is certainly feasible – but does not appear to be useful. How come?

Exercises and problems 3.5

1. Devise symbols for the configurations in Figure 3.5.9.

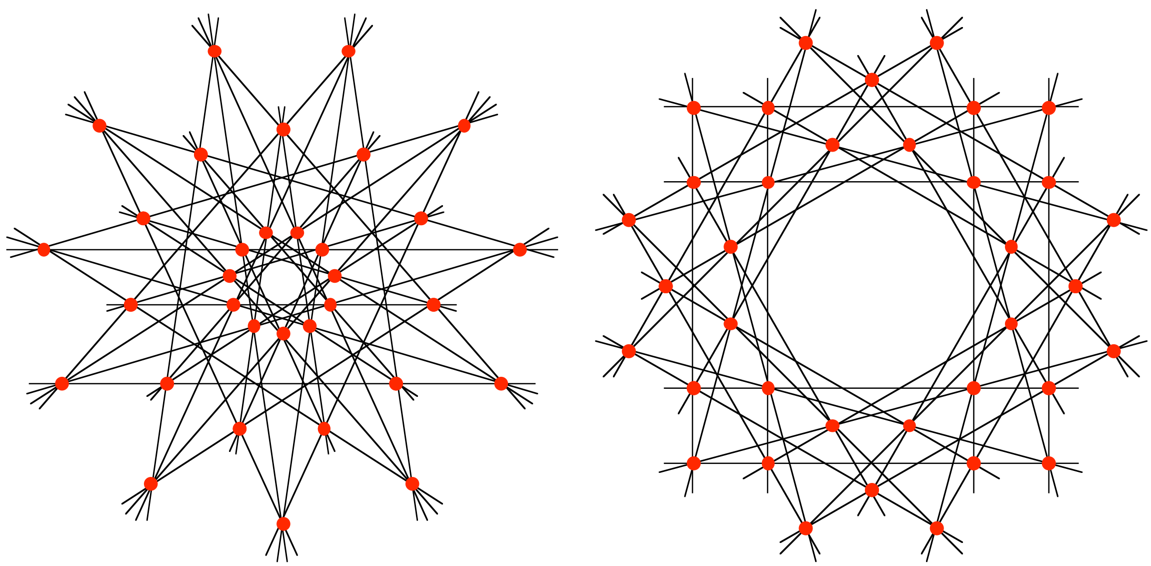


Figure 3.5.9. Two 3-astral 4-configurations.

2. What are the symbols for the objects in Figure 3.5.10. Devise a characteristic path in each and find out.

3. Superimpose each object in Figure 3.5.10 with a copy rotated 12° about the center. What is the result? Can you find a symbol for it?

4. Find the dual configurations of the ones in Figure 3.5.9.

5. Find a symbol for the 4-configuration in Figure 3.5.11.

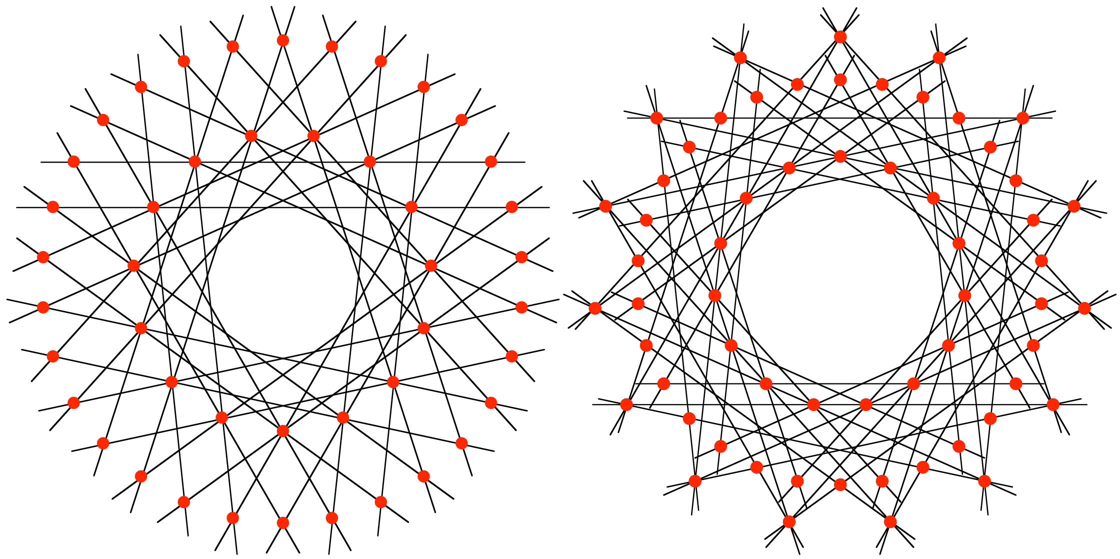


Figure 3.5.10. Not configurations!

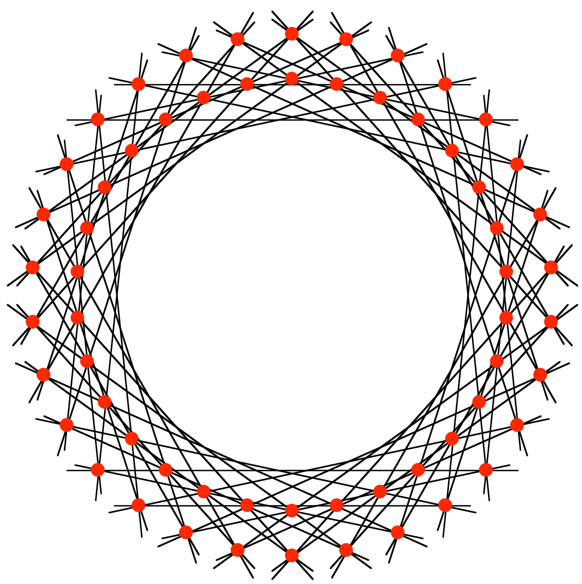


Figure 3.5.11. A configuration ((60_4)).

3.6 2-ASTRAL 4-CONFIGURATIONS

Following the terminology introduced in Section 3.5, a geometric 4-configuration (that is, an (n_4) configuration for some integer n) is called **2-astral** provided there are precisely two orbits of points and two orbits of lines under the symmetry group of the configurations, and the other conditions spelled out in Section 3.5 are satisfied. Since k = 2 is the smallest value of k possible in a 4-configuration, following the convention proposed in Section 1.5 we shall call such configurations **astral** for short. An astral 4-configuration cannot have points at infinity, since any line through such a point would have to have three points of a single orbit in the finite part of the plane. Hence we need to consider only what happen in the Euclidean plane.

The astral 4-configurations have been completely characterized. To present this characterization we need an appropriate notation; this was set up in Section 3.5. Here we shall present the list of these astral configurations (Theorem 3.6.1). Before giving the proof that our list is complete we have to digress into explanations of some of the detailed results about intersection of diagonals in regular polygons — a topic that has its own interesting and convoluted history. Finally, a proof of completeness of the list will be given; the first such proof is that of L. Berman [B3], [B4].

The notation for astral 4-configurations has evolved in several stages since the first publication on the topic in [G39]. The notation used here, introduced in Section 3.5, is the one that was found most suitable for the present purpose as well as for the generalization to k-astral 4-configurations that we shall consider in Section 3.7. The notation is explained by the example of a (48_4) astral configuration shown in Figure 3.5.5. One of its symbols is 24#(8,7;2,5); the configuration belongs to the cohort 24#{{8,2},{7,5}}; this cohort contains only one other configuration, with symbol 24#(8,5;2,7). Both configurations are shown in Figure 3.6.1.

In the next two figures we show the smallest astral 4-configurations. The unique (24_4) is shown in Figure 3.6.2, while the six configurations (36_4) appear in Figure 3.6.3. Additional illustrations appear in several other sections, but in particular in Section 5.9.

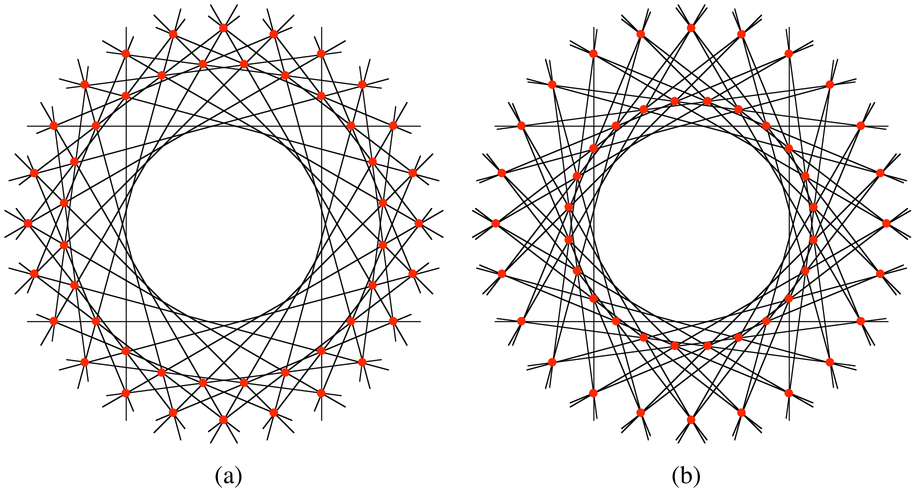


Figure 3.6.1. The only two 2-astral configurations (48_4) in the cohort 24#{{8,2},{7,5}}. (a) The configuration 24#(8,7;2,5). (b) The configuration 24#(8,5;2,7).

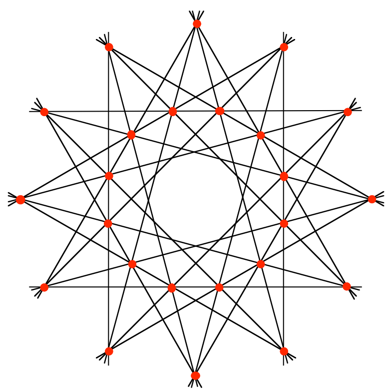
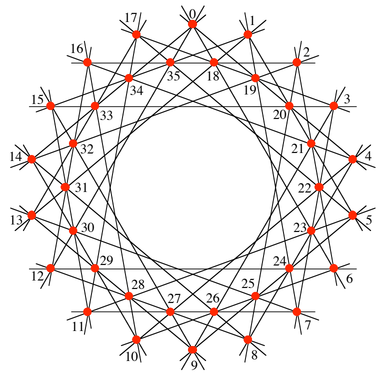
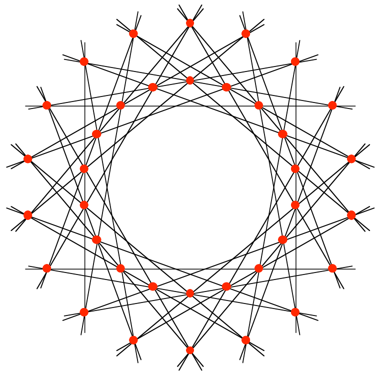


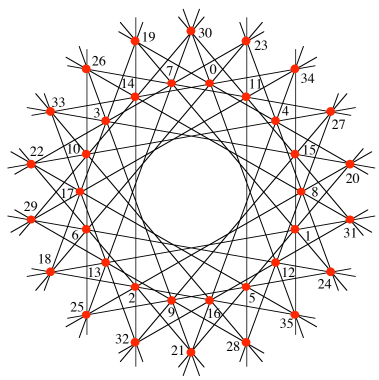
Figure 3.6.2. The smallest astral 4-configuration. It is a sporadic and selfdual (24_4), with symbol 12#(5, 4; 1, 4).



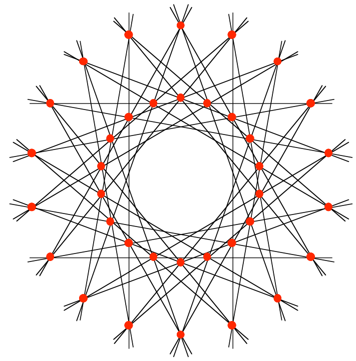
18#(6, 5; 1, 4)



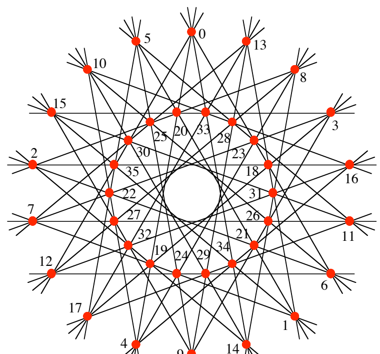
18#(6, 4; 1, 5)



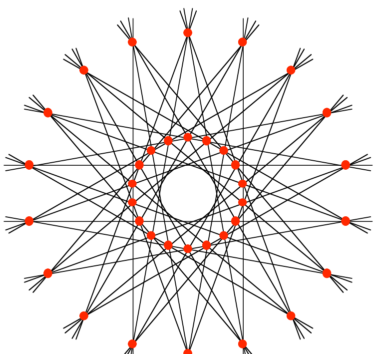
18#(7,6;2,5)



18#(7,6;2,5)



18#(8,7;1,6)



18#(8,6;1,7)

Figure 3.6.3. The six configurations (36_4) belong to three cohorts: 18#{{6,1},{5,4}}, 18#{{7,2},{6, 5}, 18#{{8,1},{7,6}}. Near each configuration we show the lexico-

graphically highest among its symbols. Although it is not obvious from the symbols or the diagrams, the three configurations at left are isomorphic. This isomorphism is established by the labels near their vertices. Since these configurations are isomorphic, their polars (shown at right) are also isomorphic to each other; they are not isomorphic to the other three configurations.

After these preliminaries, here is the detailed result.

Theorem 3.6.1.[1] Astral 4-configurations $m\#(s_1, t_1; s_2, t_2)$ must satisfy all the conditions from Section 3.5, and in particular the equation (A7):

$$\cos(s_1\pi/m)\cdot\cos(s_2\pi/m) = \cos(t_1\pi/m)\cdot\cos(t_2\pi/m).$$

The symbols of these configurations are:

(i) The *systematic configurations* with symbols (6k)#(3k-j, 2k; j, 3k-2j) for $k \geq 2$, $1 \leq j < 3k/2$, with $j \neq k$.

(ii) The *systematic configurations* with symbols (6k)#(2k, j; 3k-2j, 3k-j) for $k \geq 2$, $1 \leq j < 3k/2$, with $j \neq k$. By the general results of Section 3.5, these configurations are polar to the ones in (i) with the same values of k and j.

For even k and j = k/2, the configurations in (i) and (ii) are selfpolar, hence coincide. If $k = f\cdot g$ and $j = f\cdot h$, with $f \geq 2$, $g \geq 2$, then both (6k)#(3k-j, 2k; j, 3k-2j) and (6k)#(2k, j; 3k-2j, 3k-j) are disconnected. Each consists of f equidistributed copies of (6g)#(3g-h, 2g; h, 3g-2h) or (6g)#(2g, h; 3g-2h, 3g-h) and is denoted by (f) (6g)#(3g-h, 2g; h, 3g-2h) or (f) (6g)#(2g, h; 3g-2h, 3g-h), respectively.

For simpler formulation, we can say that the configurations in (i) and (ii) are in the cohorts of (6k)#{{3k-j},{3k-2,2k}} for $k \geq 2$, $1 \leq j < 3k/2$, with $j \neq k$.

(iii) The 27 symbols of the *sporadic configurations* listed in Table 3.6.1, and their multiples.

---

[1] I am indebted to L. Berman and T. Pisanski for a number of comments and corrections. These led to the present formulation, which I hope is more informative and useful than the statements in previous publications.

Branko Grünbaum 6/2/08 8:33 AM
Comment: P. 3.6.6, L. 1: 3k-j

Branko Grünbaum 6/2/08 8:32 AM
Comment:

30#(7,6;1,4)	30#(7,4;1,6)
30#(8,6;2,6)	
30#(11,10;1,6)	30#(11,6;1,10)
30#(12,10;6,10)	
30#(12,11;2,7)	30#(12,7;2,11)
30#(13,12;1,8)	30#(13,8;1,12)
30#(13,12;7,10)	30#(13,10;7,12)
30#(14,12;4,12)	
30#(14,13;6,11)	30#(14,11;6,13)
42#(13,12;1,6)	42#(13,6;1,12)
42#(18,17;6,11)	42#(18,11;6,17)
42#(19,18,5,12)	42#(19,12;5,18)
60#(22,21;2,9)	60#(22,9;2,21)
60#(25,24;5,12)	60#(25,12;5,24)
60#(27,26;3,14)	60#(27,14;3,26)

Table 3.6.1. The complete list of connected sporadic astral 4-configurations. The three stand-alone symbols denote selfpolar configurations, the paired symbols correspond to configurations polar to each other.

Here too, the cohorts notation allows a more condensed listing:
30#{{7,1}{6,4}}, 30#{{8,2},{6,6}}, 30#{{11,1},{10,6}}, 30#{{12,6},{10,10}}, 30#{{12,2},{11,7}}, 30#{{13,1}12,8}}, 30#{{13,7},{12,10}}, 30#{{14,4},{12,12}}, 30#{{14,6},{13,11}},
42#{{13,1},{12,6}}, 42#{{18,6},{17,11}}, 42#{{19,5},{18,12}},
60#{{22,2},{21,9}}, 60#{{25,5},{24,12}}, 60#{{27,3},{26,14}}.

The proof of the theorem will be interwoven with an account of the history of its development. In view of all the interest in configurations during the last quarter of the 19th century (as well as the sporadic interest later), it is hard to understand that *no graphical representation* of **any** 4-configuration appeared in print prior to [G50] in 1990. The configuration shown above as Figure 3.6.2 was one of the configurations shown in that paper. Another was the (21_4) configuration that gave the paper its title; we shall encounter it again in Section 3.7.

In the early 1990 I found several 4-configurations in addition to the ones in [G50], with two or three orbits of points (and of lines); these were found by drawing with such software as was available to me (mainly MacDraw), until I was initiated to Mathematica® through friendly persuasion by Stan Wagon. (A few other k-astral 4-configurations with various k's were communicated to my by J. F. Rigby.) With programs in Mathematica it was possible to "experimentally" find all possible astral configurations with reasonably small numbers of vertices. This led to the understanding that there are systematic infinite families of such configurations, as well as an apparently finite number of sporadic configurations. I became convinced that I have a complete description, and presented this in seminars and courses during the 1990s; the results were published in 2000 [G40], together with formal demonstrations of the geometric realizability of these configurations. This covers the existence aspect of Theorem 3.6.1.

The main tool for the proof of completeness was the observation that an astral configuration $m\#(s_1, t_1; s_2, t_2)$ has a realization by straight lines if and only if the same points are reached starting from one of the regular polygons regardless of which of two diagonals we are following. In other words, the points described by $(s_1//t_1)$ must coincide with the points $(t_2//s_2)$. (Note that the designation $(s_2//t_2)$ used in determining the symbol of the configuration refers to the diagonals as looked from the other polygon.) This leads to the following necessary condition for the existence of an astral configuration $m\#(s_1,t_1;s_2,t_2)$

(1) $$\cos(s_1\pi/m)\cdot\cos(s_2\pi/m) = \cos(t_1\pi/m)\cdot\cos(t_2\pi/m).$$

Due to the dihedral symmetry of such configurations, this is also a sufficient condition for the existence. Moreover, criterion (1) is easily implemented for computational searches; the results of these calculations led to the classes listed in Theorem 3.6.1.

For the convenience of use of Theorem 3.6.1 we list in Table 3.6.2 the cohort symbols of the systematic astral configurations (n_4) with $n \le 100$.

12#{{5,1},{4,4}.

18#{{8,1},{7,6}}, 18#{{7,2},{6,5}}, 18#{{6,1},{5,4}}.

24#{{11,1},{10,8}}, *24#{{10,2},{8,8}},* 24#{{9,3},{8,6}}, 24#{{8,2},{7,5}}.

30#{{14,1},{13,10}}, 30#{{13,2},{11,10}}, 30#{{12,3},{10,9}},
30#{{11,4},{10,7}}, 30#{{10,3},{9,6}}, 30#{{10,1},{8,7}}.

36#{{17,1},{16,12}}, *36#{{16,2},{14,12}}*, *36#{{15,3},{12,12}}*, *36#{{14,4},{12,10}},*
36#{{13,5},{12,6}}, 36#{(12,4},{11,7}}, *36#{{12,2},{10,8}}.*

42#{{20,1},{19,14}}, 42#{{19,2},{20,16}}, 42#{{18,3},{15,14}},
42#{{17,4},{14,13}}, 42#{{16,5},{14,11}}, 42#{{15,6},{14,9}},
42#{{14,5},{13,8}}, 42#{{14,3},{12,9}}, 42#{{14,1},{11,10}}.

48#{{23,1},{22,16}}, *48#{{22,2},{20,16}}*, 48#{{21,3},{18,16}},
48#{{20,4},{16,16}}, 48#{{19,5},{16,14}}, *48#{{18,6},{16,12}}*,
48#{{17,7},{16,10}}, 48#{{16,6},{15,9}}, *48#{{16,4},{14,10}},*
48#{{16,2},{13,11}}.

Table 3.6.2. The cohort symbols of all systematic astral (n_4) configurations with $n \leq 100$. Disconnected configurations are in italics.

Once the characterization of the astral configurations has been guessed, it is easy to see that the symbols listed above correspond to actual geometric configurations, and are not results of an approximation error in the computations.

Indeed, for the symbols in part (i) we have to show that

(2) $$\cos((3k-j)\pi/(6k))\cdot\cos(\pi(6k)) = \cos(2k\pi/(6k))\cdot\cos((3k-2j)\pi/(6k)).$$

In view of the trigonometric identity

(3) $$(\cos a)\cdot(\cos b) = \tfrac{1}{2}\,(\cos(a+b) + \cos(a-b))$$

validity of (2) is equivalent to

$$\tfrac{1}{2}\,(\cos 3k\pi/(6k) + \cos((3k-2j)\pi/(6k))) = (\cos \pi/3)\cdot\cos((3k-2j)\pi/(6k)).$$

Since $\cos \pi/2 = 0$ and $\cos \pi/3 = \tfrac{1}{2}$, this is valid for all k and j; hence (2) is correct. The same calculation shows that the symbols in (ii) correspond to astral geometric configurations as well. The fact that the above arguments did not rely on particular values of the cosines involved shows that (i) and (ii) are symbols of systematic configurations.

The existence of the sporadic configurations proceeds somewhat analogously, but needs to rely on information specific to the angles involved. For example, concerning the configuration 30#(8,6;2,6) we note that (1) becomes

$$\cos 8\pi/30 \cdot \cos 2\pi/30 = (\cos 6\pi/30)^2,$$

which by (3) is equivalent to

$$\tfrac{1}{2} (\cos 10\pi/30 + \cos 6\pi/30) = (\cos 6\pi/30)^2$$

Since $\cos \pi/3 = \tfrac{1}{2}$ and $\cos \pi/5 = \tfrac{1}{4} (1 + \sqrt{5})$, this reduces to

$$(12 + 4\sqrt{5})/16 = (6 + 2\sqrt{5})/8,$$

which is obviously true.

Using other explicit algebraic values for cosines, similar arguments can be made for the other sporadic configurations with symbols that start with 30 or 60. Among the values that can be used are

$$\cos 2\pi/30 = (-1 + \sqrt{5} + \sqrt{6(5 + \sqrt{5})})/8,$$

$$\cos 4\pi/30 = (1 + \sqrt{5} + \sqrt{6(5 - \sqrt{5})})/8,$$

$$\cos 8\pi/30 = (1 - \sqrt{5} + \sqrt{6(5 + \sqrt{5})})/8,$$

and so on.

For the symbols that involve 42 it is convenient to follow a slightly different path. The validity of the first of these symbols, 42#(13,12;1,6), is by (1) and (3) equivalent to

$$\cos \pi/3 + \cos 2\pi/7 = \cos 3\pi/7 + \cos \pi/7$$

that is

$$1 + 2\cos 2\pi/7 + 2\cos 4\pi/7 + 2\cos 6\pi/7 = 0.$$

But this is simply an expression of the fact that the centroid of a regular heptagon, centered at the origin and with one vertex at (1,0), is itself at the origin. An completely analogous reasoning shows the validity of the other symbols involving 42.

What is still missing is a proof that there are no other astral 4-configurations. Since these configurations are determined by intersections of diagonals of regular polygons, and since these have been extensively studied and completely determined, in the late 1990's it seemed to me that it should be very easy to supply the proof of completeness.

In reality this task proved far from simple, and it was first successfully carried out in 2001 in the PhD work of L. Berman [B4], [B3]. Berman's rather complicated argumentation relied on the complete description of intersections of diagonals of regular polygons, given by Poonen and Rubinstein [P8] in 1998. Theirs was a new proof (and a much more convenient presentation) of material that has been contained, to a large extent, in earlier publications of G. Bol [B26] in 1933 (with some misprints) and J. F. Rigby [R3] in 1980[2]. For regular n-gons with prime n, or with any odd n, it has been proved by many authors that there are no intersections of more than two diagonals; references to these papers and other related material can be found in [R3], and especially in [P8].

However, independently of these developments, an approach that is easier to apply for our purposes was published by G. Myerson [M21] in 1993; it came to my attention only recently. Myerson's result (his Theorem 4) that is relevant to our proof can be formulated as follows.

Theorem 3.6.2. (Myerson [M21]) . The equation

$$\sin \pi/6 \cdot \sin t = \sin(t/2) \sin(\pi/2 - t/2)$$

is valid for all t. The only other solutions of the equation

(4) $$\sin x_1\pi \cdot \sin x_2\pi = \sin x_3\pi \cdot \sin x_4\pi$$

in rational x_1, x_2, x_3, x_4 with $0 < x_1 < x_3 \le x_4 < x_2 \le ½$ are given in Table 3.6.3.

---

2 In contrast to other writers on the topic, Rigby considers the multiple intersections of diagonals outside the n-gon as well. However, his intended [R3, p. 222] investigation of outside intersections of four or more diagonals seems not to have been published, and remains an open problem.

Label	x_1	x_2	x_3	x_4
1	1/21	8/21	1/14	3/14
2	1/14	5/14	2/21	5/21
3	4/21	10/21	3/14	5/14
4	1/20	9/20	1/15	4/15
5	2/15	7/15	3/20	7/20
6	1/30	3/10	1/15	2/15
7	1/15	7/15	1/10	7/30
8	1/10	13/30	2/15	4/15
9	4/15	7/15	3/10	11/30
10	1/30	11/30	1/10	1/10
11	7/30	13/30	3/10	3/10
12	1/15	4/15	1/10	1/6
13	2/15	8/15	1/6	3/10
14	1/12	5/12	1/10	3/10
15	1/10	3/10	1/6	1/6

Table 3.6.3. The complete list of sporadic solutions of equation (4) as given by Myerson in [M21].

The result of Theorem 3.6.2 gives an immediate solution to the completeness question of Table 3.6.1. Indeed, we only have to recall that $\sin \alpha = \cos(\pi/2 - \alpha)$ in order to see that the rows of Table 3.6.3 correspond (in an appropriate permutation) to the rows of Table 3.6.1. For example, rows with labels 1, 2, 3 correspond to the entries involving 42 of the earlier table, while those labeled 4, 5, and 14 correspond to the last three rows of table 3.6.1.

This completes the proof of Theorem 3.6.1.

Exercises and problems 3.6.

1. Verify the complete correspondence between Myerson's list in Table 3.7.3 and the list of sporadic symbols in Table 3.6.1.

2. Verify the validity of the existence claims made above for all sporadic configurations.

3. Draw the configuration 36#(15,12; 3,12) = (3) 12#(5,4;1,4). Is it selfpolar?

4. Prove that the configurations 18#(6,5;1,4) and 18#(6,4;1,5) shown in Figure 3.6.3 are not isomorphic.

5. Determine whether the pairs of polar configurations in Figure 3.6.3 are in appropriate orientation to exhibit the polarity, or does one member of the pair have to be rotated.

3.7 3-ASTRAL 4-CONFIGURATIONS

The 3-astral 4-configurations have a lot in common with the 2-astral configurations we studied in Section 3.6, but they also have many properties and peculiarities that are not present in the earlier case. This is the main reason for treating them in a separate section.

It seems to me that the 3-astral configurations are arguably the most interesting type of configurations. The reason for this assessment is that they are more general in the opportunities for investigation than the 2-astral 4-configurations considered in the preceding section, but are still experimentally quite accessible. As we shall show, there are many open problems that seem very attractive, as well as tractable with an appropriate investment of effort. Naturally, k-astral configurations with $k \geq 4$ have their own attraction and appeal, but with increasing k they are harder to investigate and, in any case, much less is known about them.

The first graphic presentation in a published paper of any 4-configuration[1] was that of a 3-astral (21_4) in [G50]; here we show it in Figure 3.7.1. As with 2-astral configurations, 3-astral ones will be illustrated in various section; several are shown in Section 5.9.

The notation we use for the k-astral configurations is the one introduced in Section 3.5, based on characteristic paths. In the case of 3-astral configurations there are, in general, 6 different characteristic paths, leading to distinct symbols. We preferentially choose the symbol that is lexicographically the highest.

For 3-astral configurations our approach is completely analogous to the treatment of 2-astral configurations in Section 3.6. It would be nice if at this stage we could formulate a theorem analogous to Theorem 3.6.1, and give necessary and sufficient conditions for symbols corresponding to geometric 3-astral 4-configurations. However, we have

1 Kárteszi [K5] in 1986 and Zeitler [Z8] in 1987 came very close to finding such 4-configurations. In the diagrams they show one only has to delete some lines, and make all copies of one of the shown lines, to get a 3-astral 4-configuration, with m = 10 in the former and m = 12 in the other.

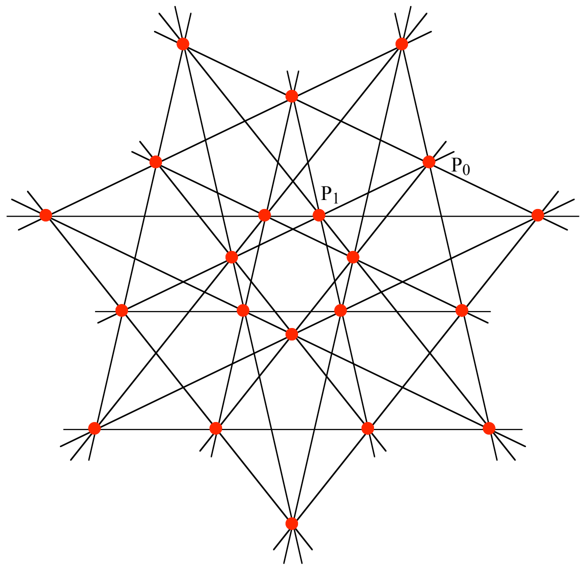


Figure 3.7.1. The 3-astral 4-configuration (21_4). This configuration is 3-astral with symbol 7#(3,2;1,3;2,1) obtained from the characteristic path that starts at P_0 and has as its next point P_1. This configuration belongs to the "trivial" type.

only partial information; the most important criterion is the condition (A7) from Section 3.5, the analog of the trigonometric condition in Theorem 3.6.1:

Theorem 3.7.1. If $m\#(s_1,t_1;s_2,t_2;s_3,t_3)$ is the symbol of a geometric 3-astral 4-configuration then

(*) $$\cos s_1\pi/m \cdot \cos s_2\pi/m \cdot \cos s_3\pi/m = \cos t_1\pi/m \cdot \cos t_2\pi/m \cdot \cos t_3\pi/m.$$

This is an expression of the fact that each characteristic path has its endpoint in the same orbit as its starting point. Conversely, if a symbol satisfies (*) and the natural conditions listed in Section 3.5 then *in general* there exists a geometric 3-astral 4-configuration with the symbol in question.

The natural conditions just mentioned are:

(**) No entry is equal to either of the two adjacent entries, the first and last considered as adjacent;

(***) Each entry is smaller than m/2.

(****) The sum of all s_j and t_j entries in the symbol is even.

There are two deep deficiencies in this theorem. One unsatisfactory aspect of Theorem 3.7.1 (and of the analogous statements one can make for $k \geq 4$) is that we do not have any analogue of Myerson's Theorem 3.6.2, hence we cannot devise a list of all the configurations in question. As is stated in [M21], Myerson's methods could probably lead to a complete, explicit list of solutions of (*), but this appears to be a momentous task — a task that has not been carried out. This is the first big problem concerning 3-astral 4-configurations.

The other problem is euphemistically covered by the italicized words "in general". We shall discuss later in the section the known and the unknown results in this direction.

For the presentation of *known* solutions of (*) satisfying all the necessary conditions it is convenient to distinguish three kinds of symbols $m\#(s_1,t_1;s_2,t_2;s_3,t_3)$ or the corresponding cohorts $m\#\{\{s_1,s_2,s_3\},\{t_1,t_2,t_3\}\}$, which we shall call "trivial", "systematic", and "sporadic". The terminology was introduced in Section 3.5, and here we only briefly remind the reader of the meaning of these terms.

- In accordance with this terminology, **trivial** symbols (and 3-astral configurations) have the form m#(b,c;d,b;c,d), where b, c, d are different positive integers, each less than m/2. Since the terms on the two sides cancel each other without any calculations and the other conditions are automatically satisfied, the label "trivial" seems appropriate — not in any derogatory sense but as describing the mathematically simplest case. In other words, trivial are those astral configurations for which the cohort symbol $m\#\{s,t\}$ is of the special form $m\#\{s,s\}$. Figure 3.7.1 shown an example of a trivial 3-astral configuration. From the general properties of equivalent symbols and symbols of dual configuration discussed in Section 3.5 it follows that all trivial 3-astral configurations are selfdual; the same applies to all trivial k-astral configurations with odd k. Indeed, m#(b,c,d,b,c,d) has as dual m#(c,d,b,c,d,b), which is the same as the original; the argument is analogous for other odd k. Obviously, the polar of a trivial configuration is itself trivial.

- **Systematic** symbols are those that contain infinite families for which the validity can be verified by *formal* trigonometric calculations, without the need to determine *val-*

ues of the trigonometric functions that depend on specific parameters. At present, four such families m#{s,t} of 3-astral 4-configurations are known, mostly through unpublished work of L. Berman.

(1) m = 2q, s = {q-p,q-2r,p}, t = {q-2p,q-r,r}

(2) m = 3q, s = {q+p,q-p,p}, t = {q,q,3p}

(3) m = 6q, s = {3q-p,r,p}, t = {3q-2p,2q,r}

(4) m = 10q, s = {5q-p,2p,p}, t = {|5q-4p|,4q,2q}.

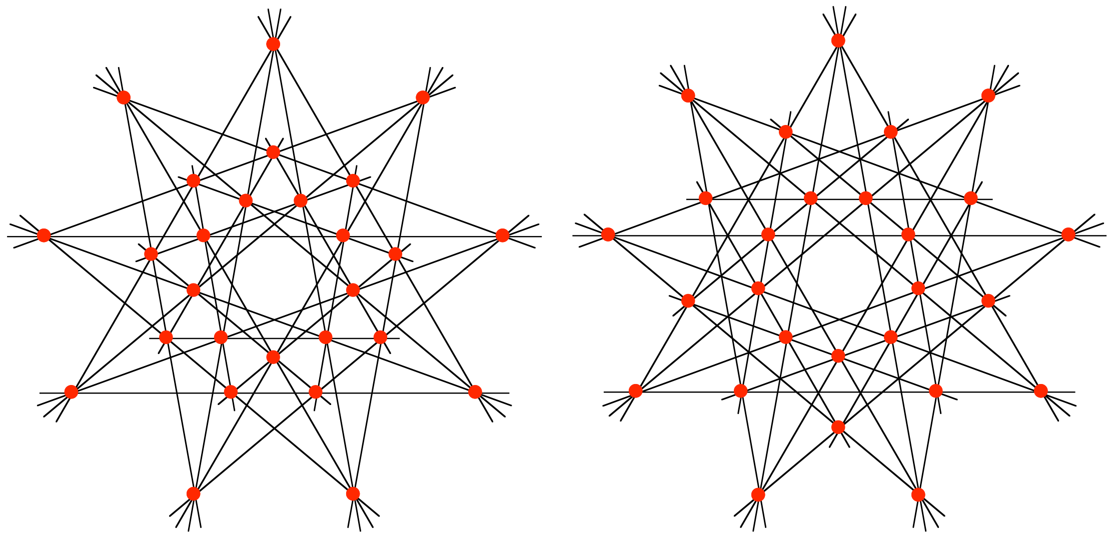
Here p, q and r are any positive integers, and the possibilities have to be understood in the sense of cohorts, that is, all permutations within s and within t, as well as interchanging s and t, are allowed provided conditions (**) and (***) are satisfied; condition (****) is fulfilled automatically. If all the entries are distinct the cohort contains 12 distinct configurations; equality of some entries reduces this number.

For example, in family (4) with m = 20, q = 2. For p = 1 we have s = {9,2,1} and t = {8,6,4}; this leads to a total of 12 distinct symbols: 20#(9,8;2,6;1,4), 20#(9,8;2,4;1,6), 20#(9,6;2,8;1,4), 20#(9,6;2,4;1,8), 20#(9,4;2,8;1,6), 20#(9,4;2,6;1,8), and six more in which the positions of 1 and 2 are interchanged. Switching the two sets of parameters yields additional 12 symbols, but no new configurations, since these symbols are equivalent to the earlier dozen. For p = 3 we have s = {7,6,3} and t= {8,4,2} leading again to 12 different symbols. For p = 2 and p = 4 we get trivial symbols only, while $p \geq 5$ exceeds the bound in (***). On the other hand, the cohort of 9#{{4,2,1},{3,3,3}} consists of just two configurations, shown in Figure 3.7.2. As mentioned above, in some cases the resulting symbols become trivial.

The verification that the above symbols of the four families satisfy condition (*) is quite straightforward. We illustrate this only for family (2), for which condition (*) reduces to the verification of

$$\cos(q+p)\pi/m \cdot \cos(q-p)\pi/m \cdot \cos p\pi/m = (\cos q\pi/m)^2 \cdot \cos 3p\pi/m.$$

Since q = m/3, the value of $\cos q\pi/m = \cos \pi/3 = ½$, and standard trigonometric identities yield



(a) (b)

Figure 3.7.2. The only two distinct members in the cohort 9#{{4,2,1},{3,3,3}} of family (2) for q = 3, p = 1. (a) 9#(4,3,2,3,1,3); (b) 9#(4,3,1,3,2,3).

$$2 \cos (q+p)\pi/m \cdot (\cos q\pi/m + \cos (q-2p)\pi/m) = \cos 3p\pi/m \text{ , so}$$

$$\cos (q+p)\pi/m + 2 \cos (q+p)\pi/m \cdot \cos (q-2p)\pi/m = \cos 3p\pi/m \text{ ,}$$

$$\cos (p\pi/m + \pi/3) + \cos (2q-p)\pi/m + \cos 3p\pi/m = \cos 3p\pi/m \text{ , and finally}$$

$$\cos (p\pi/m + \pi/3) + \cos (-p\pi/m + 2\pi/3) = 0,$$

which is obviously true.

Similar calculations validate the other families of symbols.

- **Sporadic** symbols and configurations are those that do not belong to any of these two families. For example, 18#(5,4;1,3;1,2) shown in Figure 3.7.3 is sporadic — *at least for the time being*. The reason for this qualification is that although the symbol is neither trivial nor belongs to one of the four families in (ii), it may well be part of a still not discovered infinite (systematic) family.

In Table 3.7.1 we give a list of cohorts of the sporadic 3-astral configurations (n_4) with $n \leq 108$. It was obtained by numerically solving equation (*), and eliminating duplicates and symbols that correspond to trivial or systematic 3-astral configuration. An

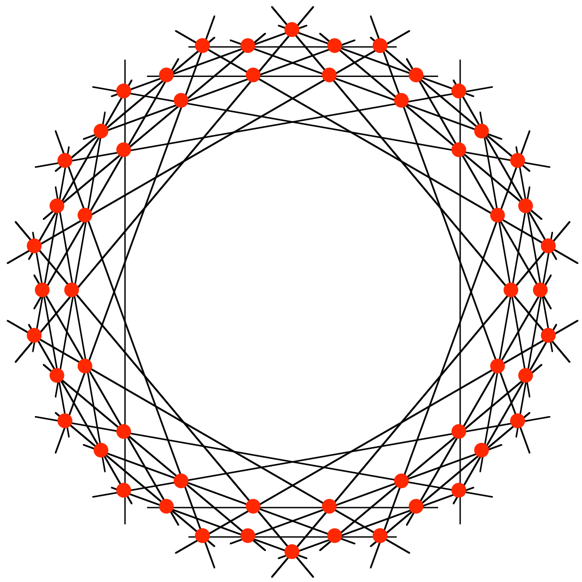


Figure 3.7.3. The sporadic configuration 18#(5,4;1,3;1,2).

unexpected result of these computations is that such sporadic configurations exist only for n that are multiples of 12.

One additional comment concerning Table 3.7.1. Some of the cohort symbols contain the same symbol in both parts. This implies that the crucial relation (*) will be satisfied even if the symbol is deleted from both parts. In such a case the **reduced** cohort symbol belongs to a 2-astral 4-configuration.This brings us to other open problems:

- Do there exist any other systematic families besides the ones in (ii) above?
- Is the list of connected sporadic 3-astral configurations finite?

It is worth noting that there is no known visible cue in a given 3-astral configuration whether it is trivial, systematic, or sporadic. It takes working out its symbol and looking at the criteria in order to decide where it belongs.

We turn now to the second deficiency in Theorem 3.7.1. It parallels the problems with the Steinitz theorem on 3-configurations encountered in Chapter 2, and did not arise for 2-astral 4-configurations.

m = 18

{5, 1, 1}, {4, 3, 2}
{6, 3, 2}, {5, 5, 1}
{7, 4, 2}, {6, 6, 1}
{8, 2, 1}, {6, 6, 5}a
{8, 4, 3}, {7, 7, 1}
{8, 6, 1}, {8, 5, 4}*bA
{8, 7, 2}, {8, 6, 5}*a
{6, 2, 1}, {5, 4, 2}*
{7, 1, 1}, {6, 4, 3}
{7, 6, 1}, {7, 5, 4}*A
{8, 3, 2}, {7, 5, 5}
{8, 5, 4}, {7, 6, 6}b
{8, 6, 3}, {7, 7, 5}

m = 24
{6, 2, 1}, {5, 3, 3}
{8, 3, 2}, {7, 5, 3}*A
{8, 6, 2}, {7, 6, 5}*A
{9, 8, 2}, {9, 7, 5}*A
{10, 5, 3}, {9, 8, 1}
{10, 7, 5}, {8, 8, 8}
{10, 9, 3}, {10, 8, 6}*B
{11, 3, 2}, {9, 8, 7}
{11, 5, 3}, {10, 9, 2}
{11, 6, 5}, {9, 9, 8}
{11, 8, 3}, {10, 9, 7}
{11, 10, 2}, {11, 8, 8}*
{8, 2, 1}, {7, 5, 1}*A
{8, 3, 3}, {7, 6, 1}
{9, 2, 1}, {8, 5, 3}
{10, 3, 2}, {9, 7, 1}
{10, 6, 5}, {9, 9, 1}
{10, 8, 2}, {10, 7, 5}*A
{11, 2, 1}, {8, 8, 8}
{11, 3, 3}, {10, 7, 6}
{11, 6, 2}, {9, 9, 7}
{11, 8, 2}, {11, 7, 5}*A
{11, 9, 3}, {11, 8, 6}*B

m = 30
{7, 2, 1}, {6, 4, 2}*A
{7, 5, 1}, {6, 5, 4}*A
{8, 3, 2}, {6, 6, 3}*B
{8, 4, 2}, {7, 6, 1}a
{8, 7, 1}, {8, 6, 4}*A
{9, 1, 1}, {8, 4, 3}
{9, 4, 2}, {7, 7, 3}
{9, 8, 2}, {9, 6, 6}*cB
{10, 2, 1}, {7, 6, 6}b
{10, 3, 1}, {8, 7, 3}*dC
{10, 3, 2}, {9, 6, 2}*D
{10, 4, 2}, {7, 7, 6}
{10, 4, 3}, {9, 7, 1}e
{10, 6, 3}, {9, 8, 2}c
{10, 7, 1}, {10, 6, 4}*A
{10, 8, 2}, {10, 6, 6}*B
{10, 9, 1}, {9, 8, 7}*C
{11, 2, 1}, {10, 6, 2}*E
{11, 3, 1}, {9, 8, 2}c
{11, 4, 1}, {10, 6, 4}*E
{7, 3, 1}, {6, 4, 3}*A
{8, 2, 1}, {6, 6, 1}*B
{8, 4, 2}, {6, 6, 4}*aB
{8, 5, 2}, {6, 6, 5}*
{8, 7, 2}, {7, 6, 6}*bB
{9, 2, 1}, {7, 6, 3}
{9, 7, 1}, {9, 6, 4}*eA
{10, 1, 1}, {8, 6, 4}
{10, 2, 1}, {8, 7, 2}*bC
{10, 3, 1}, {9, 6, 1}*dD
{10, 4, 1}, {8, 7, 4}*C
{10, 4, 3}, {9, 6, 4}*eD
{10, 6, 1}, {8, 7, 6}*C
{10, 6, 4}, {8, 7, 7}
{10, 7, 3}, {9, 7, 6}*D
{10, 8, 3}, {9, 8, 6}*D
{11, 1, 1}, {8, 7, 6}
{11, 3, 1}, {9, 6, 6}c
{11, 3, 1}, {10, 6, 3}*cE
{11, 5, 1}, {10, 6, 5}* E

{11, 6, 1}, {10, 8, 2}
{11, 7, 1}, {11, 6, 4}*fA
{11, 7, 6}, {10, 10, 2}j
{11, 8, 2}, {11, 6, 6}*B
{11, 8, 7}, {10, 10, 6}
{11, 10, 1}, {11, 8, 7}*gC
{12, 1, 1}, {10, 8, 4}
{12, 2, 1}, {11, 6, 4}f
{12, 3, 2}, {11, 7, 3}*F
{12, 4, 2}, {11, 7, 4}*hF
{12, 6, 1}, {10, 8, 7}i
{12, 6, 2}, {10, 10, 2}*jG
{12, 6, 3}, {10, 10, 3}*kG
{12, 6, 4}, {10, 10, 4}*lG
{12, 6, 6}, {11, 10, 1}g
{12, 7, 1}, {12, 6, 4}*lA
{12, 7, 6}, {11, 10, 4}m
{12, 8, 2}, {11, 8, 7}*gF
{12, 8, 6}, {10, 10, 8}*G
{12, 9, 6}, {10, 10, 9}*nG
{12, 10, 2}, {11, 10, 7}oF
{12, 10, 3}, {12, 9, 6}*nD
{12, 11, 1}, {12, 10, 6}*pE
{12, 11, 6}, {11, 10, 10}*G
{13, 2, 1}, {12, 6, 6}r
{13, 3, 1}, {10, 9, 8}s
{13, 4, 1}, {12, 8, 4}*J
{13, 4, 3}, {12, 9, 1}
{13, 6, 1}, {10, 10, 8}t
{13, 6, 2}, {11, 11, 1}
{13, 6, 4}, {12, 8, 7}q
{13, 6, 6}, {11, 10, 8}u
{13, 7, 1}, {12, 10, 1}*qK
{13, 7, 2}, {12, 10, 2}*K
{13, 7, 3}, {12, 10, 3}*nK
{13, 7, 5}, {12, 10, 5}*K
{13, 7, 6}, {12, 10, 6}*pK
{13, 7, 7}, {12, 11, 4}
{13, 8, 7}, {12, 10, 8}*vK
{13, 9, 6}, {13, 10, 3}*D
{13, 9, 7}, {12, 12, 3}w
{13, 10, 1}, {13, 8, 7}*vC
{13, 10, 3}, {13, 9, 6}*D
{13, 10, 6}, {12, 11, 8}y
{13, 11, 1}, {12, 11, 8}*yJ

{11, 7, 1}, {10, 7, 6}*fE
{11, 7, 3}, {10, 9, 2}
{11, 8, 1}, {10, 8, 6}*E
{11, 8, 4}, {10, 10, 1}
{11, 9, 1}, {10, 9, 6}*E
{11, 10, 3}, {11, 9, 6}*D
{12, 2, 1}, {10, 7, 6}f
{12, 2, 1}, {11, 7, 1}*fF
{12, 4, 2}, {10, 7, 7}h
{12, 5, 2}, {11, 7, 5}*F
{12, 6, 1}, {10, 10, 1}*iG
{12, 6, 2}, {11, 7, 6}*jF
{12, 6, 3}, {11, 9, 1}k
{12, 6, 6}, {11, 8, 7}g
{12, 7, 1}, {10, 10, 4}l
{12, 7, 6}, {10, 10, 7}*mG
{12, 8, 2}, {12, 6, 6}*gB
{12, 8, 2}, {11, 10, 1}g
{12, 9, 2}, {11, 9, 7}*F
{12, 10, 1}, {12, 8, 7}*qC
{12, 10, 2}, {11, 11, 4}o
{12, 11, 1}, {10, 10, 10}p
{12, 11, 4}, {12, 10, 7}*H
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{13, 2, 1}, {12, 8, 2}*rJ
{13, 3, 1}, {12, 8, 3}*sJ
{13, 4, 2}, {10, 10, 7}
{13, 5, 1}, {12, 8, 5}*J
{13, 6, 1}, {12, 8, 6}*tJ
{13, 6, 3}, {11, 9, 8}
{13, 6, 4}, {12, 10, 1}q
{13, 7, 1}, {12, 8, 7}*qJ
{13, 7, 1}, {13, 6, 4}*qA
{13, 7, 3}, {12, 9, 6}n
{13, 7, 4}, {12, 10, 4}*K
{13, 7, 6}, {10, 10, 10}p
{13, 7, 6}, {12, 11, 1}p
{13, 8, 2}, {13, 6, 6}*uB
{13, 9, 1}, {12, 9, 8}*J
{13, 9, 7}, {12, 10, 9}*wK
{13, 10, 1}, {12, 10, 8}*vJ
{13, 10, 2}, {12, 11, 6}
{13, 10, 4}, {12, 12, 1}
{13, 10, 7}, {12, 12, 6}x
{13, 11, 1}, {13, 10, 6}*yE

{13, 11, 4}, {12, 10, 10}x
{13, 11, 4}, {13, 10, 7}*xH
{13, 12, 2}, {13, 11, 7}*zF
{13, 12, 6}, {13, 10, 10}*G
{14, 2, 1}, {11, 10, 10}
{14, 4, 1}, {12, 12, 1}*M
{14, 4, 2}, {12, 12, 2}*aaM
{14, 4, 3}, {12, 12, 3}*wM
{14, 5, 4}, {12, 12, 5}*M
{14, 6, 1}, {13, 11, 1}*yN
{14, 6, 2}, {13, 11, 2}*bbN
{14, 6, 4}, {12, 12, 6}*ccM
{14, 6, 4}, {13, 11, 4}*ccN
{14, 7, 1}, {12, 10, 10}cc
{14, 7, 1}, {14, 6, 4}ccA
{14, 7, 4}, {12, 12, 7}*M
{14, 7, 6}, {13, 11, 7}*zN
{14, 8, 2}, {13, 11, 6}
{14, 8, 4}, {12, 12, 8}*M
{14, 8, 6}, {13, 11, 8}*N
{14, 9, 4}, {12, 12, 9}*M
{14, 9, 6}, {14, 10, 3}
{14, 10, 1},{13, 12, 6}dd
{14, 10, 2},{12, 11, 11}
{14, 10, 3},{14, 9, 6}*D
{14, 10, 4},{13, 12, 7}ii
{14, 10, 6},{13, 13, 2}ee
{14, 10, 8},{13, 13, 6}
{14, 11, 1},{14, 10, 6}*eeE
{14, 11, 4},{14, 10, 7}*ffH
{14, 12, 2},{14, 11, 7}*F
{14, 12, 6},{13, 12, 11}*ggN
{14, 12, 8},{13, 13, 10}hh
{14, 13, 2},{14, 11, 10}*
{14, 13, 7},{14, 12, 10}*K
{13, 11, 4}, {12, 12, 6}x
{13, 11, 7}, {12, 11, 10}*zK
{13, 12, 3}, {13, 10, 9}*L
{14, 1, 1}, {12, 10, 8}
{14, 3, 2}, {11, 11, 9}
{14, 4, 2}, {12, 11, 7}aa
{14, 4, 3}, {12, 10, 9}w
{14, 4, 3}, {13, 9, 7}w
{14, 6, 1}, {12, 11, 8}y
{14, 6, 2}, {11, 11, 10}bb
{14, 6, 3}, {13, 11, 3}*N
{14, 6, 4}, {13, 10, 7}cc
{14, 6, 5}, {13, 11, 5}*N
{14, 7, 1}, {12, 12, 6}cc
{14, 7, 3}, {12, 11, 9}
{14, 7, 6}, {12, 11, 10}z
{14, 7, 6}, {13, 12, 2}z
{14, 8, 2}, {14, 6, 6}*B
{14, 8, 4}, {13, 12, 1}
{14, 8, 7}, {13, 10, 10}dd
{14, 9, 6}, {13, 11, 9}*N
{14, 9, 8}, {13, 13, 3}
{14, 10, 1}, {14, 8, 7}*ddC
{14, 10, 3}, {13, 11, 9}
{14, 10, 4}, {12, 12, 10}*iiM
{14, 10, 6}, {13, 11, 10}*eeN
{14, 10, 7}, {12, 12, 11}ff
{14, 10, 10},{13, 12, 11}gg
{14, 11, 4}, {12, 12, 11}*ffM
{14, 12, 1}, {13, 13, 7}
{14, 12, 3}, {14, 10, 9}*L
{14, 12, 6}, {14, 10, 10}*ggG
{14, 13, 1}, {14, 12, 8}*hhJ
{14, 13, 4}, {13, 12, 12}*M

m = 36
{8, 4, 1}, {7, 5, 3}
{11, 7, 6}, {10, 10, 2}
{12, 3, 2}, {10, 8, 3}*A
{12, 4, 2}, {11, 7, 2}*B
{12, 5, 2}, {10, 8, 5}*A
{12, 5, 4}, {11, 7, 5}*B
{12, 8, 4}, {11, 8, 7}*B
{12, 9, 4}, {11, 9, 7}*B
{11, 2, 1}, {10, 5, 3}
{12, 2, 1}, {10, 8, 1}*A
{12, 4, 1}, {11, 7, 1}*B
{12, 4, 3}, {11, 7, 3}*B
{12, 5, 3}, {11, 8, 1}
{12, 7, 2}, {10, 8, 7}*A
{12, 9, 2}, {10, 9, 8}*A
{12, 10, 4}, {11, 10, 7}*B

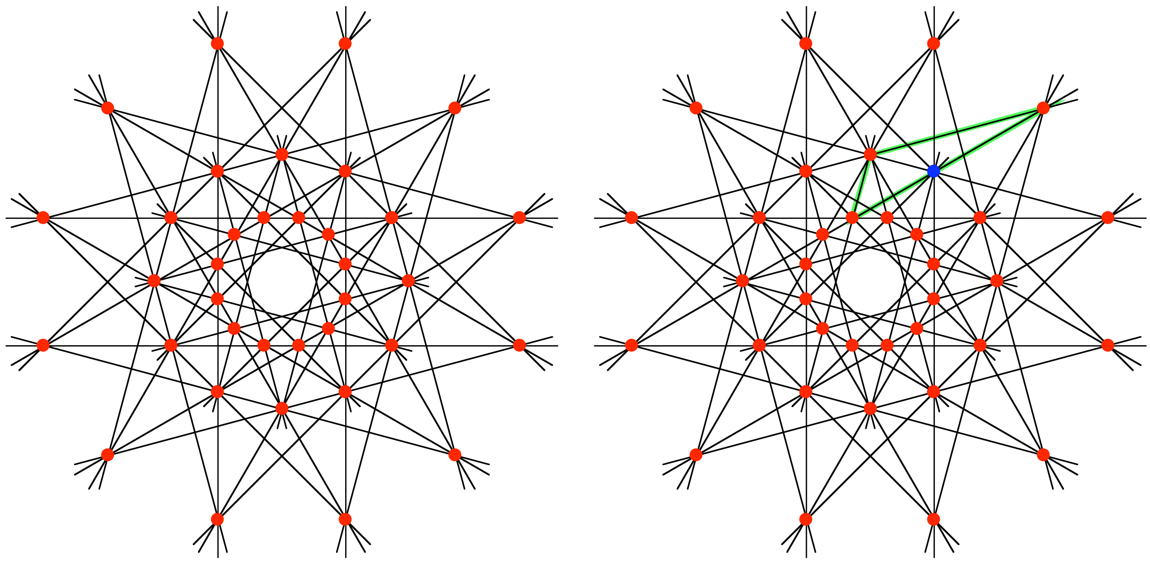
{12, 11, 2}, {11, 10, 8}*A
{13, 10, 5}, {12, 12, 2}
{13, 12, 4}, {13, 11, 7}*B
{14, 7, 3}, {13, 10, 1}
{14, 12, 4}, {14, 11, 7}*B
{15, 2, 1}, {14, 7, 5}
{15, 3, 2}, {14, 8, 4}
{15, 6, 3}, {14, 10, 2}
{15, 12, 2}, {15, 10, 8}*A
{15, 13, 5}, {15, 12, 8}*
{16, 4, 2}, {15, 10, 3}
{16, 7, 5}, {15, 12, 1}
{16, 8, 7}, {15, 13, 1}
{16, 12, 4}, {16, 11, 7}*B
{16, 15, 3}, {16, 12, 12}*C
{17, 2, 1}, {15, 14, 3}
{17, 5, 3}, {16, 11, 8}
{17, 6, 1}, {14, 14, 10}
{17, 7, 3}, {16, 13, 4}
{17, 8, 4}, {15, 13, 11}
{17, 10, 5}, {15, 14, 11}
{17, 12, 3}, {16, 13, 11}
{17, 13, 5}, {17, 12, 8}*
{17, 15, 3}, {17, 12, 12}*C

{13, 4, 1}, {12, 7, 3}
{13, 12, 2}, {13, 10, 8}*A
{14, 2, 2}, {13, 6, 5}
{14, 11, 7}, {12, 12, 10}
{14, 13, 5}, {14, 12, 8}*
{15, 3, 2}, {13, 10, 5}
{15, 3, 3}, {13, 11, 1}
{15, 10, 3}, {14, 11, 7}
{15, 12, 4}, {15, 11, 7}*B
{15, 14, 4}, {15, 12, 10}*
{16, 5, 4}, {15, 11, 1}
{16, 8, 4}, {15, 12, 3}
{16, 10, 8}, {15, 14, 3}
{16, 13, 5}, {16, 12, 8}*
{17, 2, 1}, {14, 12, 12}
{17, 3, 2}, {14, 13, 11}
{17, 5, 4}, {15, 12, 11}
{17, 7, 2}, {15, 13, 10}
{17, 7, 5}, {15, 15, 3}
{17, 8, 7}, {15, 13, 12}
{17, 12, 2}, {17, 10, 8}*A
{17, 12, 4}, {17, 11, 7}*B
{17, 14, 4}, {17, 12, 10}*
{17, 16, 2}, {17, 14, 12}*

Table 3.7.1. A list of all sporadic 3-astral 4-configurations (n_4) with $n \leq 72$. In the cohort notation used, each entry corresponds to a cohort of configurations. Reducible cohorts are indicated by an asterisk following the symbol. Equal upper-case letters indicate that the configurations reduce to the same 2-astral configuration. In case of the symbols for m = 30, some have a common factor 2; however, they are not disconnected, since in each case the symbol corresponding to entries one-half of the ones given would violate condition (****). Equal lower-case letters attached to the symbols indicate that the symbols share one of the parentheses; while it is not clear what this commonality implies, it is signaled to ease possible investigations.

The problem is that in the discussions in Sections 3.5, 3.6, and (so far) in 3.7 we did not worry about possible *unintended incidences* of points and lines. However, such incidences may well happen, as is shown by the example in Figure 3.7.4. The reason is easily discerned from the symbol of the configuration, as first pointed out in [B20]. In the language of the characteristic paths this happens when a segment of the path (or the line it determines) passes through a point that is in the same orbit as the endpoint of another segment of the path, but is not itself a vertex of the path. This is illustrated in Figure 3.7.4(b), where the characteristic path starts at the red point of the middle orbit and goes towards the innermost orbit of points, but the second segment contains the blue point of the orbit of the starting point. That causes each point of this orbit to be on six lines. This description is easily translated in the language of the configuration symbols: As we step from one entry of the general symbol $m\#(s_1,t_1;s_2,t_2; \ldots ;s_k,t_k)$ to the next, a consecutive string of entries needs to be changed in only its first or else its last terms in order to obtain a valid symbol for an h-astral configuration with $h < k$ and the same m. In the 3-astral example in Figure 3.7.4, its symbol 12#(5,4;1,5;4,1) contains the string 5,4;1,5. If the last entry is changed to 4, the resulting symbol 12#(5,4;1,4) corresponds to the 2-astral configuration we have seen in Figure 3.6.2. Since the configuration in Figure 3.7.4 is selfdual, it is clear that there necessarily are lines that meet six of its points. We formulate this in the general case of k-astral configurations by the following requirement:

(A8) The symbols of a k-astral configuration $m\#(s_1,t_1;s_2,t_2; \ldots ;s_k,t_k)$ should not contain a string such that changing one of the ends of the string results in a valid symbol for an h-astral configuration with the same m and with $h < k$.



(a) (b)

Figure 3.7.4. (a) The trivial 3-astral configuration 12#(5,4,1,5,4,1) is not a configuration at all — it is a prefiguration due to the presence of an orbit of points each incident with six lines, and an orbit of lines each incident with six points. (b) The explanation for this situation, as detailed in the text.

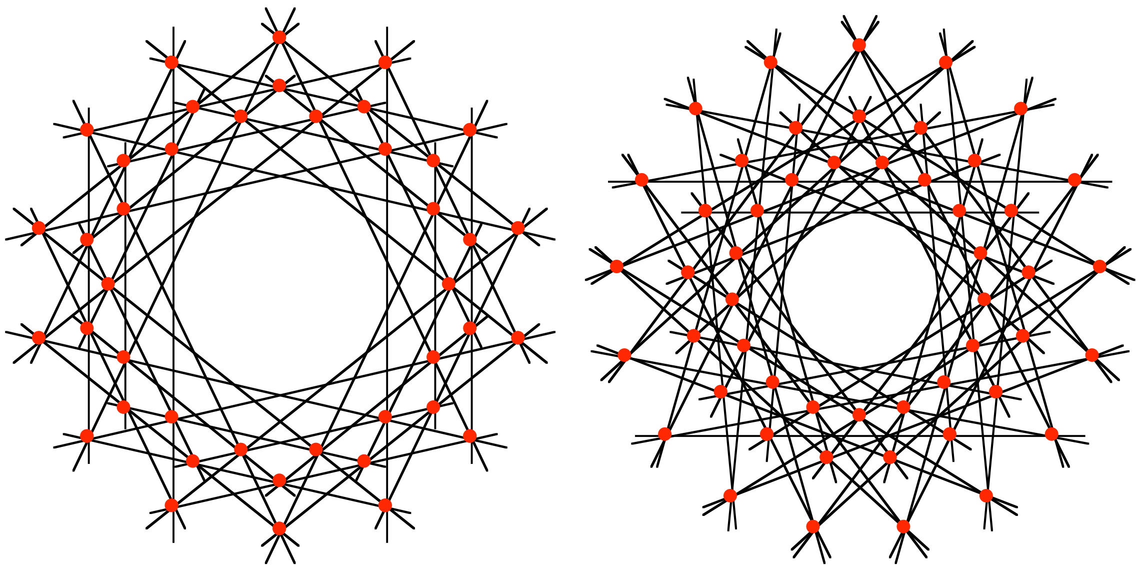
Exercises and problems 3.7

1. Find the symbols for the two configurations in Figure 3.7.5, and decide whether it is trivial, systematic or sporadic. Find all the other configurations that are in the same cohorts.

2. Find other examples of unintended incidences like the ones in Figure 3.7.4.

3. Find all systematic configurations with m = 20, and draw three of them.

4. The configuration in Figure 3.7.5a has its lines parallel in sets of six, three on each side of the center. Find a 3-astral 4-configuration in which the points appear in collinear sets of six.



(a) (b)

Figure 3.7.5. Identify these configurations.

5. Is there a trivial 3-astral configuration in which the points are in collinear sets of six?

6. Find the symbol of the configuration in Figure 3.7.6, and explain your findings.

7. Prove that the configuration (21_3) in Figure 3.7.1 is the only k-astral configuration (21_3) for any k.

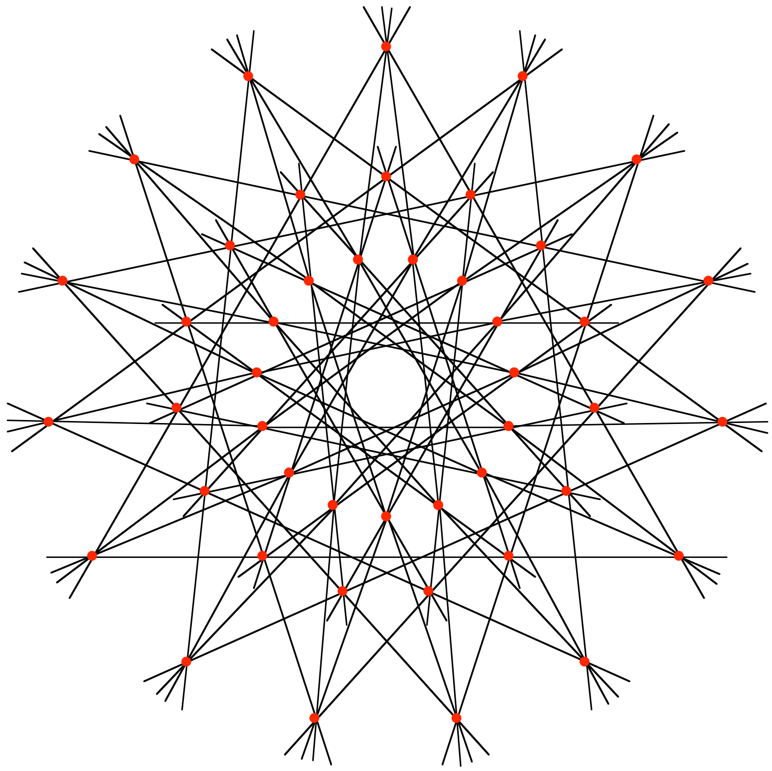


Figure 3.7.6. An interesting configuration.

3.8 k-ASTRAL CONFIGURATIONS FOR k ≥ 4

As in the preceding three sections, configurations with 4 or more orbits of points that are known in greatest detail are those k-astral ones in which a dihedral symmetry group acts transitively on the points (and lines) of k different orbits, and each orbit has the same number of points. We shall discuss these first. Although the definitions are completely analogous to the ones in previous cases, striking differences in properties led us to separate the present case from the cases of 2- and 3-astral configurations. The main change is in the possibility of various unintended incidences not encountered earlier; hence it is clear that in a number of cases we can speak only of representations and not of realizations. But before we get to that, let us review the definitions.

A **k-astral** 4-configuration (n_k), with n = k·m, is a configuration with a dihedral symmetry group d_m that operates transitively on each of k orbits of points situated at vertices of a regular m-gon, and k orbits of lines, provided all orbits have the same number of elements and each element is incident with two elements of the other kind from each of two orbits. As detailed in Section 3.5, we can attach to each k-astral configuration a well-determined set of mutually equivalent symbols, derived from the consideration of the characteristic paths possible in the configuration.

This is illustrated in Figure 3.8.1, where a characteristic path starts at P_0, and goes on to P_1, and other points that are not labeled to avoid clutter. Since P_1 has symbol (2 // 1), and the following points of the characteristic path shown have symbols (5 // 3), (4 // 5), (1 // 2) and (3 // 4), while the orbits have size 11, — the symbol of this configuration is 11#(2,1;5,3;4,5;1,2;3,4). Hence it is a 5-astral configuration.

A similar procedure leads in the case of the symbol 9#(2,1;4,2;1,3;2,3;1,3) to the diagram shown in Figure 3.8.2(a). Here we came face to face with a serious problem: Our graphics, which frequently show only (slightly elongated) segments that are necessary to connect all points that are incident according to the symbol, are misleading. Configurations consist of *lines*, not segments, — and if the segments we used are extended to the rim of the diagram, additional incidences become evident; thus, we do not have a

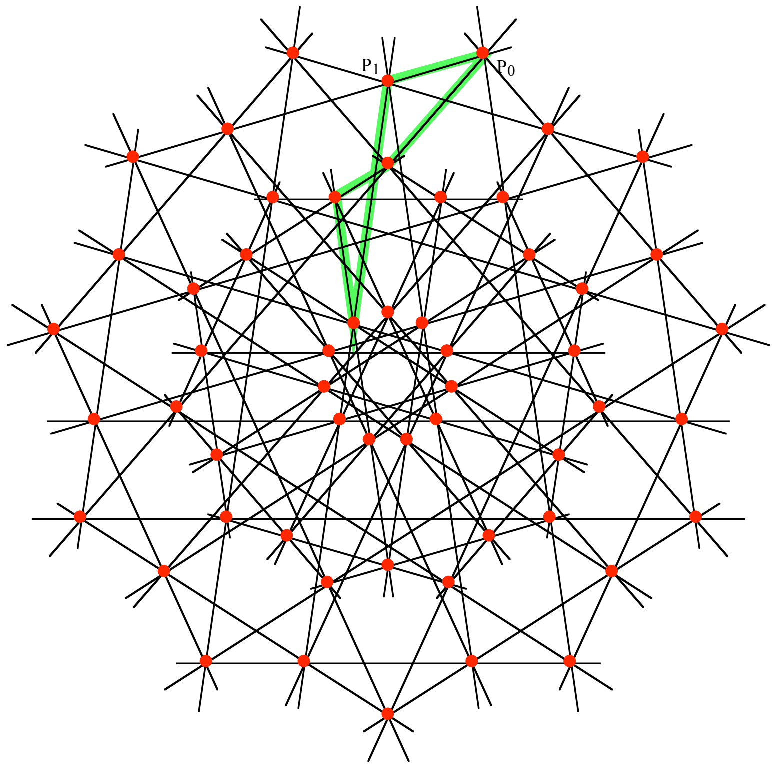
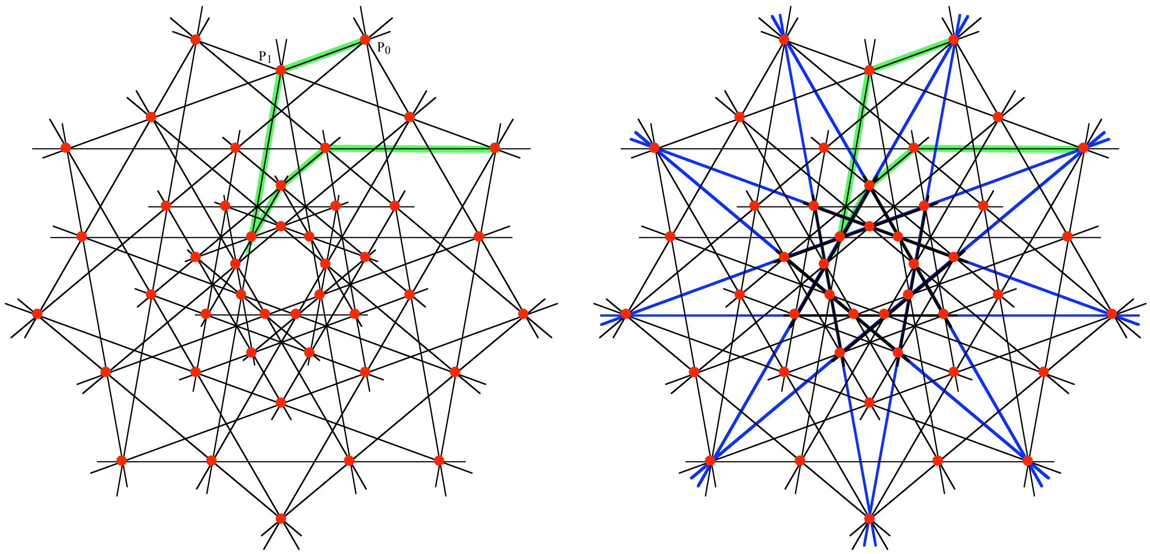


Figure 3.8.1. A 5-astral configuration (55_4) with symbol 11#(2,1;5,3;4,5;1,2;3,4) and symmetry group d_{11}.

configuration at all. Instead, we have a prefiguration, that can be interpreted as a *representation* of the abstract configuration 9#(2,1;4,2;1,3;2,3;1,3).

As mentioned already in Section 3.7, the explanation of the problem is that the characteristic path essentially crosses itself at a configuration point. This is detectable from the symbol of the configuration, and leads to the following condition we repeat here from Section 3.7 in a slight reformulation; the condition was signaled by Boben and Pisanski in [B20]:

(A8) The symbols of a k-astral configuration m#(s_1,t_1;s_2,t_2; ... ;s_k,t_k) should not contain a string of even length such that changing at most one of the ends of the string results in a valid symbol for an h-astral configuration with the same m and with h < k.



(a) (b)

Figure 3.8.2. Problems with realization of 9#(2,1;4,2;1,3;2,3;1,3).

If condition (A8) is not fulfilled, the symbol m#(s_1,t_1;s_2,t_2; ... ;s_k,t_k) encodes for a *representation* by a prefiguration, and not for a *realization* by a configuration. In the example of Figure 3.8.2, the symbol can be written in the equivalent form 9#(1,3;2,1;4,2;1,3;2,3). Then the string 3,2,1,4,2,1 can be replaced by 4,2,1,4,2,1, which leads to the trivial 3-astral configuration 9#(4,2;1,4;2,1).

The change in (A8) consists in the words "at most", which really mean that the string itself should be usable in a configuration symbol. This could not have happened with 3-astral configurations, but can happen in the situation considered here.

As an example we show in Figure 3.8.3 the result of drawing the configuration that corresponds to the symbol 12#(5,4;1,4;1,4;5,4). The expected (48_4) configuration did not come about. Instead, we obtained a prefiguration that looks as having three orbits of points and lines but points in one orbit are incident with six lines while lines of one orbit are incident with six points.

The explanation for the (mis)behavior of the symbols in these cases is the rather obvious failure of (A8): The characteristic path returns to one orbit three times; in other

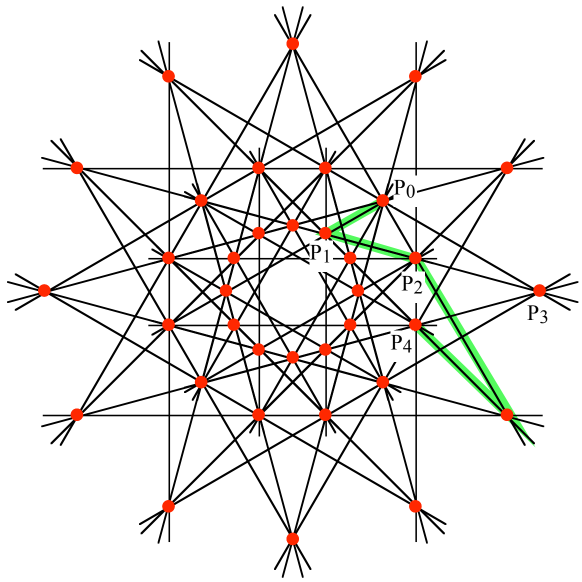


Figure 3.8.3. Given the symbol 12#(5,4;1,4;1,4;5,4) for a 4-configuration (48_4), the software produced this result which has three orbits of points and three of lines; it would be a (36_4) configuration, if it were not for the middle orbit of points: Each appears to be on *six* lines! In fact, these are two coinciding points of the configuration, and the lines with span 5 also represent two lines of the configuration.

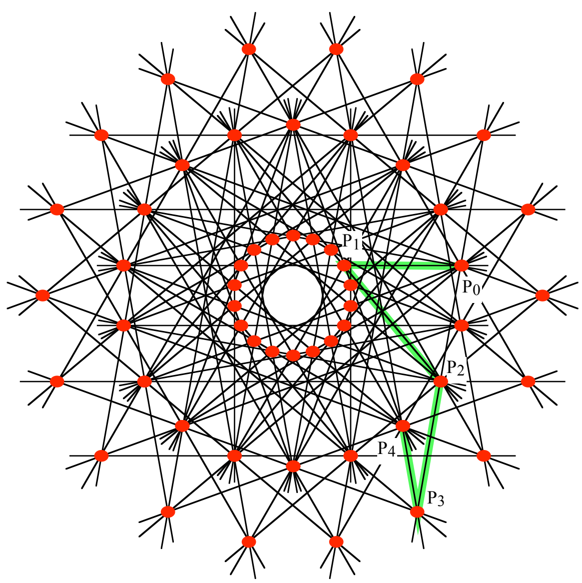


Figure 3.8.4. For the symbol 18#(8,6;1,7;2,5;7,6) we obtain the prefigurations shown. It does have four orbits of lines, each incident with two points from each of two point orbits; but there appear to be only three point orbits, one of which consists of points incident with eight lines. These actually represent pairs of coinciding points of th configuration.

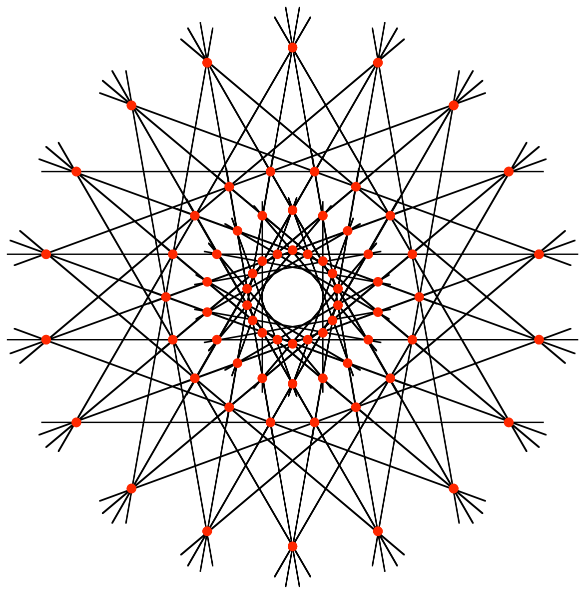


Figure 3.8.5. The symbol 18#(8,6;7,5;2,7;1,6) leads to a prefiguration (polar to the one in Figure 3.8.4) in which each line of one orbit is incident with eight points (in four different orbits).

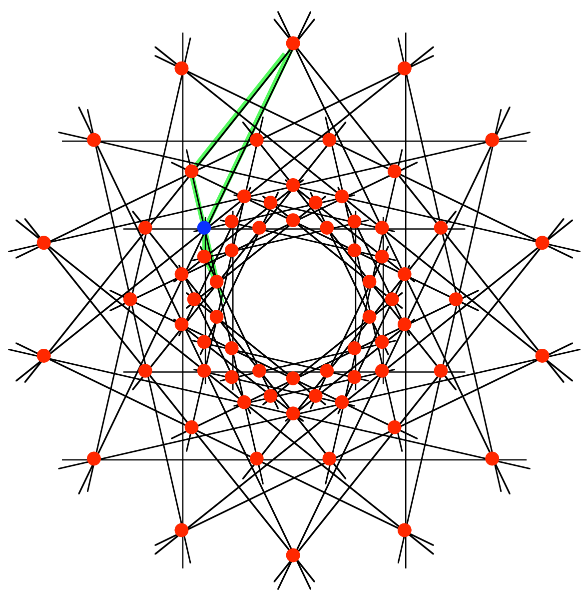


Figure 3.8.6. The characteristic path of the prefiguration corresponding to the 5-astral symbol 14#(5,1;3,2;4,3;2,5;1,4) contains the string 3,2,4,3,2,5; replacing its last entry by 4 we get the symbol of the 3-astral trivial configuration 14#(3,2;4,3;2,4). The blue point is a vertex of the characteristic path, but is also in the relative interior of a different segment of this path; hence an orbit of points that are on six lines each. The line of that segment is clearly incident with six points, as are all lines in its orbit.

words, a proper string of the symbol already codes for a 4-configuration, and so does the remaining part, hence there are six or more lines through each point of the appropriate orbit. In fact, the points of the middle orbit are doubled-up, and so are the span-5 lines. Other examples are given in Figures 3.8.4 and 3.8.5.

The example in Figure 3.8.6 shows that if the relative interior of a segment of the characteristic path contains the endpoint of another segment, unexpected incidences occur as well.

There is one more set of circumstances in which unexpected incidences of a different kind occur; it was also signaled by Boben and Pisanski in [B20]. It is illustrated by Figure 3.8.7, in which points of one orbit are on five lines while lines of one orbit contain five points. To avoid such incidences, the following condition is imposed by Boben and Pisanski beyond the ones we already require:

(A9) The symbol of a k-astral configuration $m\#(s_1,t_1;s_2,t_2; \dots ;s_k,t_k)$ should not contain a string of odd length, such as $s_i,t_i;s_{i+1},t_{i+2}; \dots ;s_j$, such that

$s_i + t_i + s_{i+1} + t_{i+2} + \dots + s_j$ is an even integer and

$$\prod_{i\le g\le j} \cos(s_g\cdot\pi/m) = \prod_{i\le g\le j-1} \cos(t_g\cdot\pi/m) .$$

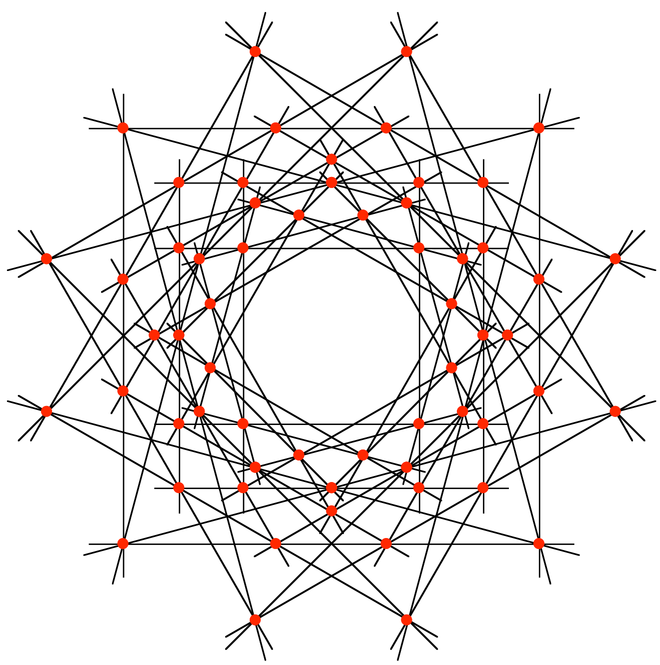


Figure 3.8.7. 12#(3,2,3,4,2,3,1,3,4,1) is not a 4-configuration.

Clearly, the 5-astral configuration 12#(3,2,3,4,2,3,1,3,4,1) in Figure 3.8.7 violates this condition: The sum of the string of the first five entries is 14, and

$$\cos \pi/4 \cos \pi/4 \cos \pi/6 = \cos \pi/6 \cos \pi/3.$$

Hence the unintended incidences.

Exercises and problems 3.8

1. The symbol of the prefiguration in Figure 3.8.7 contains many pairs of equal entries. Explain why canceling any such pair would not yield an example violating condition (A9).

2. The string s_1,t_1,s_2 with $m = 12$ is in a sense the only *known* source of examples violating condition (A9). By this is meant that one can obviously add the same numbers to the even and odd position (2 was added in the example of Figure 3.8.7), and one can use multiples of the string with the appropriate multiples of m. Decide whether there are any essentially different strings.

3. Prove that no 4-astral configuration can violate condition (A9).

4. The symbol that yielded the example in Figure 3.8.2 belongs to the cohort m#{{4,2,2,1,1},{3,3,3,2,1}} with $m = 9$, while the one in Figure 3.8.1 corresponds to $m = 11$. Why are the results different in the two cases?

5. Does the cohort 9#{{4,2,2,1,1},{3,3,3,2,1}} contain any geometrically realizable configurations?

6. List all the configuration symbols for 4-astral configurations $\{28_4\}$.

7. Draw the (potential) configuration 7#(3,2;1,3;1,3;2,1;3,1) and describe what happens.

8. Find some systematic families for 4-astral configurations, other than the ones that arise from an h-astral configuration with $h < k$ by insertion of matched pairs.

3.9 OPEN PROBLEMS

Among the most intriguing open problems concerning 4-configurations are the following.

1. Is there *any* analog for 4-configurations of Steinitz's theorem (Theorem 2.6.1)? This theorem can be interpreted as saying that every connected combinatorial 3-configuration can be represented geometrically in the plane if one incidence is disregarded. How much of a combinatorial 4-configuration can be realized?

2. Can any of the cyclic configurations (n_4) with generating lines {0,1,6,-3} or {0,1,5,-2}, described in Section 3.1, can be geometrically realized (for any $n \geq 18$)? Cany *any* cyclic 4-configurations be geometrically realized?

3. It is clear that there are some 4-configurations that can be geometrically realized in *rational* plane; as an example we may take the configuration $\mathcal{LC}(4)$ introduced in Section 1.1, and other similarly built configurations. Can any astral (or k-astral) configuration that can be geometric realized (or represented) in the Euclidean plane be realized (or represented) in the rational plane? (It is well known that this cannot be done in a k-astral way.)

4. The astral configuration (24_4) in Figure 3.6.2 and the 3-astral configuration (21_4) in Figure 3.7.1 have the property that the underlying combinatorial configurations have groups of automorphisms that act transitively on the flags of the configuration. (A *flag* consists of a line and a point, incident with each other.) The configuration $\mathcal{LC}(4)$ mentioned above has the same property. Do there exist any other 4-configurations with a single orbit of flags (under automorphisms)?