CHAPTER 1. BEGINNINGS

1.1 INTRODUCTION.

The word "configuration" has many meanings in both colloquial and technical use. In the present work, however, it will be used in one meaning only, although with several nuances which will be explained soon. By a *k-configuration*, specifically an (n_k) configuration, we shall always mean a set of n *points* and n *lines*, such that every point lies on precisely k of these lines, and every line contains precisely k of the points. The variants of the meaning will concern the interpretation of "point" and "line", with additional distinctions regarding the space in which the points and lines are taken. However, in this Introduction it is simplest to interpret the words at just their most naïve meaning – – points and lines in the Euclidean plane. It is probably surprising that even with this simple interpretation there is sufficient material to consider writing a book about, and that there are many problems that are easily stated but are still unsolved.

For a quick orientation (see Figure 1.1.1), here are three examples of well-known

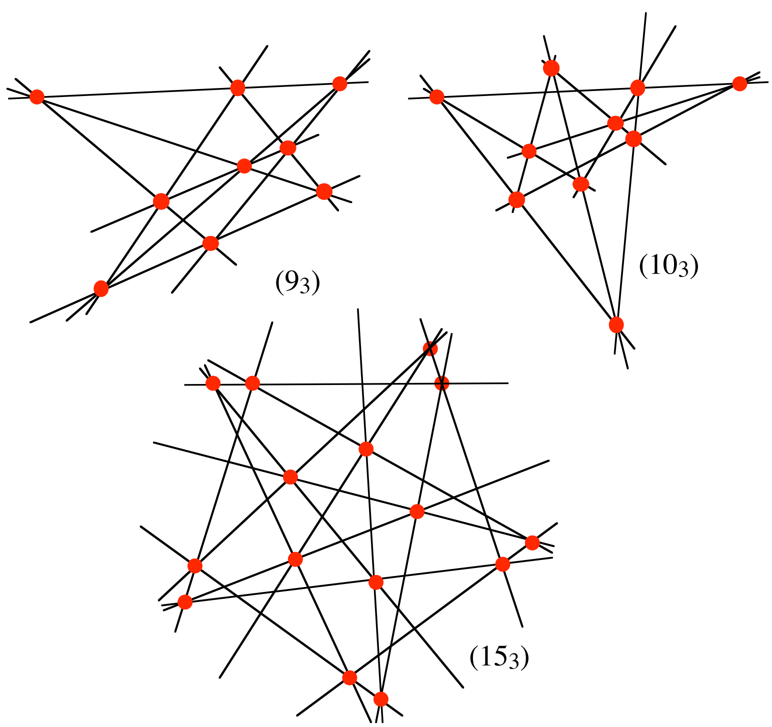

Figure 1.1.1. The 3-configurations of Pappus (9_3), Desargues (10_3), and Cremona-Richmond (15_3).

configurations (n_3), about which much has been written and which are known by the names of specific mathematicians. Each will appear several times in our discussions. Much less known are 4-configurations; three examples are shown in Figure 1.1.2.

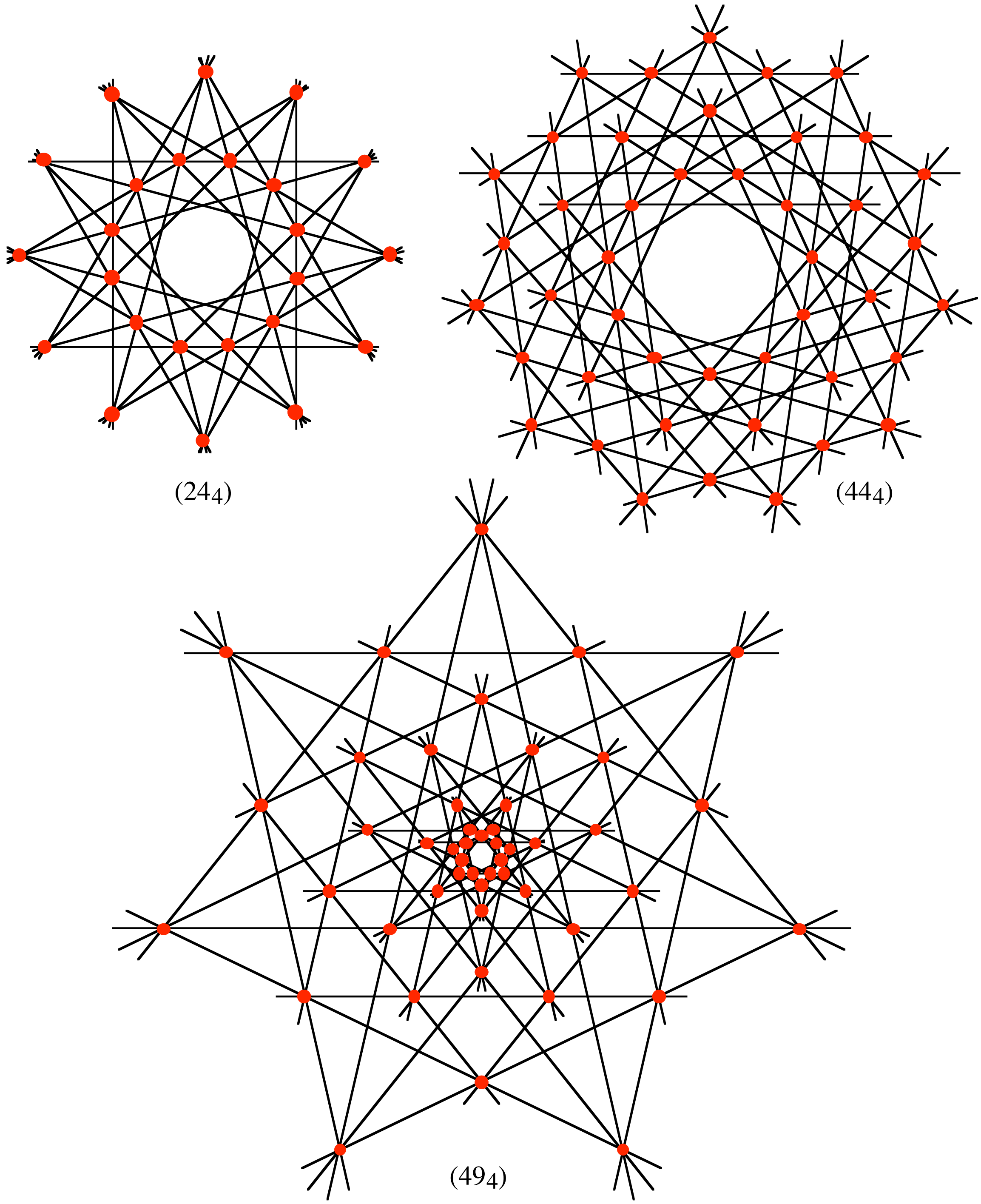

Figure 1.1.2. Three examples of 4-configurations.

A configuration (50_5) is illustrated in Figure 1.1.3.

It is both clear and natural that, with increasing k, the images of k-configurations become more complicated. In fact, the smallest n for which a configuration (n_6) is known to exist has a value of n = 110. (This topic will be discussed in detail in Chapter 4.) One concern that can be answered easily is whether for an arbitrary integer k there exists a configuration (n_k). Indeed, taking in the k-dimensional Euclidean space a "box" consisting of $n = k^k$ points of the integer lattice, together with the $n = k^k$ lines through them that are parallel to the coordinate axes, we see that there exists a configuration (n_k) with points and lines in the k-dimensional space. But then an appropriate projection onto a suitable plane yields the required configuration in the plane. We shall have repeated use of this configuration, hence we give it a special symbol LC(k). In [P5], T. Pisanski calls these the "generalized Gray configurations". The drawback of this construction is, obviously, that already for k = 7 this yields $n = 7^7 = 823{,}543$ — a rather unwieldy number. One may expect that with some ingenuity this number can be reduced, just as the corresponding $6^6 = 46{,}656$ has been reduced to 130. A different construction of some k-configurations with arbitrary k was proposed by Kantor [K2].

Some very specific question that can be asked for any k, but which we shall here illustrate for k = 3 only, are:

(A) For which n do configurations (n_3) exist?

(B) For each n such that configurations (n_3) exist, determine all distinct ones.

(C) How can given geometric configurations be represented symbolically? Given a type of symbolical representation, how can one decide whether it corresponds to a geometric configuration, and if it does, how can one draw it?

As we shall see in Section 2.1, question (A) has a simple answer: Configurations (n_3) exist if and only if $n \geq 9$. However, the question becomes much harder for configurations (n_k) with $k > 3$; in that form it was first posed by Reye [R2] in 1882, and is listed as Problem 12 in Section 7.2 of [B28]. We shall consider these cases in Chapters 3 and 4.

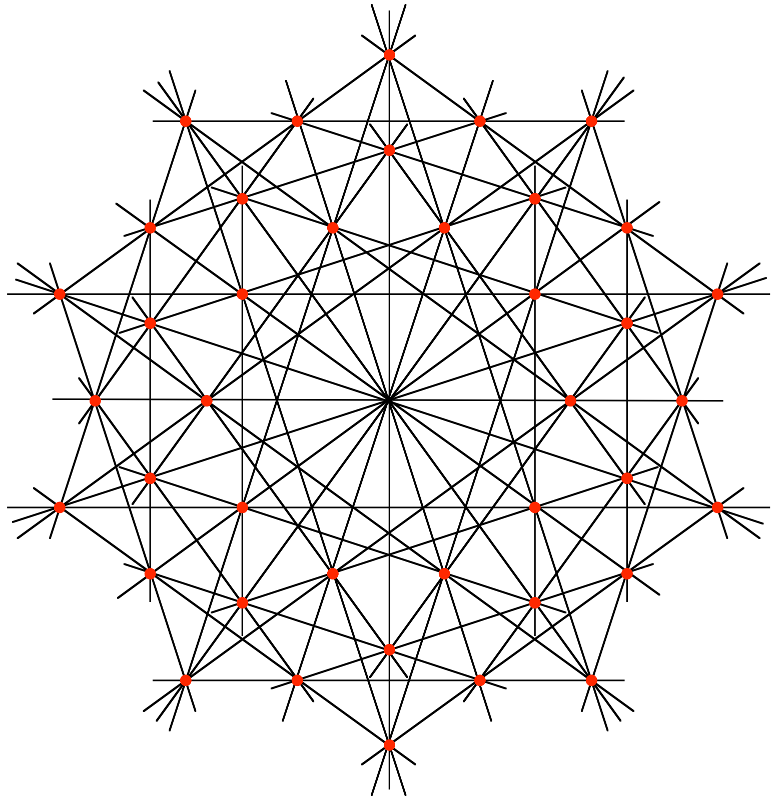

Figure 1.1.3. A configuration (50_5) in the projective (extended Euclidean) plane. There are ten points at infinity, in the direction of the sets of five parallel lines in the diagram. The smallest (n_5) configurations known are two (48_5) shown in Figure 4.1.4.

In contrast, to answer question (B) we first have to decide under what circumstances are two configurations considered to be the same, that is, not to be distinguished from each other for the purposes of the intended classification. As it turns out, in analogy to many other geometric topics, there are several sensible ways of classification, each leading to its own answer to question (B).

As an illustration of these differences we consider the case of configurations (9_3), which include the Pappus configuration from Figure 1.1.1. The three configurations in Figure 1.1.4 are obtained from each other by a simple affine transformation. They are considered the same under the so-called *projective* equivalence, which assigns to the same class configurations obtained as affine (or, more generally, projective) images of each other.

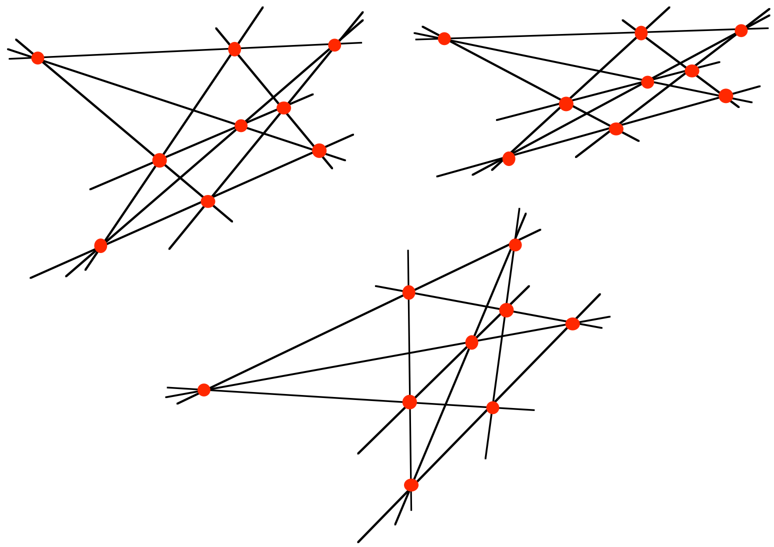

Figure 1.1.4. Three affinely equivalent configurations (9_3).

The three configurations in Figure 1.1.5 are not projectively equivalent, but have the same *incidences*.

Here and until further notice, incidences are defined between points and lines, and an incidence means that the point lies on the line or, equivalently, that the line passes through the point. Two configurations have the same incidences provided their points and lines can be given such labels that a point and a line are incident in one of them if and only if they are incident in the other. Since affine (and projective) transformations preserve lines and incidences, it is obvious that classification by incidences is coarser that the projective classification. The labels attached to the configurations show that they have the same incidences.

Concerning (C) we shall see that the available resources are rather modest. Some of the approaches will be discussed in the appropriate sections of the book. However, there is practically nothing relevant to configuration (n_k) with $k \geq 5$.

In Figure 1.1.6 are shown three configurations (9_3) that do not have the same incidences. While it may appear that proving their difference may be a staggering task, we shall see in Section 2.2 that — using appropriate tools — it can be accomplished in a few seconds. In fact, with just slightly greater effort, it can be shown that in this classification there are precisely three distinct configurations; one of each equivalence class is shown in Figure 1.1.6.

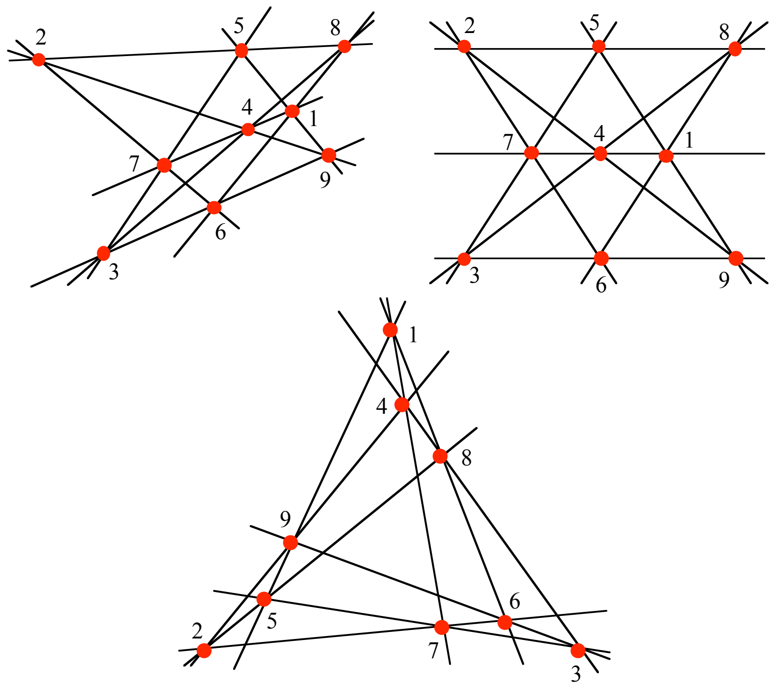

Figure 1.1.5. Three configurations (9_3) that have the same incidences but are not projectively equivalent.

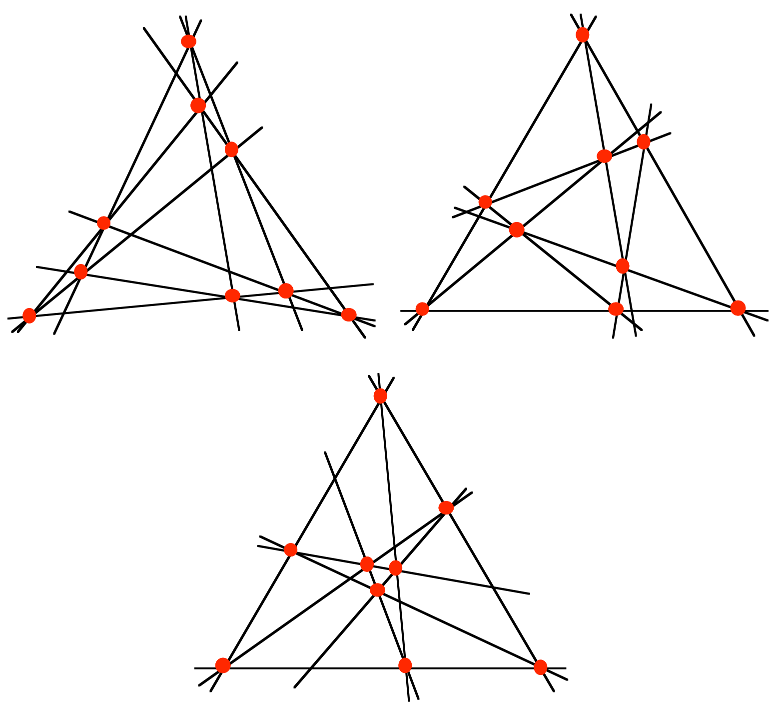

Figure 1.1.6. Three configurations (9_3) that have different incidences.

To simplify the expressions used in the sequel, we shall say that two configurations are *isomorphic* (or *combinatorially equivalent*, or of the *same combinatorial type*) if and only if they have the same incidences. In this terminology, all configurations in Figures 1.1.4, 1.1.5 and 1.1.6(a) have the same combinatorial type, different from the types in Figure 1.1.6(b) and (c).

In the next section we shall give an informal historical survey of the theory of configurations. In Section 1.3 we shall give formal definitions of the various concepts that are used in the book. Later sections of this chapter will present a selection of tools that have been found useful in the study of configurations. Chapter 2 will be devoted to a detailed study of 3-configurations, and Chapter 3 will deal with 4-configurations. Chapter 4 will present information about k-configurations for $k \geq 5$, as well as some other kinds of configurations. Chapter 5 will discuss known results on various properties of configurations (among them connectivity, Hamiltonicity, movability). In many sections we shall also pay attention to *combinatorial configurations* and on *configurations of pseudolines*. (The italicized concepts will be explained in the following sections.)

* * * * * *

A number of other directions of investigation start with families of lines and/or points in the plane, but have distinct aims from the study of configurations. The closest one to configuration has only recently been named, although some of its results go back close to two centuries. In an attempt to find a common framework for the various results that are more-or-less well known the term "aggregate of lines (or points)" has been proposed. The topics covered in this discipline deal – like configurations – with incidences of lines and points, but without the assumption of equal numbers for all lines and all points. The most famous among them are known as "orchard problems" and "Sylvester's problem". The former typically asks to locate a certain given number of points so that a maximal possible number of lines are incident with precisely 3 (or some other chosen number) of these points. For references see [B28, Chapter 7], [B27], [C13, Section F12], and [I1].

Sylvester's problem is to show that if a family of n lines is such that they are not in a pencil (that is, all incident with a single point), then there is an *ordinary point* -- a point incident with precisely two of the lines. In fact, the number of points is always greater than one, and a longstanding conjecture is that there are at least [n/2] such points. For more details about this problem and related ones see [B27], [B28, Chapter 7], or [C13, Section F12].

*   *   *   *   *   *

A guiding principle of this book is the conviction that mathematics should be both interesting and attractive, and that it should give pleasure while working on it or reading about it. That is why the pace of the presentation is rather leisurely, eschewing acronyms and *ad hoc* abbreviations. It is also the reason for the inclusion of many diagrams – even in situations in which a formal argument could have been supplied. It is my hope that the reader will find this approach inviting, and the appeal to geometric intuition useful and stimulating.

Exercises and problems 1.1.

In the following sections we shall present lots of exercises that deal with configurations. As a warm-up, here are some questions that are only marginally relevant to configurations, but have much in common with the spirit that permeates the study of configurations.

1. The *orchard problem* is: For sets of p points in the plane, find t(p), the maximal possible number of lines containing precisely three of these points; see [B33]. It is known that t(12) = 19. Can you find a set of 12 points with this property? Concerning t(13) it is known only that $22 \le t(13) \le 24$. Can you improve on this?
2. Particular case of the generalized *Sylvester problem*. Let s(p) denote the minimal possible number of points incident with precisely two of a set of p lines, not all concurrent and no two parallel. It is known only that $7 \le s(15) \le 9$. What is the correct value?
3. Given n lines in the plane, no three concurrent and no two parallel. Show that among the bounded regions they determine in the plane there are at least n-2 triangular regions. This is known as *Roberts' theorem* (see [G36, p. 398]).

1.2 AN INFORMAL HISTORY OF CONFIGURATIONS

Somewhat parallel to the history of Western civilization, the development of the theory of configurations can be assigned to distinct periods. However, in the case of configurations it is possible to assign precise dates to each of these periods.

We begin with the **prehistory**. By this I mean the relevant results developed by various mathematicians prior to the year 1876, at which time the concept of configurations was first formulated by T. Reye [R1]; see also [R2], where the notation n_k was introduced[1]. The results in question were formulated as theorems pertaining to certain sets of points and lines; in retrospect we can see that the results can be interpreted as configurations, or as implying the existence of certain configurations. Typical for this "prehistoric" results is the theorem of Pappus (see, for example [C7, p. 67], [C12, p. 231], which is usually formulated as follows (see Figure 1.2.1):

If alternate vertices of a hexagon [2,6,8,3,5,9] lie on two lines, then the three intersections 1, 4, 7 of opposite sides of the hexagon are collinear.

It should be noted that this formulation requires additional explanation and modifications in some special cases, but this is not really relevant to the present discussion.

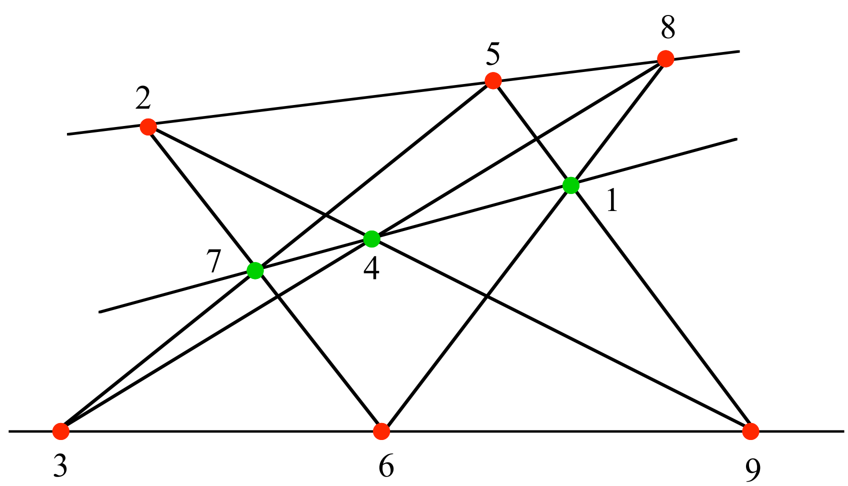

Figure 1.2.1. An illustration of the theorem of Pappus. Solid red dots are the two triplets of collinear vertices of a hexagon, solid green dots are the collinear intersection points of the three pairs of opposite sides of the hexagon.

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1 This notation, or the slight modification (n_k), are in general use. Since in many cases n is either irrelevant or not known, we shall also use "k-configuration" in such instances.

In our interpretation, Pappus' theorem amounts to the assertion that the incidences indicated in Figures 1.1.4 and 1.1.5 are correct. We used there the same notation as in Figure 1.2.1.

Other results from the prehistoric period can be found in works of Desargues, Steiner, Moebius, Cayley, and others. We shall mention them in due course.

* * * * *

Next comes the **classical period**, which runs from 1876 to 1910. It starts with the publication of Reye's book [R1], and ends with the publication of the survey of configurations by Ernst Steinitz [S19] in the Encyklopädie der mathematischen Wissenschaften. The period covers the formulation of the configuration concept as well as a number of basic results, in particular those related to questions (A) and (B) in Section 1.1. This includes works by such mathematicians as Reye, Kantor, Martinetti, Schroeter, Schönflies, Brunel, Burnside, Daublebsky, Steinitz and others. For example, Kantor [K3], [K4] gives answers to question (B) for n = 8, 9, 10. Unfortunately, his results are not correct as claimed. One of his errors concerns the enumeration of the combinatorial types of configurations (10_3). Kantor claims that there are precisely ten different types of such configurations, and presents diagrams purporting to illustrate these types. In Figure 1.2.2 we reproduce Kantor's drawing of one of these configurations. However, in contrast to the situation with the Pappus configuration explained above, a configuration of this type **cannot** be drawn with **straight** lines. The reviews [R6] of [K4] by C. Rodenberg and [S10] of [K3] and [K4] by H. Schubert contain several incorrect assertions. The impossibility was first proved by Schroeter [S8], and by other methods more recently by Carver [C2], Laufer [L1], Bokowski-Sturmfels [B25], Glynn [G2], and Sternfeld et al. [S22]. Claims by Dolgachev [D8] that this result is due to Kantor [K3] (sic) and that "a modern proof" is given in [B15] are both incorrect. We shall return to this topic in Section 2.2.

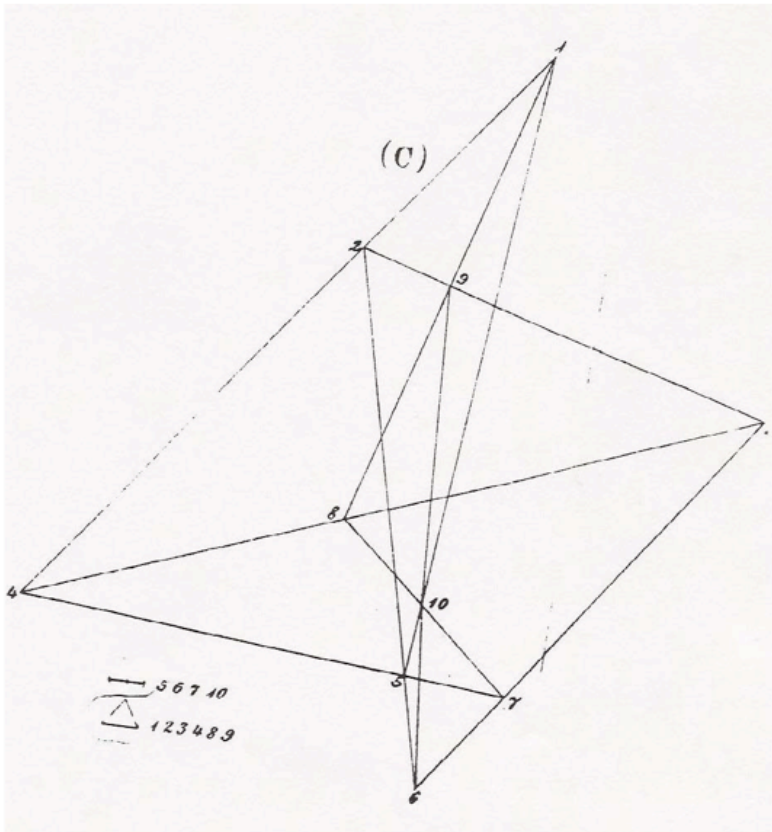

Figure 1.2.2. A diagram from Kantor [K4] which was supposed to illustrate a configuration (10_3), but which cannot be drawn by straight lines (as we shall see in Section 2.2).

While Kantor's error was discovered shortly after its publication, other deficiencies of his paper did not receive attention till very recently (see, for example, Section 2.3). The same late discovery of errors occurred with certain works of Martinetti and Steinitz, which were considered as basic for the theory of configurations throughout the twentieth century; however, the "Ernst Steinitz" article in the *MacTutor History of Mathematics* [O1] ignores all of Steinitz' work on configurations! We shall discuss Martinetti's and Steinitz's claims, and their corrected versions, in Chapter 2.

It is not clear how the various errors arose, and it is even more mysterious why they remained hidden for close to a century. A possible answer to the latter question is that for a long time apparently nobody cared enough about the topic to immerse oneself in the long and murky expositions of the original papers.

Other investigations during this period dealt with specific types of configurations, such as "mutually inscribed and circumscribed polygons". As relating to k-configurations, with the single exception of a paper by Brunel [B31], the considerations were limited to $k = 3$. We shall consider these results, as well as some more recent one on these topics, in many of the following pages.

* * * * *

The classical period, which was characterized more by enthusiasm for configurations than by solid mathematical achievement, was followed by long lasting **dark ages** of configuration theory. While configurations did not disappear completely from the mathematical horizon, in the period from 1910 till 1990 there were few significant publications on this topic. It may well have been no fault of configurations that the creative attention of mathematicians was directed to other fields. Rather it is the excitement caused by spectacular development in other mathematical disciplines that attracted most researchers; it led to general neglect of the intuitively accessible parts of geometry. The interest shifted to more technically intricate fields, leaving configurations dead in the water.

A few exceptions from this gloomy picture deserve to be mentioned. The first is the only book-length serious publication on configurations, by Levi [L3]. Unfortunately for the theory of configurations, the book had practically no influence on later works on configurations. Possible explanations for this lack of effect may be the very restricted types of questions considered (fifty pages are devoted to the consideration of the lines associated with the theorem of Pascal – a topic that was very popular in the “prehistory”, but has practically no relevance to more modern investigations of configurations), together with the dry and pedestrian tone of exposition, the pedantic discussion of topics not connected to configurations (such as the long chapter on regular polyhedra), and the almost complete absence of references to previous work on configurations. Naturally, the fact that the center of gravity of mathematicians' interests shifted to other fields must also be considered as relevant to the book's lack of influence. We shall have occasion to mention Levi's book several times in connection with some of the original results contained in it – but this is meager pickings for a book of more than 300 pages.

Three years after the publication of Levi's work an extremely well received book was published by Hilbert and Cohn-Vossen [H4] (see, for example, [F2]). Twenty years later an English translation was published, but unfortunately the editors of Mathematical Reviews did not feel it deserves more than a listing; a second German edition in 1996 did not get a even a listing. This is a grievous mistake, since many later workers (the present author included) became interested in configurations by reading the account in [H4]. This presentation of the basics of configuration theory is contained in just a part of one chapter of [H4], but presents an attractive approach to the topic. It has been often mentioned as a justification for studying configurations, by quoting the following sentence from [H4, English translation p. 95]:

> "... there was a time when the study of configurations was considered the most important branch of geometry."

I would like to conjecture that this is the greatest exaggeration of the truth that can be found in any of Hilbert's writings. While it is a fact that — as mentioned above — in the "classical period" of the history of configurations there were quite a few people interested in the topic, configurations were never a central topic of mathematical (or geometric) research.

Even so, the influence of these two books can on occasion still be discerned today. For example, the recent work [P2] by K. Petelczyc mentions only these two sources for its information about configurations — ignoring all earlier and later publications.

Another relevant publication is the book by H. L. Dorwart [D9].

Other points of light during the "dark ages" of the configurations were several papers by Coxeter. Two of his early contribution to the topic are [C5] and [C6]. The latter – reproduced in [C11] – introduced several new ideas and popularized some older ones; we shall mention it frequently in various section of this book. Coxeter's other contributions to configurations are his papers [C8], [C9] and [C10], in which he presents detailed studies of certain specific configurations.

Some other papers on configurations that were published during the "dark ages" will find mention in appropriate places. They did not amount to much, and some of them were wrong.

* * * * *

After the long "dark ages" of the theory of configurations came a **renaissance** that continues to this day. It amounted to posing questions different from the ones considered previously, as well as to application of new tools and methods. For example, the investigation of Euclidean symmetries of a configuration was found to lead to more meaningful presentations of various configurations, and also to lead to the construction of previously unimagined ones. This is evident in the illustrations of the 4-configurations in Figure 1.1.2, and will become a leading topic in several later sections. Another difference concerns the more careful attention given to questions of graphic presentation of configurations, the use of new computational methods and computer graphics in the study of configurations, and – last but not least – to precise definitions, formulation, and proofs of results.

Rather immodestly, I believe that I played a significant role in the "renaissance" of the theory of configurations. Influenced by the chapter on configurations in the Hilbert and Cohn-Vossen's book [H4], in the mid-1980's I started studying configurations and presenting some of the results and problems in various seminars at the University of Washington, and in some courses I was giving; for example, [G36], which is mentioned in [B25]. The first formal publications resulting from these actions were the publication of the Grünbaum and Rigby paper [G50] in 1990, and the Sturmfels and White paper

[S24] in the same year. The former contains the first published images of any (n_4) configurations, while the latter answers (affirmatively for $n \leq 12$) the question I posed earlier (in [G36]) whether every configuration (n_3) can be realized in the *rational* plane. These and other publications that followed led soon to a revival of interest in configurations, with significant advances in many directions. These will be discussed in Chapters 2 and 3, and seem to justify the use of the term “renaissance”.

Readers interested in the history of configurations and the individuals involved in its creation may wish to consult the various papers of H. Gropp ([G10], [G15], [G26], [G28], [G27], [G29], [G30]). However, it should be borne in mind that Gropp's attitude towards geometric configurations may be inferred from his statement in [G30]: "From the point of view of pure combinatorics the problems of realizing and drawing configurations may be artificial."

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History does not stop today. There is no doubt that the theory of configurations, and related topics, will continue to be studied and to evolve. It is not possible to predict what direction the future investigations will take, but it is possible to hope that deeper connections will be established with algebraic geometry — a rather natural home to configurations — and to topology.

Exercises and problems 1.2

1. Define what you understand by say that "two configurations are essentially the same". Use your definition to decide whether the configurations in Figure 1.2.3 fit your definition of "sameness".

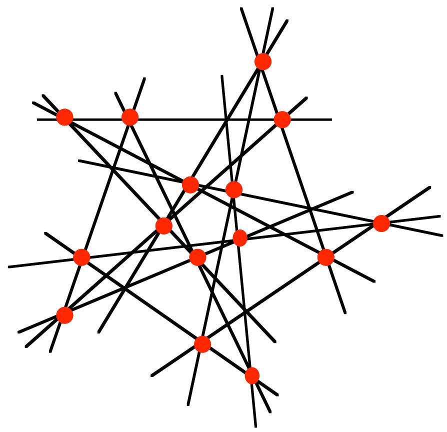


Figure 1.2.3. Two configurations (15_3). Are they "essentially the same" according to your definition?

2. Can you find a configuration that is "essentially the same" according to your definition as the ones in Figure 1.2.3 but has 3-fold rotational symmetry?

3. Find (in a book, or Google, or ...?) a formulation of the Desargues' theorem, that leads to the Desargues configuration in Figure 1.1.1.

1.3 BASIC CONCEPTS AND DEFINITIONS

In this section we shall clarify the fundamental concepts involved in the study of configurations. We shall start with very general definitions that will enable us to specialize and particularize the concepts as we find appropriate. Considerable care is required in distinguishing the various related concepts; neglect to do so led to many of the problems we mentioned in the historical account in Section 1.2. The reader who finds the plethora of words confusing, should skip the details and return to this section when the text uses terms that need explanation. This author's position in connection with the abundance of terms is that Johann Wolfgang Goethe had it all wrong when (in "Faust", lines 1995-1996) he has Mephistopheles say:

"Denn eben wo Begriffe fehlen,
Da stellt ein Wort zur rechten Zeit sich ein"[1]

On the contrary; words are needed (even if they have to be invented) when concepts appear that need to be distinguished from other concepts. So, here we go.

A *configuration* C is a family of "points" (sometimes called *vertices*) and a family of "lines" such that, for positive integers p, q, n, k each of the p "points" that constitute the family is "incident" with precisely q of the n "lines", while each of these "lines" is "incident" with precisely k of the "points". The use of quotation marks is meant to indicate that these objects can be of any nature whatsoever, as soon as the "incidence" relation satisfies what can be considered the natural conditions:

- It is a symmetric (that is, mutual) relation;
- An "incidence" can involve only a "point" and a "line", never two "points" or two "lines"; and
- Two "points" (or "lines") can be incident with at most one "line" ("point").

Moreover, it is assumed throughout the book that use of the word *configuration* implies that there is a "point" and a "line" that are not "incident" with each other. The totality of "points" and "lines" of a configuration will be called its **elements**.

[1] Freely translated: "Where concepts are missing, a word soon appears."

A configuration C with **parameters** p, q, n, k will in general be denoted by (p_q, n_k). The concept of (p_q, n_k) configurations and the notation for it were introduced by de Vries [D5] in 1888. If the particular values of p and n are not important, we shall say that we have a **[q,k]-configuration**. If q = k we shall simplify the notation by dispensing with the brackets and write **k-configuration**. In the terminology of a parallel universe, what we call a [q,k]-configuration is known as a "geometry of order (q-1, k-1)", see, for example, van Maldeghem [V2]. Note that a [q,k]-configuration is called a "slim geometry" in the terminology of that universe, and a 3-configuration is a "bislim geometry".

By counting the total number of incidences in a configuration (p_q, n_k), it follows that a necessary condition for the existence of a configuration with parameters p, q, n, k is the equation $pq = nk$.

There are additional necessary conditions. Each "point" of (p_q, n_k) is "incident" with q "lines", each of which is "incident" with k–1 other "points". Hence there are at least $p \geq q(k-1) + 1$ "points". A similar argument shows that $n \geq k(q-1) + 1$. To avoid trivialities we shall generally assume that $q \geq 2$ and $k \geq 2$, which implies that $p \geq 3$ and $n \geq 3$. (Exceptions will be signaled explicitly.) Although these necessary conditions are in many cases sufficient for the existence of some configuration (p_q, n_k), we shall see in the following sections that this is not always the case.

Much of the time we are interested in configurations (p_q, n_k) with p = n (and therefore q = k). As already mentioned, it is customary to simplify the notation for such configurations, and designate them as $\mathbf{(n_k)}$ **configurations**, or **k-configurations**. The (n_k) notation (or, more precisely, the very similar n_k notation) was introduced by Reye [R2] in 1882; it is convenient in many contexts, but in cases the number of points and lines is not known or is nor relevant, it seems illogical to insert the letter n that has no meaning in such a situation; hence the alternative "k-configuration" notation.

In the literature, k-configurations are often called "symmetric"; but this term is highly unsuitable since the configurations in question may fail to have any symmetry whatsoever, geometric or combinatorial. (The ill-advised use of "symmetric" in this context seems to go back to erroneous understandings in some late 19th century writings on configurations, and possibly to its use in the theory of BIBDs – balanced incomplete block designs; the latter are totally irrelevant to the theory of configurations. More about this in Section 1.5.) The main objection to the use of "symmetric" in this meaning is that, as we shall see throughout the text, configurations exhibiting certain genuine symmetries – combinatorial or geometric -- are very important. With much justification it may be claimed that configurations with *geometric symmetries* have been the motivating factor in the recent great expansion of knowledge about configurations. Hence bestowing the descriptor "symmetric" to configurations that may be totally devoid of symmetries is downright misleading. In the present book we shall say that k-configurations are **balanced**; it is hoped that this term will become the accepted designation for k-configurations. (The use of "balanced" in the BIBD theory is not compromised by this use for configurations, just as the use of "symmetric" for BIBDs raises no problems for configurations.)

We shall be concerned with configurations at three levels of generality. In the most restricted sense, “points” and “lines” are interpreted as being the points and lines in some space in which these concepts are defined, so as to satisfy the first two groups of Hilbert’s axioms. (For these axioms see, for example, Hilbert [H3], Noronha [N1], Sibley [S14], Stahl [S16].) In particular, this interpretation includes the traditional Euclidean plane and higher-dimensional Euclidean spaces, as well as the real projective plane and higher-dimensional projective spaces. Unless specifically stated otherwise, we shall consider configurations at this level of generality to be in the real Euclidean plane, or in the real projective plane, and call such configurations **geometric**. In Appendix A the necessary facts about the Euclidean and projective planes are collected. The usual way of presenting geometric configurations is by **diagrams**, in which “points” are represented by solid dots and “lines” by straight lines. Naturally, it must be true (and possible to verify) that the presumed lines are straight and that the incidences actually occur; some diagrams can be misleading, as illustrated by Figure 1.2.2. More details on this topic will be found in Section 2.2. For general discussions of the topic of drawing configurations see [G21], [G26].

In the most general sense we shall consider **combinatorial** (or **abstract**) configurations; we shall use the term **set-configurations** as well. In this setting “points” are interpreted as any symbols (usually letters or integers), and “lines” are families of such symbols; “incidence” means that “point” is an element of a “line”. It follows that combinatorial configurations are special kinds of general incidence structures. Occasionally, in order to simplify and clarify the language, we shall for "points" use the name **marks**, and for "lines" we shall use **blocks**. The main property of geometric configurations that is preserved in the generalization to set-configurations (and that characterizes such configurations) is that two marks are incident with at most one block, and two blocks with at most one mark. The usual way of presenting set-configurations is by **configuration tables**. In such a table the marks of each block are listed in a column that represents the block. If necessary or convenient, a label for the block may be indicated at the head of each column, but in other cases this may not be needed. An example of a set-configuration $(16_3, 12_4)$ is shown in Table 1.3.1 (with block labels); in Table 1.3.3 we dispensed with the block labels. Sometimes a combinatorial configuration (p_q, n_k) is presented simply as a family of n k-tuples formed from p marks, with appropriate restrictions. In [G11] and other papers, H. Gropp faults Hilbert and Cohn-Vossen [H4] and Coxeter [C6] for being interested in structures "realized in real geometry" and not in the "more modern 'schematical' configurations". This seems to the present author to be quite inappropriate, especially since in most of his papers Gropp does not warn the reader that his use of "point" and "line" is meant in purely combinatorial sense.

a	b	c	d	e	f	g	h	i	j	k	l
A	A	A	H	B	B	B	I	C	C	J	D
H	N	G	N	H	N	I	P	O	J	O	K
I	O	L	P	G	M	J	M	M	K	P	L
C	D	E	K	F	E	D	L	F	E	G	F

Table 1.3.1. A configuration table for a set-configuration $(16_3, 12_4)$.

It is immediate from the definitions that every geometric configuration gives rise to a set-configuration: Just label the points of the geometric configuration, and use these labels as marks to construct a configuration table. As can be checked very easily, the set-configuration in Table 1.3.1 corresponds in this way to the $(16_3, 12_4)$ geometric configuration in Figure 1.3.1. However, as we shall illustrate in Section 2.1, the converse is not valid: there are set-configurations that do not correspond to any geometric configuration.

The relationship between the configurations in Table 1.3.1 and Figure 1.3.1 is sometimes formulated by saying that the former **underlies** the latter, and that Figure 1.3.1 is a geometric **realization** of the set-configuration in Table 1.3.1.

A remarkable result goes back to Steinitz [S17], in the classical period of the theory of configurations. In one formulation, it states that every combinatorial k-configuration can be presented in an **orderly** configuration table. A configuration table is orderly if each point (mark) appears in each row of the table once and only once. We shall discuss this result and some of its ramifications in Section 2.5.

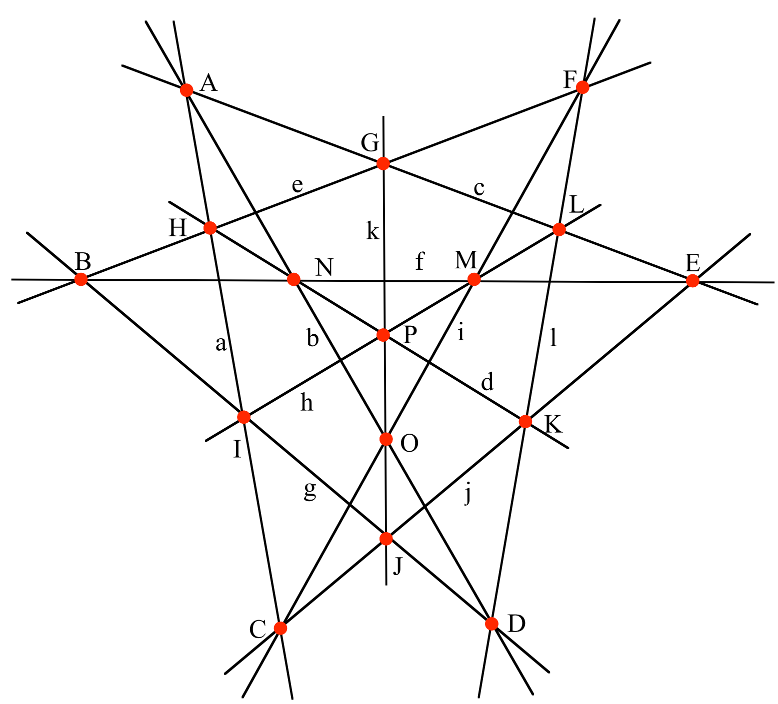


Figure 1.3.1. A geometric configuration $(16_3, 12_4)$, with points labeled in such a way as to yield the configuration table in Table 1.3.1.

If the points and lines of two configurations C' and C" admit labels such that a 1-to-1 correspondence τ of points to points (and lines to lines) preserves incidences, we shall say that C' and C" are **isomorphic** (or **combinatorially equivalent**, or of the same **combinatorial type**); sometime we shall also wish to make explicit the correspondence τ. With appropriate interpretation, this terminology applies to set-configurations as well. For example, the set-configuration in Table 1.3.1 is isomorphic to the geometric configuration in Figure 1.3.1 with τ the identity transformation ι of the labels. If an incidence-preserving correspondence maps points to lines (and vice versa), it is said to be a **duality**, and the configurations are said to be **dual** to each other.[2] A configuration table of the configuration dual to the one in Table 1.3.1 is shown in Table 1.3.2 and in Figure 1.3.2 is a geometric configuration dual to the one in Figure 1.3.1.

Here again a warning seems necessary. Some authors (for example, van Maldeghem [V2]) use the word "realization" in a different meaning. In particular, they allow (geometric) realizations of set-configurations to have additional incidences, not among the ones in the underlying set-configurations. Put differently, they use "realization" for a different concept, which we shall encounter below under the designation "representation". Thus, for us, a geometric realization of a set-configuration is isomorphic to it, while a representation that is not a realization is not isomorphic to it.

Some configurations are isomorphic to configurations dual to them. In such cases we call the configuration **selfdual**. In other words, a configuration C is selfdual if there is an incidence-preserving correspondence τ that maps the points of C onto its lines and vice versa; such correspondence τ is called a **selfduality**, and it is obvious that in this case the inverse τ^{-1} is a selfduality as well. In many cases (but not always) τ^2 is the identity map ι. An example of a selfdual configuration with $\tau^2 = \iota$ is shown in Figure 1.3.3. We shall discuss this topic in much more detail in Sections 2.10 and 5.8.

2 It is unfortunate that Coxeter [C6] uses "dual configurations" to mean any pair consisting of a (p_q,n_k) configuration and a (n_k,p_q) configuration; in this terminology any (n_k) configuration is selfdual. This error has been copied by Evans [E1*].

There is often a need to consider families of points and lines (or marks and blocks) that fail — by a "few" incidences — to be configurations in the sense we use here. We say that such a family is a **prefiguration**. We shall often encounter two types of prefigurations, although other types occur at times.

A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P
a	e	a	b	c	e	c	a	a	g	d	c	f	b	b	d
b	f	i	g	f	i	e	d	g	j	j	h	h	d	i	h
c	g	j	l	j	l	k	e	h	k	l	l	i	f	k	k

Table 1.3.2. A configuration table of the dual of the configuration in Table 1.3.1 and in Figure 1.3.1.

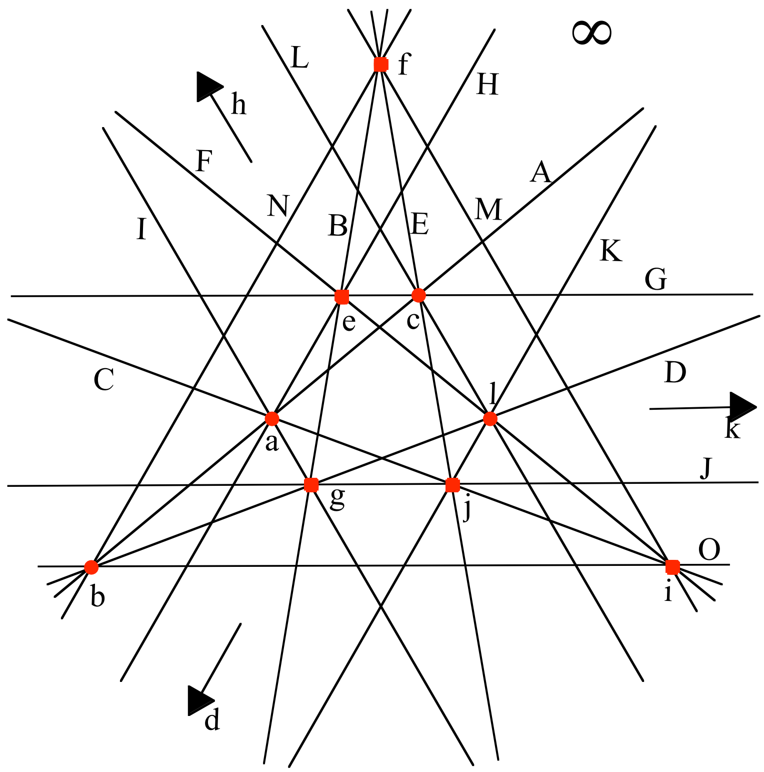


Figure 1.3.2. A geometric configuration that is a realization of the set-configuration in Table 1.3.2. It is dual to the configuration in Figure 1.3.1. The arrows indicate “points-at-infinity”, and the “line-at-infinity”, indicated by the infinity symbol, is also part of the configuration. Explanations of this kind of diagrams in the projective (or extended Euclidean) plane are given in Appendix A.

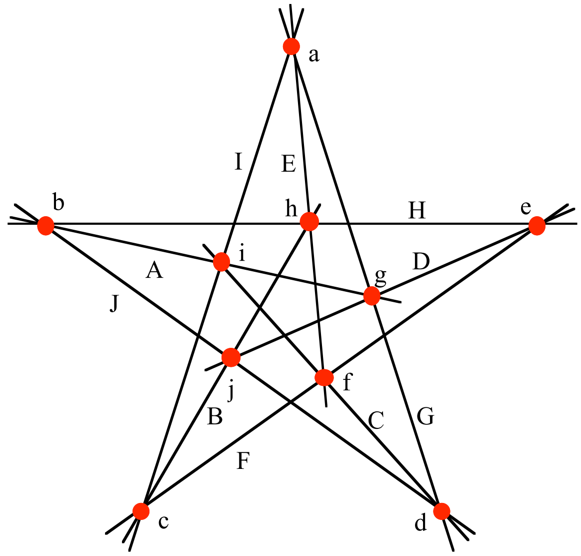


Figure 1.3.3. An example of a selfdual configuration (10_3). The selfduality mapping τ interchanges the upper- and lower-case letters. Hence clearly $\tau^2 = \iota$, the identity.

A **superfiguration** (p_q, n_k) is a family of "points" and "lines" with "incidences" as in the definition of configurations, such that each of the p "points" is "incident" with **at least** q of the n "lines", and each "line" is "incident" with **at least** k of the "points". If the number of incidences exceeding that in a [q,k]-configuration is s, we shall sometime say that it is an **#s-superfiguration**. An example of a remarkable (and quite useful – see Section 2.11) superfiguration is shown in Figure 1.3.4; it is a #2-superfiguration of a 3-configuration, since there is one line incident with four points and one point incident with four lines. (This is taken from [G36]; it also appears in [B25] and [M18], and probably in other places as well.) Superfigurations often arise through "accidental" incidences in configurations that have realizations depending on variable parameters.

In another way of looking at this situation is that sometimes it is necessary or convenient to consider certain superfigurations as *representing* combinatorial *configurations.* As already mentioned, by a **representation** we mean a family of points and lines such that all the combinatorial incidences are satisfied but some points may be on lines with which they are not incident in the combinatorial configuration, or some pairs of distinct points (or lines) of the combinatorial configuration may be represented by single points (or lines). A typical example of such a superfiguration is shown in Figure 1.3.5, which is a *representation* of the set-configuration specified in Table 1.3.3. The point 1 in Figure 1.3.5 lies on the line 089 but is not incident with it according to Table 1.3.3.

1	1	1	2	2	3	3	0	0	0
2	4	6	4	7	5	6	4	5	8
3	5	7	8	9	9	8	6	7	9

Table 1.3.3. A configuration table of a set-configuration (10_3). We shall encounter this configuration again in Section 2.2.

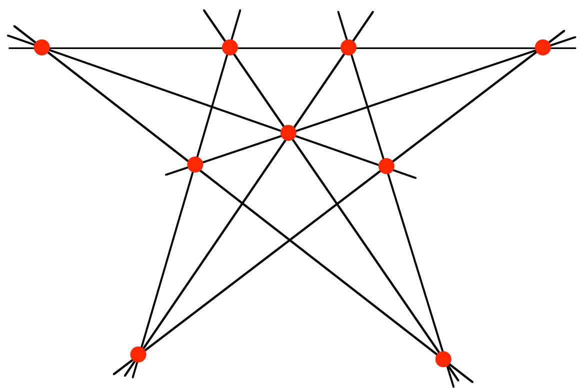


Figure 1.3.4. An example of a geometric #2-superfiguration (9_3), in which one point is incident with four lines, and one line is incident with four points.

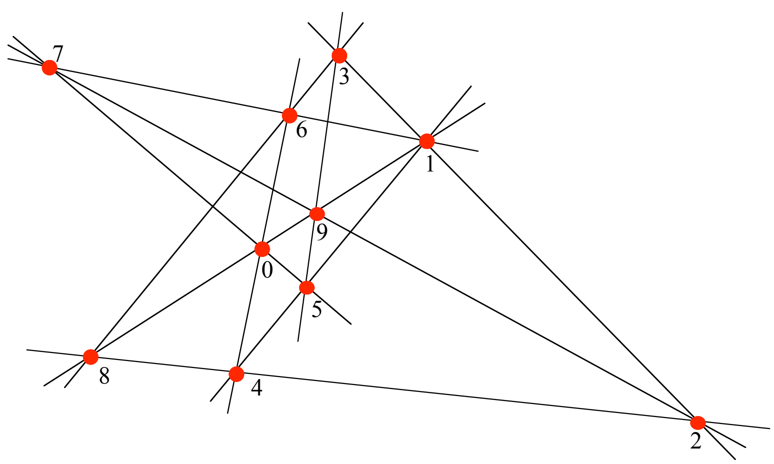


Figure 1.3.5. A #2-superfiguration that is a representation (but not a realization) of the set-configuration specified by the configuration table in Table 1.3.3.

In the same vein we call **subfiguration** (or #**s-subfiguration**) a family that is short by a number of incidences (specifically, s incidences) to be a [q,k]-configuration. An illustration of this concept is provided in Figure 1.3.6. Other examples will become very important in Section 2.5.

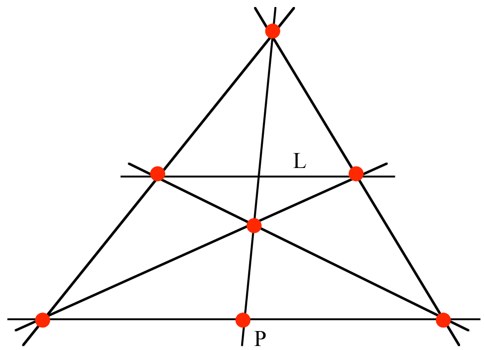


Figure 1.3.6. A subfiguration of a (7_3) configuration. If the line L were incident with the point P, this would be a configuration. However, as we shall see in Section 2.1, there exists no geometric configuration (7_3), although a set-configuration of this type is possible.

* * * * *

Intermediate in generality between set-configurations and geometric configurations are **topological configurations**, also known as **configurations of pseudolines**. A family of simple curves in the (projective or Euclidean) plane is a family of **pseudolines** provided each curve differs from a straight line in at most one segment of the Euclidean (or projective) line, and any two curves have at most one point in common, at which they cross each other. We shall discuss pseudolines in more detail in several of the following sections. The definition implies that any two points are incident with at most one pseudoline.

Configurations of pseudolines are defined in complete analogy to configurations of lines, with pseudolines taking the place of “lines”. As an example of a topological configuration we may take Kantor’s presumed configuration shown in Figure 1.2.2, which – as we mentioned in Section 1.2 and will prove in Section 2.2 — is not realizable as a configuration of lines. Other examples are shown in Figure 1.3.7. Clearly, any geometric configuration can be understood as a topological configuration, and each of the latter has an underlying set-configuration. It is easily seen that each topological configuration is isomorphic to a configuration in which each pseudoline consists of a finite number of (straight) segments (including rays, considered as segments). For a detailed discussion of the various concepts of pseudolines that appear in the literature see [E4]. Several problems of combinatorial geometry that involve pseudolines are discussed in [G37].

It is clear that if a set-configuration is geometrically *representable* by a superfiguration C, it is also *realizable* by a topological configuration. The pseudolines may be "bent" to avoid offending incidences. However, not every superfiguration can be realized by a topological configuration. For example, the superfiguration in Figure 1.3.4 is not a representation of any set-configuration, and cannot be realized by a topological configuration.

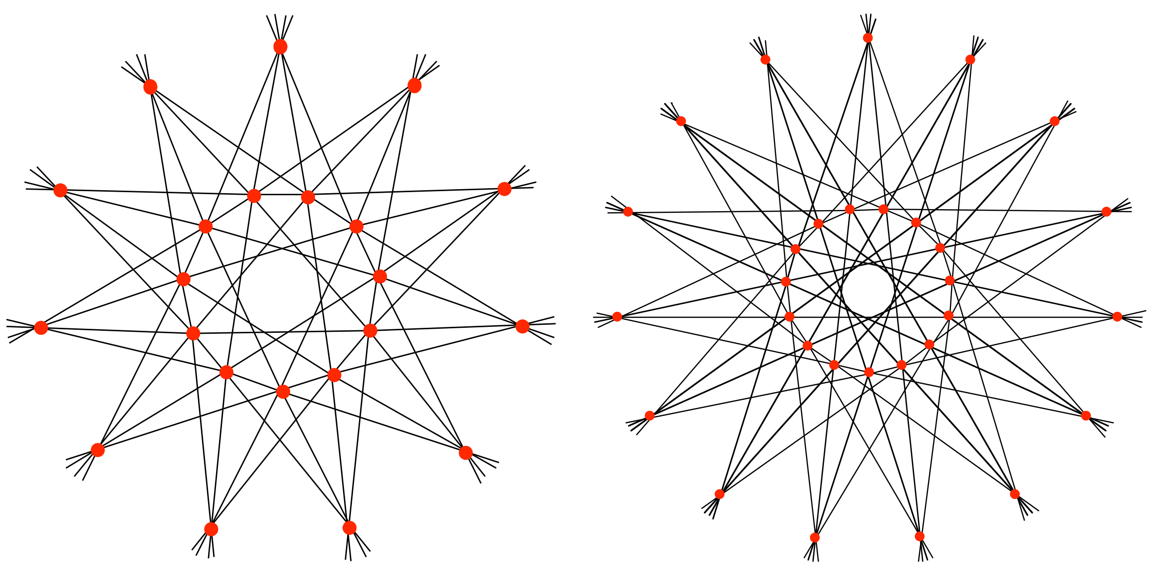


Figure 1.3.7. A (22_4) and a (30_4) configurations of pseudolines. Note how small is the departure from straight lines in these examples. It may be conjectured that these configurations are not isomorphic to any geometric configuration; however, this has not been established.

* * * * *

The terminology of the theory of configurations is very unsettled. Different writers, and "schools" use terms that are often quite unrelated to each other, and sometimes even carry a different meaning while using the same words. An example of the former is the use of terms like *slim* and *bislim geometries* to indicate what we call configurations.

On the other hand, some writers discuss at length "configurations" without bothering to mention that they have **combinatorial** – and *not* geometric – configurations in mind. This was to some extent excusable in the nineteenth century, when the essential difference between the concepts had not yet been recognized. A hundred year later this is exemplary carelessness, or at the least, total disregard for traditional terminology and the work of earlier authors. Still other writers state that configurations are "partial linear spaces with constant and equal point rank and line rank"; despite the use of explicit geometric terms (including "n-gons") the configurations they consider are only combinatorial (see, for example, [P2], [K1]).

The internet also abounds with vague, misleading, or wrong entries. This applies, in particular, to the frequently consulted Wikipedia (see [W3], as modified 9 November 2007) and Mathworld (see [W1], quoted from version dated November 30, 2007). In the latter one find, among other inaccurate assertions, that the (7_3) and (8_3) configurations are realizable with a "point at infinity".

* * * * *

It is easy to see that every realization of a set-configuration by points and lines in any Euclidean or projective space (of any dimension) can lead by suitable projection to a geometric configuration in the Euclidean plane. However, the converse question — can a given geometric configuration be realized in a higher-dimensional space in such a way that it is not contained in a subspace — has a negative answer in some cases. For example, each of the three configurations (9_3) in Figure 1.1.6 is easily seen to be contained in the plane spanned by some three of its points. With this example in view, it is meaningful to define the **dimension** of a configuration as the largest dimension of a space that is spanned by the configuration. We shall discuss this topic in Section 5.6.

In many questions about configurations one is concerned with what in the literature is often called "polygons". However, this is a misnomer since in most cases it is not segments that are relevant as "sides" of the "polygons"; instead, the intention is to deal with the lines of the configuration. We shall call **multilateral** any sequence of points and lines of a configuration that can be written as $P_0, L_0, P_1, L_1, \dots, P_{r-1}, L_{r-1}, P_r = P_0$, with each L_i incident with P_i and P_{i+1} (all subscripts understood mod r). For example, in the prefiguration shown in Figure 1.3.5, the sequence of points 1, 2, 8, 0, 4, 5, 7, 9, 3, 6, 1 (and the lines determined by adjacent pairs) determines a multilateral that involves all points and all lines. A **multilateral path** satisfies the same conditions except the coincidence of the first and last elements. A **Hamiltonian multilateral** passes through all points and uses all lines, each precisely once; hence the example just given is a Hamiltonian multilateral. We shall encounter multilaterals in several sections, and in particular Section 5.2 is devoted to Hamiltonian multilaterals.

Exercises and problem 1.3.

1. Show that the superfiguration in Figure 1.3.4 is selfdual. Find a selfduality map τ such that $\tau^2 = \iota$.

2. Decide whether any of the (12_3) configurations in Figure 1.3.8 are isomorphic. (As a practical matter, to show that two configurations are isomorphic it is sufficient to find an isomorphism. To show that they are not isomorphic, it is often simplest to find a property that is invariant under all isomorphisms but regarding which the two configurations behave differently. Neither is all that simple to actually carry out, even for rather small configurations.)

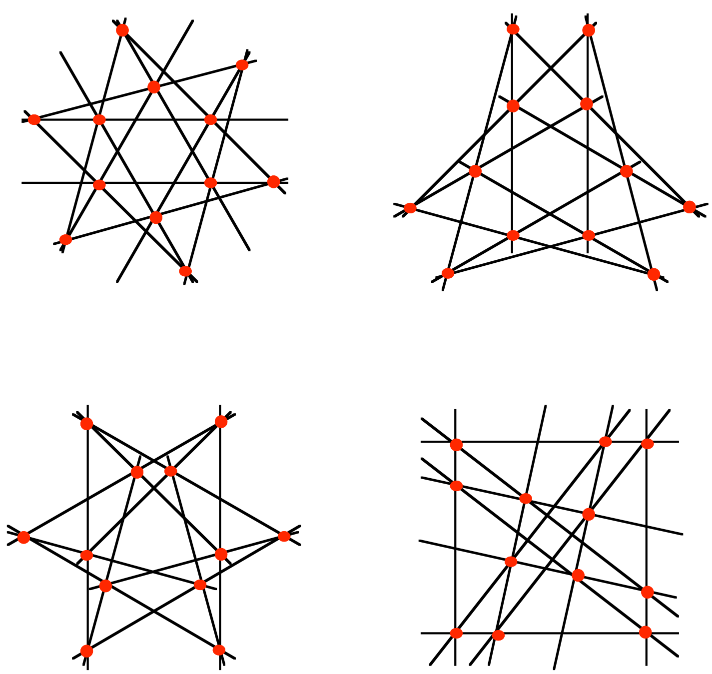


Figure 1.3.8. Four configurations (12_3). Are any isomorphic?

3. What is the smallest n required for the existence of a combinatorial configuration (n_4) ? Can you find a configuration table for it? Can you decide whether it is unique (up to isomorphism) ?

4. Do the topological configurations in Figure 1.3.7 admit Hamiltonian multilaterals? What about the configurations in Figure 1.3.8?

5. Determine the dimension of each of the configurations in Figure 1.3.8, and of the configuration in Figure 1.3.3.

1.4 TOOLS FOR THE STUDY OF CONFIGURATIONS

There are many different ways to relate configurations of points and lines to other mathematical objects; these are often useful in investigating configurations. In Section 1.3 we encountered one example of such a tool — the underlying set-configuration and its *configuration table*. A related concept, which is helpful in the context of more general combinatorial structures as well, is the *incidence matrix* of the configuration. Specifically for configurations, incidence matrices seem to have been introduced by Levi [L3], and we shall usually call them "Levi incidence matrices". Levi makes the rows of a matrix correspond to the points of the configuration, and the columns to the lines (or vice versa). An incidence between a point and a line is indicated by a marking of the corresponding element of the matrix. This can be done by assigning to such matrix elements the value 1, and to the others 0, or by some other specification. Levi's own preference is to use an array of small squares, and an X marked in each square that represents an incidence. As an illustration, consider the set-configuration we shall encounter in Section 2.2, and denote there by $(10_3)_4$. A configuration table is shown in Table 1.4.1, and a Levi incidence matrix in Figure 1.4.1(a). By a suitable permutation of the rows and columns of that matrix, we obtain the Levi incidence matrix in Figure 1.4.1(b). This latter form demonstrates one of the uses of incidence matrices: It shows at a glance that the configuration in question is selfdual, since the matrix is symmetric with respect to the main diagonal.

a	b	c	d	e	f	g	h	i	j
1	1	1	8	2	3	2	3	4	5
2	4	6	9	4	6	5	7	6	7
3	5	7	0	8	8	9	9	0	0

Table 1.4.1. A configuration table for a configuration discussed in Section 2.2, and denoted there by $(10_3)_4$. The notation is adapted from Schroeter [S8]. Two versions of the Levi incidence matrix of this configuration are shown in Figure 1.4.1, and its configuration graph in Figure 1.4.2. A realization with pseudolines appears in Figure 1.2.2.

	a	b	c	d	e	f	g	h	i	j
1	X	X	X							
2	X				X		X			
3	X					X		X		
4		X			X				X	
5		X					X			X
6			X			X			X	
7			X					X		X
8				X	X	X				
9				X			X	X		
0				X					X	X

(a)

	c	b	i	j	e	f	a	h	g	d
3						X	X	X		
2					X		X		X	
8					X	X				X
9								X	X	X
4		X	X		X					
6	X		X			X				
1	X	X					X			
7	X			X				X		
5		X		X					X	
0			X	X						X

(b)

Figure 1.4.1. Two versions of the Levi incidence matrix of the configuration table in Table 1.4.1. In (a) the matrix is formed in the obvious way, while the rearranged form in (b) illustrates the selfduality of the configuration. Part (b) is adapted from Laufer [L1].

* * * * *

Several graphs have been attached to configurations; it is probably simplest to describe them in terms of geometric configurations, although the concepts apply to topological and combinatorial configurations (with appropriate changes in wording), and even to more general combinatorial systems.

The earliest of these graphs was proposed by K. Menger in a course on projective geometry at Notre Dame University in 1945. It seems that he never published on the topic; the first publication discussing it is a paper (doctoral thesis under Menger's supervision) of M. P. van Straten [V3] in 1949. The name "Menger graph" appears to have been introduced by Coxeter in [C5] and [C6]; it has been used in other works as well, for example in van Maldeghem [V2]. It will also appear later in this book, in Section 5.1. The **Menger graph** M(C) of a configuration C is the graph with vertices corresponding to those of C; an edge of M(C) connects two of its vertices if and only if the corresponding points of the configuration are collinear on a line of the configuration. There seem to have been only few uses of this kind of graph. One mention of the Menger graph of a configuration is in the paper by Di Paola and Gropp [D6], in connection with their definition of the "configuration graph" of a configuration. It will also appear later in this book, in Section 5.1. For a configuration C, the **configuration graph** is the graph with the same vertices as C, and with an edge connecting two vertices of the graph if and only if the corresponding points are not on any line of C; Gropp [G32] calls it the "Martinetti graph", Mendelsohn *et al.* [M5] call it the **deficiency graph**. We shall use the latter term. Obviously, in graph theoretic terminology the deficiency graph of a configuration is the complement of its Menger graph. As we shall see in Section 1.7 and in Chapter 2, constructs related to deficiency graphs have been used in some of the earliest papers on configurations, under the name "Restfigur" ("remainder figure") in enumerations of configurations (n_3) for small values of n. We shall enlarge on these remainder figures below. An interesting recent application is also given in Section 1.7.

The major shortcoming of both the Menger and the deficiency graphs is that they do not uniquely determine the configuration. This was noted already in [M5] and [D6]. The simplest example of distinct configurations having the same configuration graphs is that of the two (10_3) configurations which we shall denote (in Section 2.2) by $(10_3)_1$ and $(10_3)_4$; this is illustrated in Figure 1.4.2. In [M5, p. 96] a family of such pairs is indicated.

Of far greater importance is the third graph associated with a configuration, its "Levi graph". This concept was introduced by Levi in [L4], and the name was first used by Coxeter in [C6]; see also van Maldeghem [V2]. The **Levi graph** L(C) of a configuration C is the bipartite graph, with "black" points corresponding to the points of C and

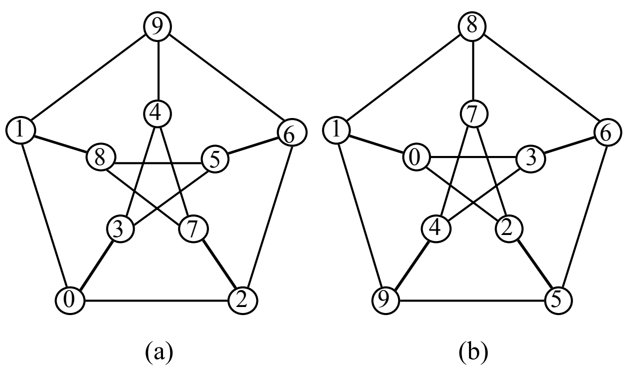


Figure 1.4.2. The configuration graph of the configuration $(10_3)_4$ specified by the configuration table given in Table 1.4.1 and by the Levi incidence matrices in Figure 1.4.1 is shown in (a). In (b) is the isomorphic configuration graph of the Desargues configuration, denoted by $(10_3)_1$ in Section 2.2, and using the label shown there.

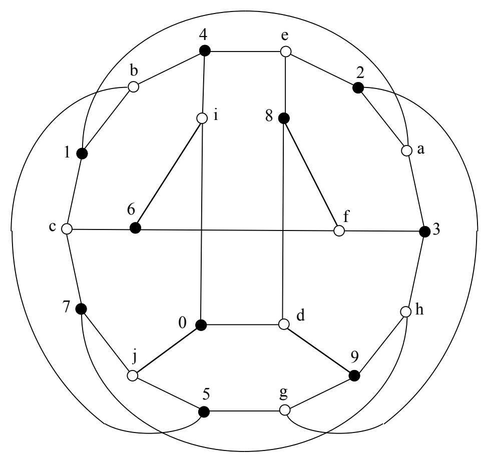


Figure 1.4.3. The Levi graph L(C) of the topological configuration C = $(10_3)_4$. The color-reversing mirror symmetry of the graph shows that the configuration $(10_3)_4$ is self-dual, under the correspondence implied by the symmetry.

"white" points to the lines of C; two points of the Levi graph determine an edge if and only if one represents a point and the other a line incident with that point. As an illustration we show in Figure 1.4.3 the Levi graph L(C) of the (combinatorial and topological) configuration $(10_3)_4$.

The importance of Levi graphs derives from their one-to-one correspondence with combinatorial configurations. More specifically, we have the following widely used result:

A bipartite graph G is the Levi graph L(C) of a [q,k]-configuration C if and only if:

- All black vertices are q-valent, all white vertices are k-valent;
- G has *girth* at least 6, that is, all circuits in G have length at least 6. In particular, G has no loops or digons.

The correspondence between G = L(C) and C is one-to-one, up to isomorphism in each class of objects.

Various graph-theoretic concepts can be transferred to configurations by the use of Levi graphs. For example, we say that a configuration is **connected** or **c-connected** for some integer c if and only if its Levi graph is connected resp. c-connected. Translated into the terminology of configurations, connectedness means that any two elements can be included in a multilateral path; a configuration C is c-connected if on selecting any c–1 elements of C, any pair of the remaining elements can be connected by a multilateral path that does not use any of the selected elements.

A not entirely trivial result, easily provable using Levi graphs, is that any k-connected configuration of is 2-connected. We shall see this in Section 5.1, together with a discussion of some related questions.

As mentioned earlier, a concept related to configuration graphs are the "remainder figures". We shall put it in a more general setting, and we define:

Definition 1.4.1. Given a configuration C and a point P of C, the **complementary graph of C at P** consists of the points of C that are not collinear with P on any line of C, and the segments connecting pairs of these points. The **complementary graph complex** of C is the family consisting of complementary graphs of C at all its points P. It is usually more convenient to take, instead of the family, the union of the members of the family.

1	2	3	4	5	6	7	8
2	3	4	5	6	7	8	1
4	5	6	7	8	1	2	3

Table 1.4.2. The configuration table of a set configuration (8_3).

For example, for the configuration (8_3) given by the configuration table in Table 1.4.2, the complementary graph at each point is a singleton vertex, and the complementary graph complex consists of eight isolated vertices. Analogously, the complementary graph complex of the combinatorial configuration (14_4) shown in Table 1.4.3 consists of 14 isolated vertices. For each of the combinatorial configurations (9_3) (see Figure 1.1.5) and (15_4) (see section 3.1) the complementary graph complex consists of one or more circuits that cover all the vertices of these configurations; more about these cases in Sections 2.2 and 3.1. It should be noted that in general, the isomorphism of the complementary graph complexes of two configurations does not imply the isomorphism of the configurations; this is illustrated in Section 2.3.

A	A	A	A	B	B	B	C	C	C	D	D	D	E
B	F	G	H	G	H	E	H	E	F	E	F	G	F
C	L	N	P	L	M	P	L	M	N	L	Q	M	G
D	M	R	Q	Q	N	R	R	Q	P	N	R	P	H

Table 1.4.3. A configuration table (adapted from [M8]) of a combinatorial configuration (14_4).

* * * * *

Each given geometric configuration C determines an **arrangement** A(C) of the Euclidean or projective plane. By this is meant the 2-complex consisting of the intersection points of the lines of C (**vertices** of A(C)), of the open segments (**edges** of A(C)) of each line constituting the complement of the points of the line, and the 2-dimensional open convex polygons (**cells** of A(C)) that constitute the connected components of the complement of the union of the lines. For example, in Figure 1.4.4 we show another drawing of the (10_3) configuration C from Figure 1.3.3, in which we made visible all the intersection points of its lines; the points that are not configuration points are shown by hollow circles. It is easy to count that, in the Euclidean plane, A(C) has 25 points, 70 edges (twenty unbounded), and 46 cells (20 unbounded). If considered in the projective plane (that is, the extended Euclidean plane), then it has 60 edges and 36 cells. In either case, one can apply the appropriate Euler relation, or other results that have been established for arrangements to deduce properties of configurations. The concept of arrangement associated with a configuration can be applied to topological configurations as well. This will be useful in Sections 2.1 and 3.2.

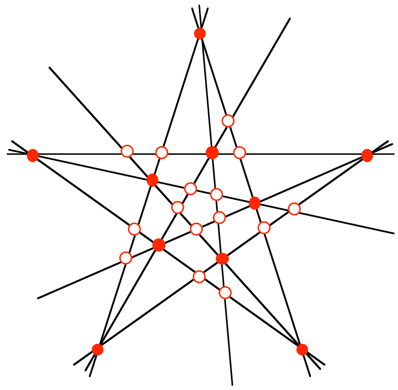


Figure 1.4.4. The arrangement associated with the (10_3) configuration from Figure 1.3.3.

* * * * * * *

Other tools have been found very useful in some questions concerning configurations are of a more algebraic character. They have been explained and applied in several widely quoted and studied works, in particular those by Bokowski and Sturmfels [B25] and Bokowski [B21]. We shall not have any occasion to use them here, so we advise the reader interested in seeing how this approach works to consult with the literature.

Another topic we are not going into in this book are programs for computers and computer graphics. Many of the enumeration results (especially those of a purely combinatorial character) have been obtained by using computers. We shall mention a number of such cases, but do not find it appropriate to enter into details beyond giving credit and references to the original works.

Much of the work presented in this book could not have been done at all without the use of widely available software. My main tools were the various consecutive versions of Mathematica®, Geometer's Sketchpad®, and ClarisDraw®, as implemented on several generations of Apple® computers. Other computer-related programs and graphics were generated by various collaborators in the joint papers we shall mention in due course, and I sometimes adapted them to the formats used here.

Exercises and problems 1.4.

1. Find a "configuration" table for the prefiguration in Figure 1.3.3, and a Levi incidence matrix for it. Find a version of the incidence matrix that exhibits the selfduality of the prefiguration.

2. Consider the graph determined by the vertices and (finite) segments in Figure 1.4.4. Does it admit a Hamiltonian circuit?

3. For each mark of the set-configuration (14_4) in Table 1.4.3 find its complementary graph. Does the complementary graph complex determine any configuration?

1.5 SYMMETRY

By a **symmetry** of an object we generally mean a mapping of the object onto itself that preserves some relevant features of the object. For configurations, by **symmetry** we shall understand that the incidence relations are preserved, but will also impose other requirements that will depend on the kind of configuration considered and on other aspects of the discussion.

More specifically, for combinatorial configurations a **symmetry** is just an incidence preserving one-to-one mapping (permutation) of the elements of the configuration onto themselves. We find it convenient to distinguish between **automorphisms**, that is symmetries that map marks to marks and blocks to blocks, and **dualities**, that map marks to blocks and vice versa. By an **automorphism** of a geometric or topological configuration we shall understand an automorphism of the underlying combinatorial configuration.

For topological configurations a **symmetry** is a *homeomorphism* of the plane onto itself that maps the configuration onto itself. However, in different contexts, this definition should be understood in one of three ways, depending on the **plane** we are considering. This can be either the Euclidean plane E^2, or the extended Euclidean plane E^{2+}, or the projective plane P^2. Although the projective plane is homeomorphic with the extended Euclidean plane, when considering symmetries of E^{2+} we require that the line-at-infinity be mapped onto itself. It follows that the symmetries of a topological configuration in E^{2+} can be a proper subset of the symmetries of such a configuration in P^2.

Analogously, **symmetries** of geometric configurations in E^2 are *isometries* of the plane that map the configuration onto itself. For geometric configurations in E^{2+} we need an isometry of E^2 that maps the finite part of the configuration onto itself and permutes the points-at-infinity.

For both topological and geometric configurations it is sometimes useful to include dualities among their symmetries. In particular, for geometric configurations a special type of duality is called **polarity** or **reciprocation**, since it arises by the polarity (also called reciprocation by some) in a circle.

As is obvious, in each of the interpretations of the term "symmetry", all symmetries of a configuration form a group, **the symmetry group** of the configuration (in the appropriate sense). Quite often it is convenient to consider only a subgroup of the symmetry group of a configuration. In such a case we shall say that the group in question is **a group of symmetries** of the configuration.

We shall soon see examples of these various interpretations of "symmetry". But first we should discuss two aspects of symmetries of geometric configurations (that apply in some cases to the other kinds of configurations as well) that lead to classifications of the appropriate configurations.

Our first concern is the collection of orbits of the configuration under *the* symmetry group of the configuration. If a configuration has h_1 orbits of points and h_2 orbits of lines we shall occasionally say that it is of **orbit type [h_1, h_2]** or **[h_1, h_2]-orbital**. We note that no *geometric* or *topological* (n_k) configuration with $k \geq 3$ can have a single orbit of points, or a single orbit of lines; in contrast, there are many such *combinatorial* configurations of this type, and even of [1,1] orbit type. More generally, if a geometric [q,k]-configuration (that is, a (p_q, n_k) configuration) is [h_1, h_2]-orbital, then $h_1 \geq [(k+1)/2]$ and $h_2 \geq [(q+1)/2]$. This is a consequence of the fact that no isometric symmetry can map the middle one of three collinear points onto one of the other points of the triplet, and analogously for lines. In case equality holds in both inequalities we shall say that the configuration is **astral**. Most interesting seem to be [h_1, h_2]-orbital configurations with $h = h_1 = h_2$; in that case we shall simplify the language by calling the configuration **h-orbital** or, more often, **h-astral**. If the values of h_1, h_2 or h are not relevany or not known, we shall speak of **multiastral** configurations. (More on this terminology at the end of the present section.)

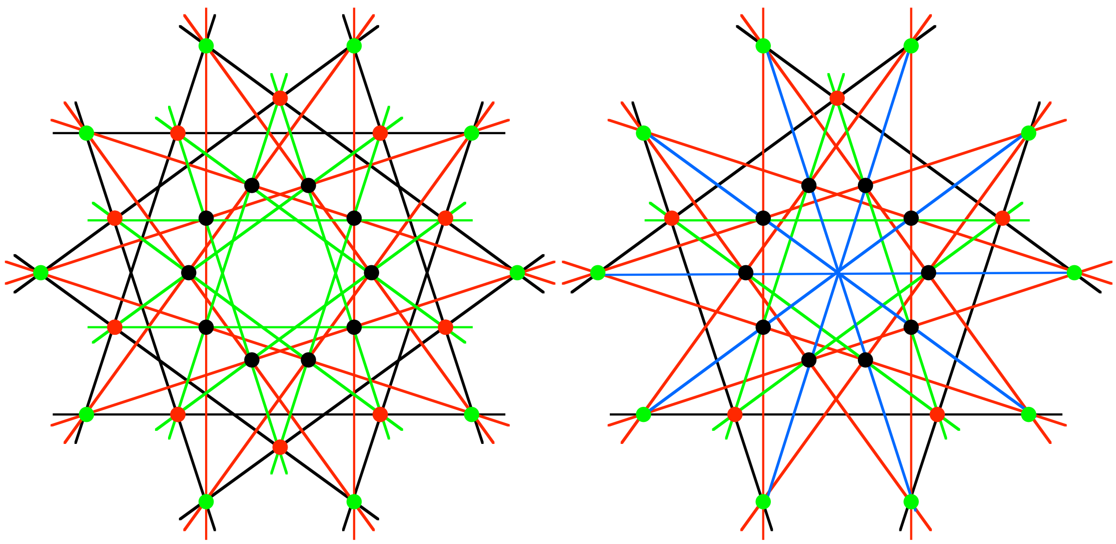
In many situations we shall be dealing with configurations in which all orbits have the same number of elements; however, some cases in which this condition is not fulfilled do have interesting features, and lead to various questions. In any case, this is not a requirement included in the definition.

The other aspect of symmetry considerations for a geometric configuration is the determination of its **symmetry group**. From the well-known classification of isometries of the Euclidean plane it follows that the symmetry group of a geometric or topological configuration is either a **cyclic group** c_r or a **dihedral group** d_r, where r is a positive integer. The group c_r consists of rotations about a **center** through integer multiples of

2p/r, the zero multiple being the identity. The group d_r consists of the same rotations as its subgroup c_r, together with r **mirrors**, that is lines of reflective isometry.

For example, the configuration in Figure 1.5.1(a) has symmetry group d_{10}, the one in (b) has symmetry group d_5. Other illustrations are given in Figure 1.5.2 and 1.5.3.

Although configurations with non-trivial symmetry group occurred in the literature from time to time, it is the recent — last twenty years or so — systematic concern with very symmetric configurations that led to the revival of interest in the whole topic of configurations. We shall investigate symmetric configurations of various kinds in several of the following sections.



(a) (b)

Figure 1.5.1. Geometric configurations (30_4) of orbit type [3,3] (that is, 3-astral) and (25_4) of orbit type [3,4]; the orbits are color-coded. The configuration in (a) has symmetry group d_{10} and all (point and line) orbits of size 10. The configuration in (b) has symmetry group d_5, two orbits of size 10 and one of size 5 for points, and one orbit of size 10 and three of size 5 for lines.

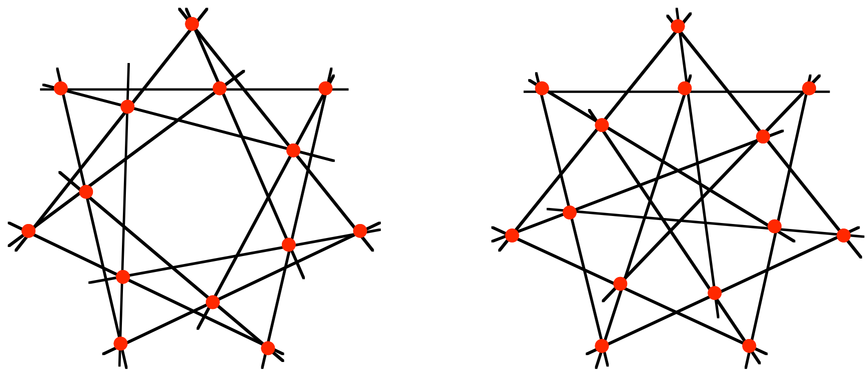


Figure 1.5.2. Two geometric configurations (14_3) of the orbit type [2,2], and with symmetry group c_7. Both are astral.

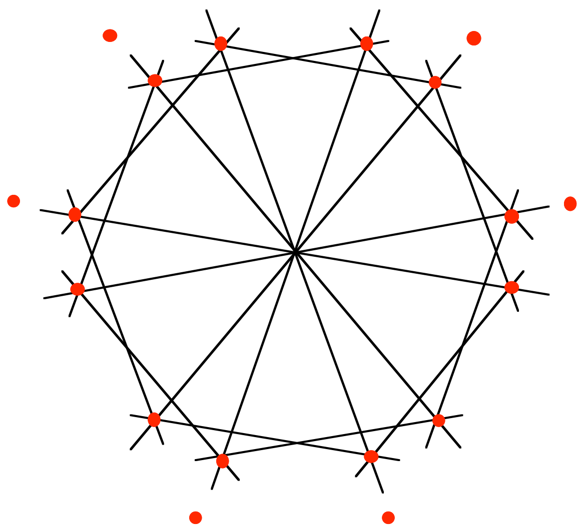


Figure 1.5.3. A geometric configuration (18_3) of the orbit type [2,2] with symmetry group d_6 in the extended Euclidean plane. This configuration cannot be represented with high symmetry in the Euclidean plane, but it is astral in the extended Euclidean plane E^{2+}.

As an example of the use of the different notions of symmetry we reproduce as Figure 1.5.4 once more Figure 1.3.3, and show its Levi graph in Figure 1.5.5. Although the symmetry group of this configuration is c_5, the symmetries of its Levi graph show that the automorphism group of this configuration is c_{10}, and the group of automorphisms and selfdualities is d_{10}.

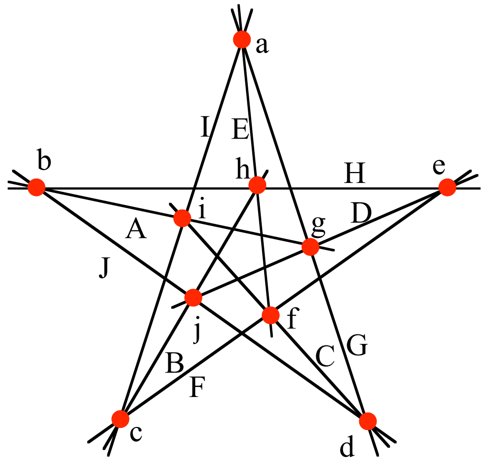


Figure 1.5.4. This (10_3) configuration has symmetry group c_5, and orbit type [2,2]; hence it is astral.

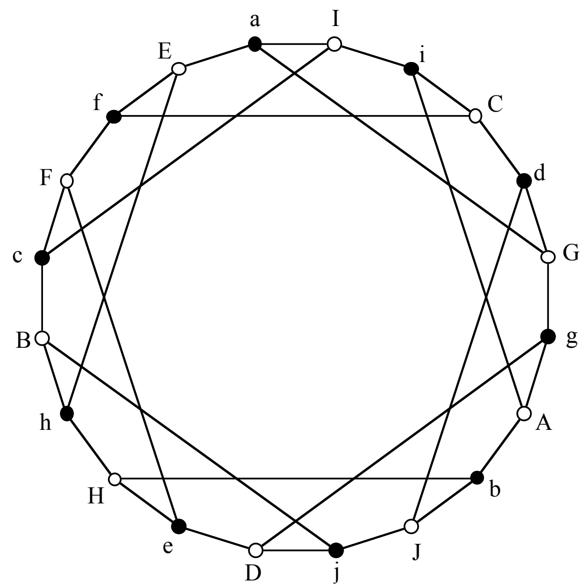


Figure 1.5.5. The Levi graph of the (10_3) configuration shown in Figure 1.5.4.

One urgent note of caution.

As is the case in many rapidly developing fields, the terminology of configurations with varying degrees of symmetry is still unsettled. One could almost claim that each author introduces separate concepts, and often even changes them from paper to paper. This has certainly been the case with the present writer — naturally, on each occasion there was some good reason for the terms introduced and used.

The astral, h-astral and $[h_1, h_2]$-astral terminology we shall use in this book is a development of the various similar concepts introduced in [G39], [G40], and [G46]. It should be stressed that although astral configurations are visually attractive and theoretically most easily investigated, for many kinds of configurations they are not the smallest possible.

In [B20] M. Boben and T. Pisanski introduced a related terminology, dealing with *polycyclic* configurations in the Euclidean plane. They call a configuration C **k-cyclic** provided there exists an automorphism α of order k of the underlying abstract configuration such that all orbits of points and lines of C under α have the same size k (that is, number of elements in each is k). In this terminology the configuration in Figure 1.5.1(a) is 10-cyclic (with three orbits of points, and three orbits of lines), while the configuration in Figure 1.5.1(b) is 5-cyclic (with five orbits each of points and lines). The two configurations in Figure 1.5.2 are 7-cyclic.

Another related concept is that of **celestial configurations**, considered by L. Berman in [B7]. We shall discuss it in Chapter 3.

Exercises and problems 1.5.

1. Decide whether the two (14_3) configurations in Figure 1.5.2 are isomorphic.
2. Find the symmetry group of each of the configurations (12_3) in Figure 1.3.8.
3. Consider the different realizations of the Pappus configuration $(9_3)_1$ in Figure 1.5.6. Maps between the different realizations establish automorphisms of the underlying combinatorial configuration. Find permutation representations for each of these mappings. Which of them correspond to geometric (isometric) symmetries?
4. Show that all points of the combinatorial configuration underlying the Pappus configuration $(9_3)_1$ form a single orbit (under automorphisms). What about the lines? What about the other two configurations (9_3) (see Figure 1.1.6)?
5. Show that there exist combinatorial configurations such that all the points are in one orbit (under automorphisms) but the lines belong to more than one orbit.

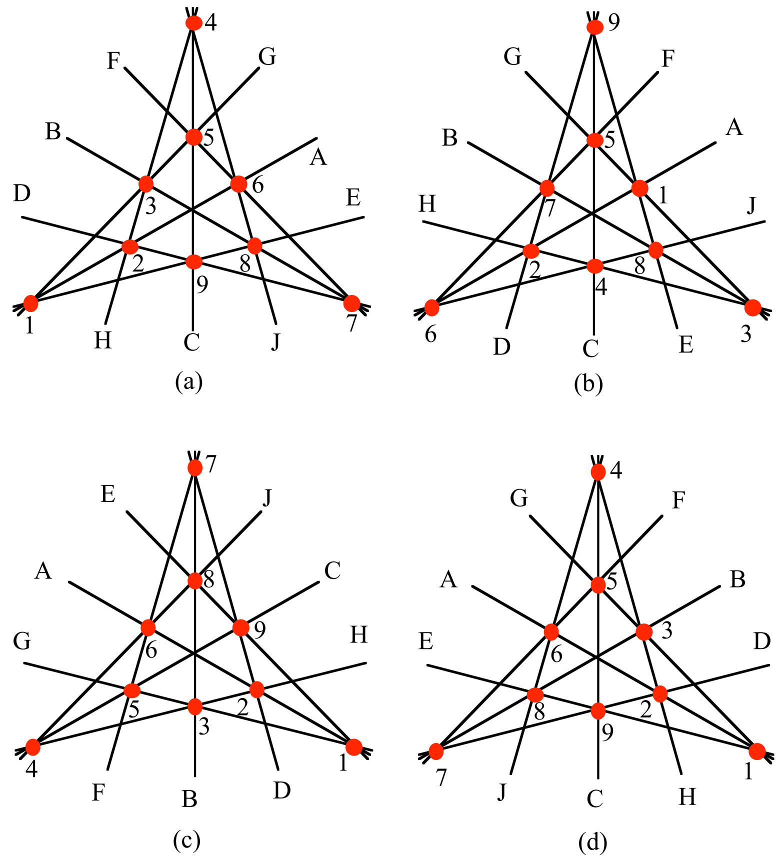


Figure 1.5.6. Four realizations of the Pappus configuration $(9_3)_1$.

1.6 REDUCED LEVI GRAPHS

We introduced Levi graphs of configurations in Section 1.4. Now, with the symmetry concepts available, we can modify Levi graphs in such a way that for symmetric geometric configurations the information appears in a much more condensed form. We call these graphs "reduced Levi graphs", and describe them separately for cyclic and for dihedral symmetry groups[1].

The **reduced Levi graph** R(C) of a geometric (or topological, or combinatorial) configuration C with a **cyclic** group c_r and of orbit type $[h_1, h_2]$ is a bipartite graph, that consists of h_1 black vertices and h_2 white ones, corresponding to the **orbits** of points and of lines of C. An edge connects two vertices (of different colors, naturally) if and only if points of the corresponding orbit are incident with the lines of the corresponding line orbit. For the (12_3) configuration C in Figure 1.6.1 this first step leads to the graph in Figure 1.6.2(a).

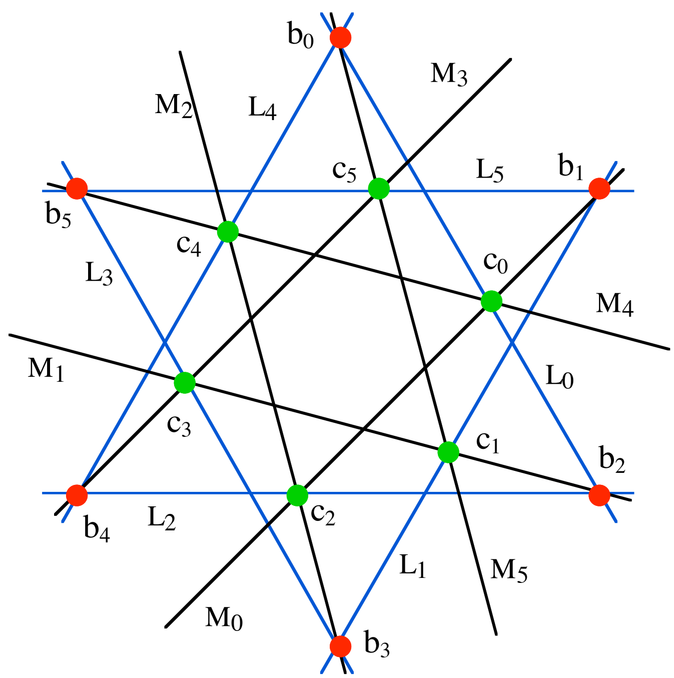


Figure 1.6.1. An astral (12_3) configuration C, with cyclic symmetry group c_6 and with labels and colors convenient in the construction of its reduced Levi graph R(C).

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1 The term "reduced Levi graph" has been used with a different meaning by R. Artzy in [A2]. We shall discuss this in Section 2.11.

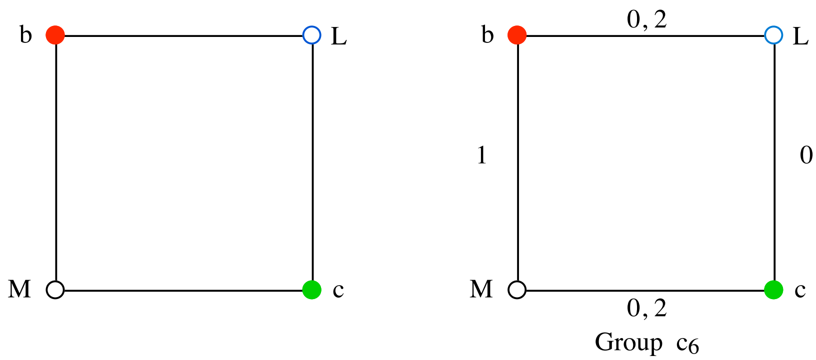


Figure 1.6.2. The formation of the reduced Levi graph R(C) of the configuration C in Figure 1.6.1. The first step is shown at left, the complete graph R(C) at right. All subscripts are understood mod 6.

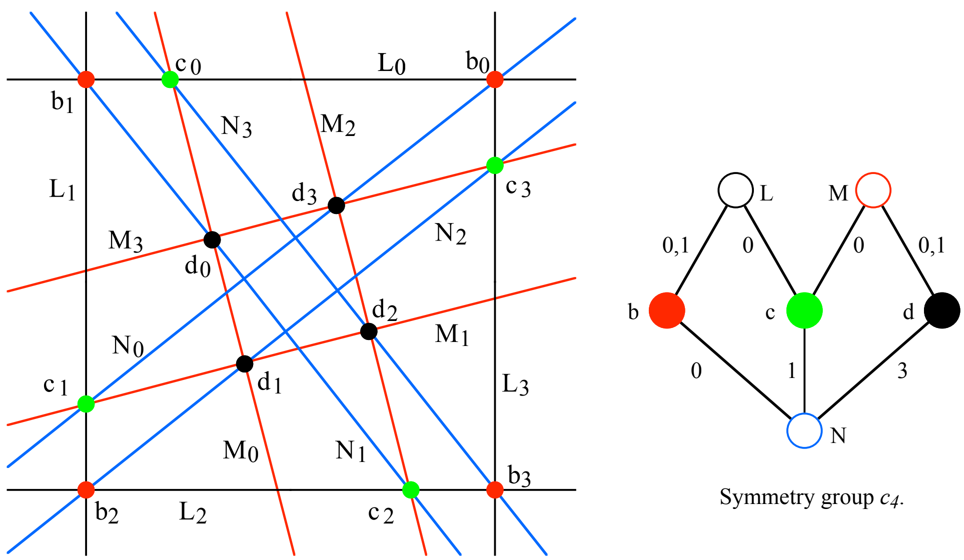
The reduced Levi graph R(C) relies on the labeling of the points and vertices in a way that corresponds to the action of (a generator of) the symmetry group. With such a labeling, an edge of the R(C) carries as labels the differences between the labels of the points that led to the particular edge. Thus in the case illustrated in Figures 1.6.1 and 1.6.2, each line L_j is incident with points b_j and b_{j+2} and c_j, leading to the labels shown on the edges connecting L to b and c in the reduced Levi graph. Similarly for the lines M_j. An indication of the relevant symmetry group completes the reduced Levi graph.

As we shall see in Section 2.8, changing only the symmetry group in this graph helps specify a whole infinite family of graphs of astral configurations.

A different example is presented in Figure 1.6.3. It is a slightly more symmetric version of a configuration shown in Figure 2 of Daublebski [D2] and specified combinatorially in his list as #88. It is interesting to note that (as stated by Daublebski) the *automorphisms* of the underlying set-configuration act transitively on the points; in fact the automorphisms are transitive on the lines, and even on the flags, see Exercise 1.6.6 below. This difference between the symmetry group of a geometric configuration and its underlying set-configuration is a frequent phenomenon; we noted it in connection with the configuration in Figure 1.5.4 and its Levi graph in Figure 1.5.5.

It is quite obvious that the reduced Levi graph R(C) of a configuration C can be used to find the configuration table or its equivalent Levi graph L(C) (as originally defined in Section 1.4). It is also clear that the construction of a reduced Levi graph can be carried out for combinatorial configurations that have a cyclic symmetry group.

A slight modification of the construction outlined above is necessary in case some lines M_j of a configurations C are mapped onto themselves by a c_2 subgroup of the cyclic group of symmetries of C. Then the order of the cyclic group c_k must be even, k = 2t, and for each j we have $M_j = M_{j+t}$. In the reduced Levi graph R(C) we indicate this fact by writing $M^{\sim}$. An illustration is given in Figure 1.6.4, in the case of a configuration $(4_3, 6_2)$ with cyclic symmetry group c_4. Similar to this is the modification required if the configuration includes points-at-infinity; such points are mapped onto themselves by a 180° rotation, that is, by a c_2 symmetry. This is illustrated by the configuration and its reduced Levi diagram in Figure 1.6.5.



Symmetry group c_4.

Figure 1.6.3. The (12_3) configuration listed as #88 in the enumeration of Daublebski [D2], and its reduced Levi graph.

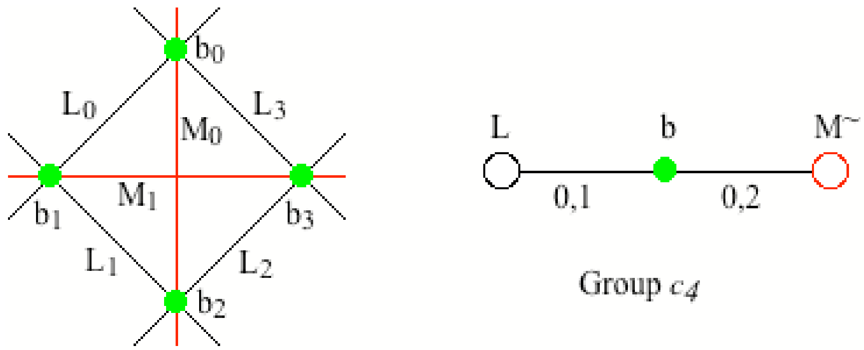


Figure 1.6.4. An example illustrating the formation of the reduced Levi diagram in the presence of lines that are mapped onto themselves by a subgroup c_2 of the symmetry group. The cyclic symmetry group of the (3,2)-configuration shown at left is c_4.

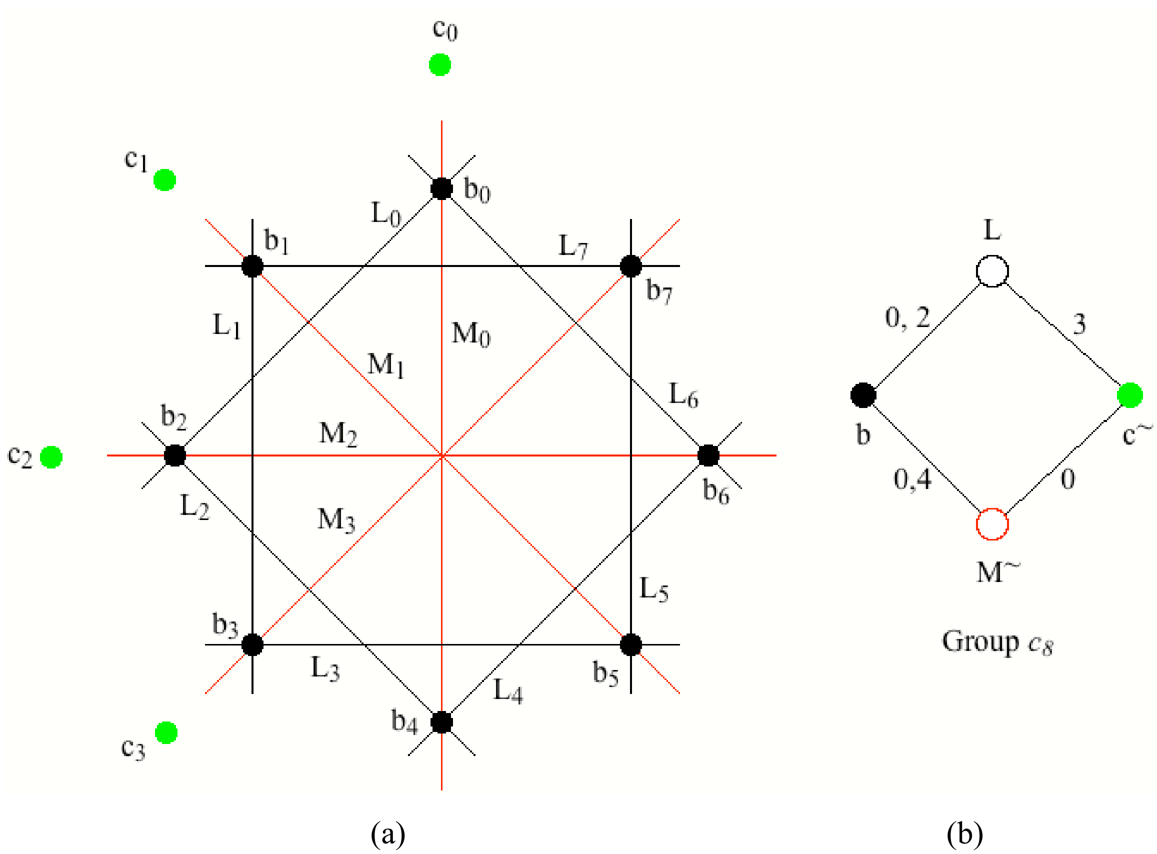


Figure 1.6.5. (a) A (12_3) configuration astral in the extended Euclidean plane, with cyclic symmetry group c_8; the points at infinity (indicated by the detached dots) and the lines M_j are individually invariant under a c_2 subgroup. (b) The reduced Levi diagram of this configuration.

A concept closely related to reduced Levi graphs of configurations with cyclic symmetry group was introduced by T. Pisanski under the name "voltage graphs", and first published in [B20]. The description presented above differs from the one used in [B20] in that our starting point is a given symmetric (geometric or topological) configuration, while Pisanski starts with a combinatorial configuration that admits a non-trivial cyclic automorphism. For general graphs the concept of "voltage" was introduced by Gross [G33], and presented in detail in several chapters of [G34] and [G35].

For configurations with dihedral symmetry the construction of reduced Levi graphs is slightly more complicated — but in some cases it allows for a commensurately more compressed encoding of the configuration. Naturally, a configuration with dihedral symmetry group d_j can also be considered under the action of the cyclic symmetry group c_j. However, this disregard of reflections as symmetry operations leads in many cases to an increase in the *number* of orbits (and decrease in the *size* of some orbits). For example, if the configuration in Figure 1.5.1(b) is considered under the symmetry group c_5, all orbits have size 5 and the orbit type is [5, 5].

The main steps to consider in the construction of a reduced Levi graph of a configuration with dihedral symmetry are:

- One reflection and its images under the cyclic part of the group are selected and kept throughout. In each orbit a representative pair of elements or a singleton is chosen. The labels of mirror-image chosen pairs carry + or – superscripts, the superscript is ± if the chosen element is mapped onto itself by the mirror. In the latter case it is often convenient to drop the superscript entirely.

- Each black point of the graph corresponds either to a pair of configuration points related by the appropriate reflection, or to a point on a mirror; white points (empty circles) correspond analogously to lines — pairs related by reflection in the mirror, or single lines invariant under the reflection chosen. Rotations carry the labels to all points and lines of the configuration.

- The edges of the graph carry the information regarding which signed labels of lines lead to which signed labels of points, as well as the actual difference in label subscripts. The positive or negative superscripts are indicated by p and n, respectively.

Details of the construction and labeling are best understood by examples. In Figure 1.6.6 we show a labeled configuration (12_3) with symmetry group d_3. Its reduced Levi diagram is shown in Figure 1.6.7. More complicated examples are shown in Figure 1.6.8 to 1.6.10. In cases where the configuration is part of a family with varying numbers of points, it is sometimes convenient to use negative integers to indicate the difference in the labels of points and lines. This is illustrated by the examples in Figure 1.6.10, that belong to families we shall discuss in Section 3.X.

As in the case of cyclic symmetry groups, for configurations with dihedral symmetry group that contain lines or points that are mapped onto themselves by halfturns a few special conventions are needed for the labeling.

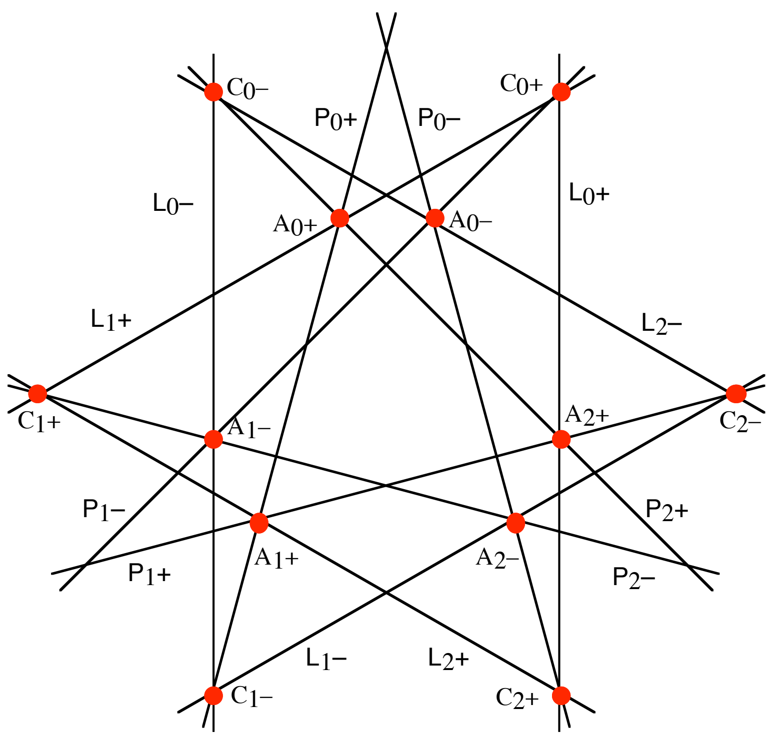


Figure 1.6.6. An astral configuration (12_3) with symmetry group d_3.

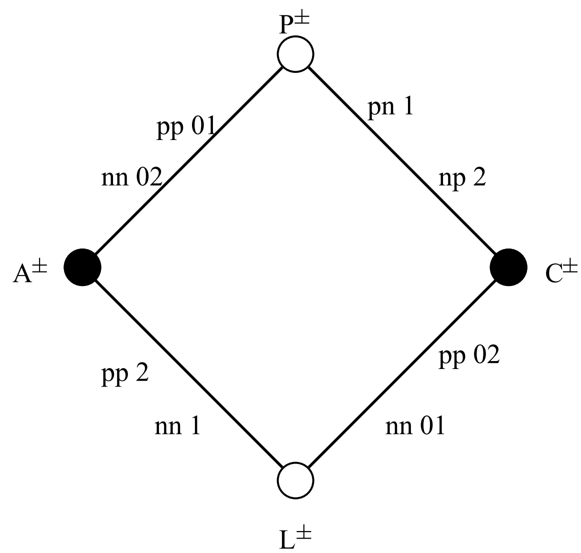


Figure 1.6.7. The reduced Levi graph of the configuration in Figure 1.6.6.

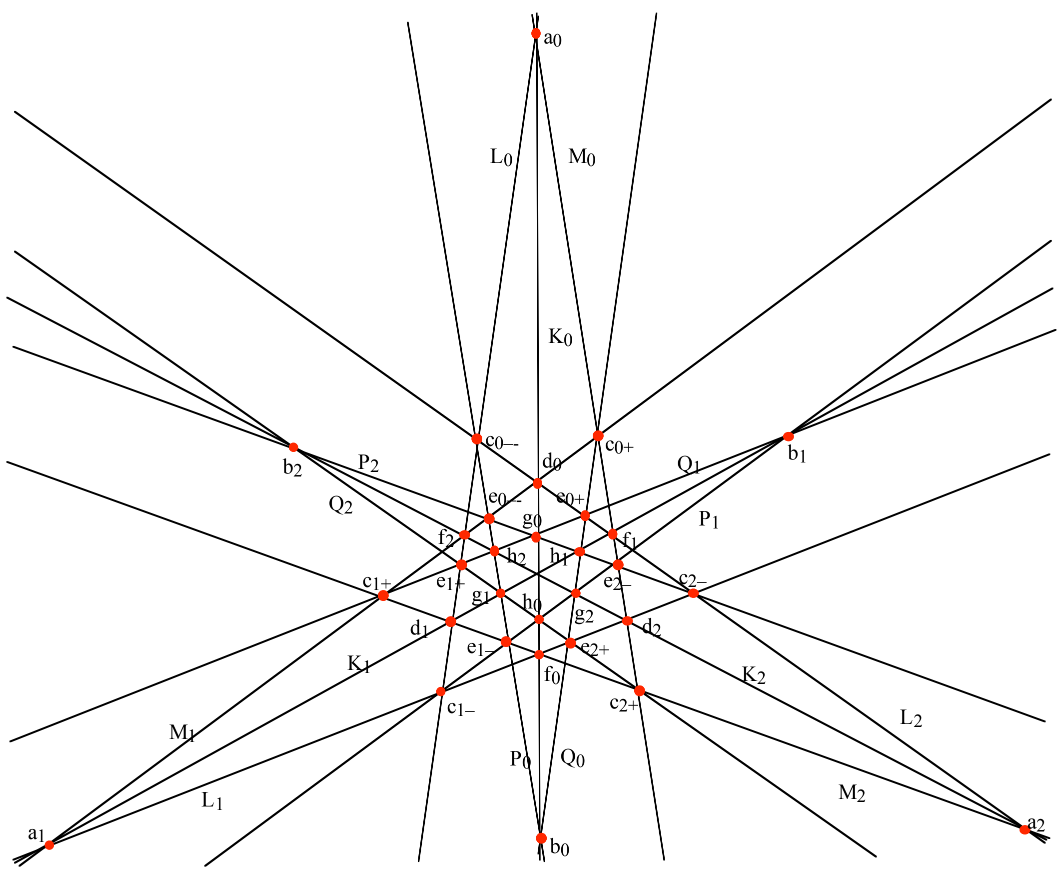


Figure 1.6.8. A $(30_3, 15_6)$ configuration with symmetry group d_3.

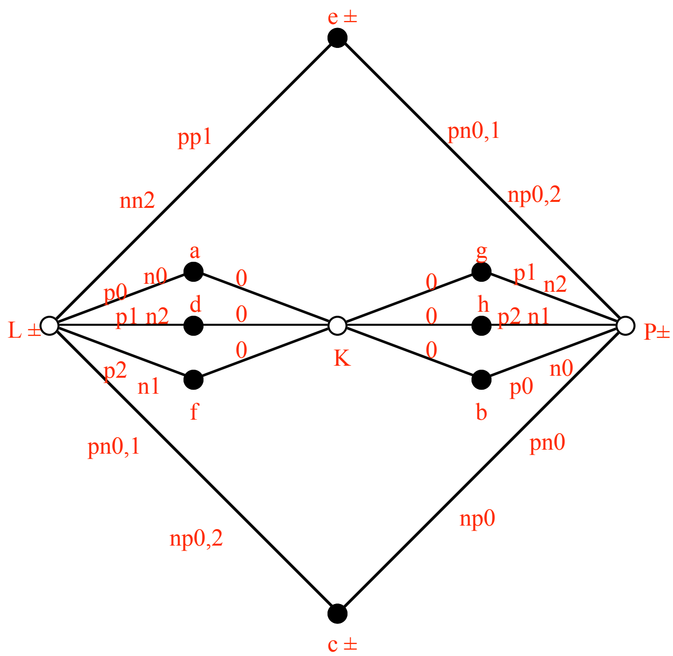
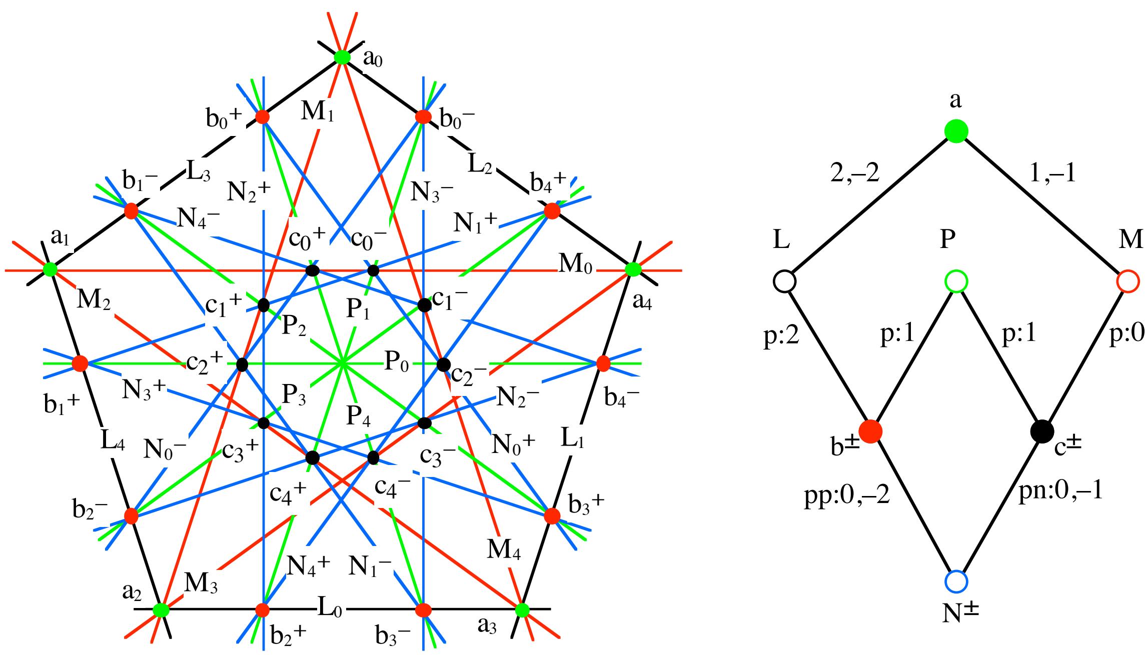
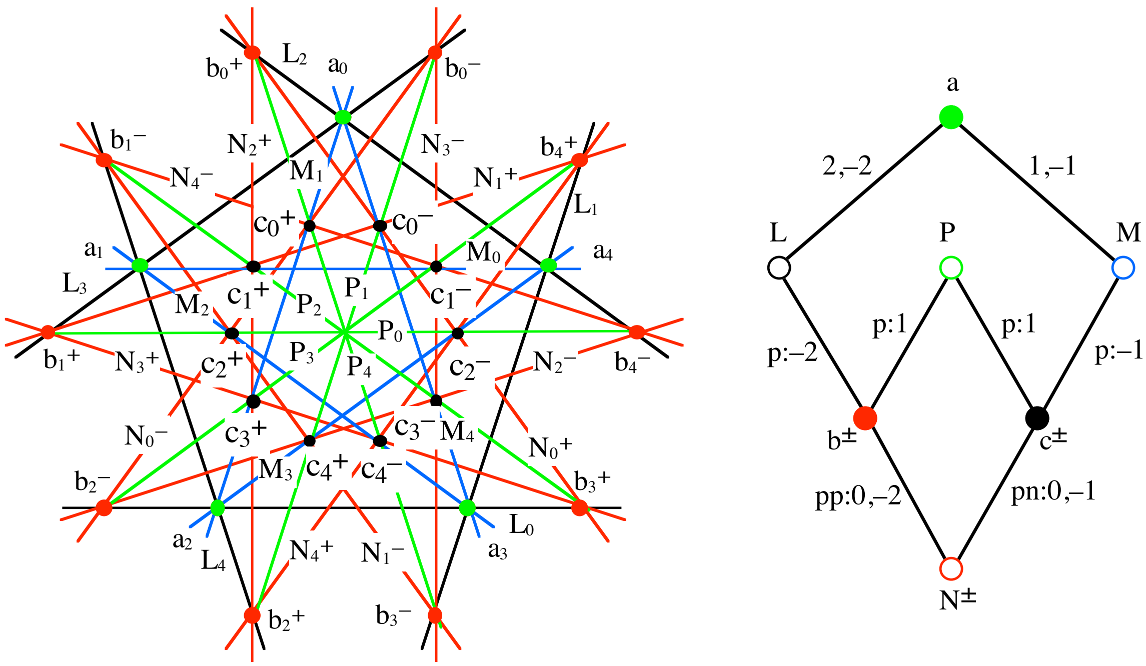


Figure 1.6.9. The reduced Levi graph of the configuration in Figure 1.6.8.



(a)



(b)

Figure 1.6.10. On the left, two configurations (25_4), each of orbit type [3, 3] and with symmetry group d_5. The second one is the same as the configuration in Figure 1.5.1(b). On the right — the reduced Levi graphs of these configurations. All subscripts are mod 5. The configuration in (a) is due to J. Bokowski, see Section XXX.

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Exercises and problems 1.6.

1. Find the reduced Levi graph of the (14_3) configuration in Figure 1.5.2a, and compare it with the reduced Levi graph of the (12_3) configuration shown in Figure 1.6.1

2. Find labels for the configuration in Figure 1.5.2b that yield the reduced Levi graph shown in Figure 1.6.11.

3. Show that the (10_3) configuration in Figure 1.6.12 is isomorphic with the Desargues configuration (10_3) shown in Figure 1.1.1.

4. Find a reduced Levi graph for the (10_3) configuration in Figure 1.6.12.

5. Find the reduced Levi graphs of the configurations in Figure 1.3.8.

6. (i) Use the labels in Figure 1.6.3 to label the Levi graph in Figure 1.6.12.

(ii) Use the Levi graph in Figure 1.6.12 to show that all points of the (12_3) configuration in Figure 1.6.3 form one orbit under its group of automorphisms.

(iii) Use the Levi graph in Figure 1.6.12 to show that the (12_3) configuration in Figure 1.6.3 is selfdual.

(iv) Use the Levi graph in Figure 1.6.12 to show that all flags of the (12_3) configuration in Figure 1.6.3 form one orbit under its group of automorphisms.

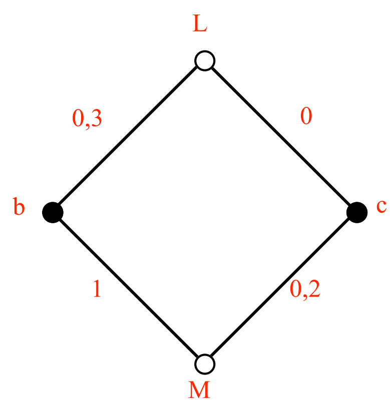


Figure 1.6.11. A reduced Levi graph used in Exercise 2.

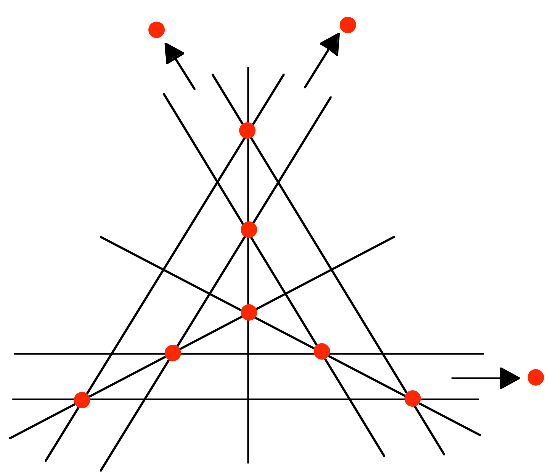


Figure 1.6.12. A configuration used in Exercises 3 and 4.

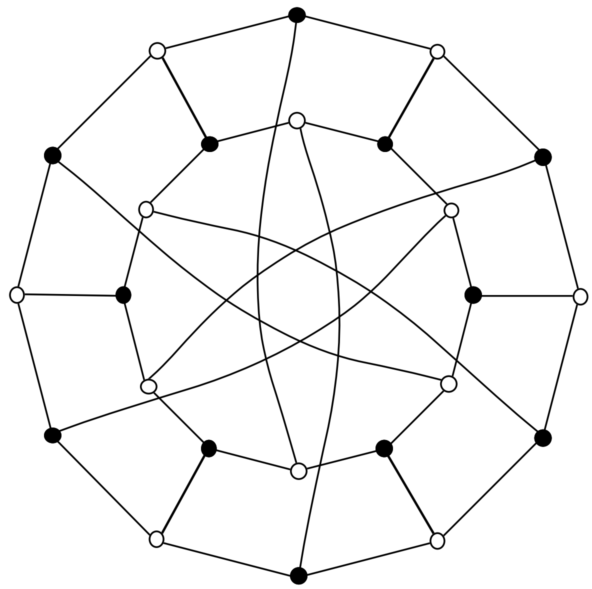


Figure 1.6.13. The Levi graph of the (12_3) configuration in figure 1.6.3.

1.7 DERIVED FIGURES AND OTHER TOOLS.

Testing whether a **given** mapping between the elements of two configurations $(p_q,\ n_k)$ is an isomorphism is straightforward, though tedious and somewhat time-consuming; its complexity is polynomial in p (and n) for given q and k. However, deciding whether **there exists** an isomorphism is much harder if only brute force is used. That is why, since the beginning of the study of configurations in the nineteenth century, variants of the idea of **derived figures** have been found useful in finding isomorphisms between configurations or establishing that they are not isomorphic. The method is, in essence, a sort of "preprocessing" and is particularly timesaving if many configurations are to be considered simultaneously, or if they are known to have only few transitivity or symmetry classes. It is also very helpful if one aims at determining the automorphism group of a configuration.

The idea underlying "derived figures" is to associate with each point (or line) of a given configuration C a small "figure", determined by the point (or line) and the incidences in C. The associated ("**derived**") figure should be easy to determine, and it should be easy to see whether two such figures isomorphic. The vagueness of the above description should not bother you too much: it is not supposed to be an algorithm, just a heuristic approach which has been found convenient, and which can best be explained by examples.

Consider the (10_3) configuration indicated in Figure 1.7.1(a), which we shall call $(10_3)_9$ as in Section 2.2. We would like to determine whether it is isomorphic to the Desargues (10_3) configuration shown in Figure 1.7.1(b). We would also like to determine its automorphism group, the transitivity classes of its elements, and the possible symmetry groups that isomorphic configurations can have.

We start by observing that since in any (10_3) configuration C each point lies on three lines which together contain six other points of C, for each point of C there are three points of C to which it is not connected by any line of C. (For example, the point 2 in Figure 1.7.1(a) is connected by no line of the configuration to any of 5, 7, 8.) One kind of "derived figure" associates to each point of C the set of the three points of C

that are not connected to it, together with any lines of C that contain two of these points or all three of them. (In the literature, this is often called the "**remainder** figure", or "Restfigur" in the German literature.) Clearly, the derived figure could be any of the five schematically indicated in Figure 1.7.2. It is easily checked that for the configuration $(10_3)_9$ of Figure 1.7.1(a) the derived figure is of type (iv) at points 0, 1, 4, 7, and is of type (iii) for the other six points. Thus the points of $(10_3)_9$ form at least two transitivity classes; since the points of the Desargues configuration form one transitivity class, these two configurations cannot be isomorphic. The same conclusion can be reached without appealing to the automorphisms group of the Desargues configuration: We note that the derived figure at each vertex of the Desargues configuration is of type (v) -- and even a single such derived figure shows the impossibility of an isomorphism to $(10_3)_9$.

A closer examination of the derived figures of $(10_3)_9$ shows that the points 3 and 8, which lie on lines determined by 0, 1, 4, 7, cannot be in the same transitivity class as 2, 5, 6, 9; hence there are at least three transitivity classes. Each of the sets {0, 1, 4, 7}, {2, 5, 6, 9}, {3, 8} is either an equivalence class of points of $(10_3)_9$ or a union of such classes. We shall try to determine which is the case. We start by looking for a permutation of the vertices that maps 0 to 1. Since they determine a line, 1 must be mapped onto 0 (since no line connects 1 to 4 or 7). Hence the permutation we are trying to find has the cycle (0, 1), and also the singleton cycle (8). The points 4 and 7 are either invariant, or else interchanged. In the former case, we would have 9 invariant as well; but then the line through 1 and 9 would be mapped on the line through 0 and 9 -- which is not a line of the configuration. On the other hand, if we assume that the permutation contains the cycle (4, 7) then we find, successively, that it must contain the cycles (3), (2,5) and (6, 9) as well. Hence the only candidate for an automorphism that maps 0 to 1 is the permutation s = (0, 1) (2, 5) (3) (4, 7) (6, 9) (8). A check reveals that $\underset{\cdot}{s}$ is indeed an automorphism of $(10_3)_9$. A similar analysis shows that there is no automorphism that maps 0 to 4, or 2 to 6, or 3 to 8. Hence the decomposition

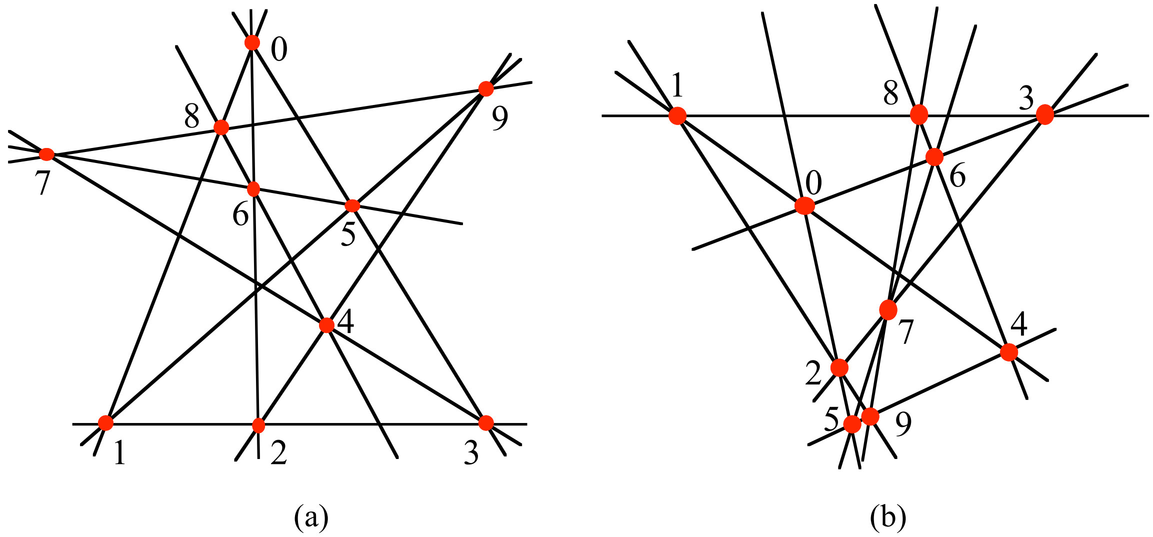


Figure 1.7.1. Two configurations (10_3).

of the vertices of $(10_3)_9$ into transitivity classes is {0, 1} {2, 5} {3} {4, 7} {6, 9} {8}. Moreover, the *automorphism* group consists of two elements, the identity and s. Thus a labeling of the points of $(10_3)_9$, more rational than the one in Figure 1.7.1(a), is as shown in Figure 1.7.3; now s interchanges the starred and double-starred versions of each letter, while keeping those without stars invariant. Concerning the *symmetry* groups of geometric configurations isomorphic to $(10_3)_9$ we can say that they either have the trivial symmetry group c_1 as in Figure 1.7.3, or else d_1 or c_2. However, since two of the points of the configuration remain invariant under s, it cannot have symmetry group c_2. By an elementary but slightly longer argument it can be shown that d_1 is impossible as well; hence no geometric realization has any nontrivial symmetry.

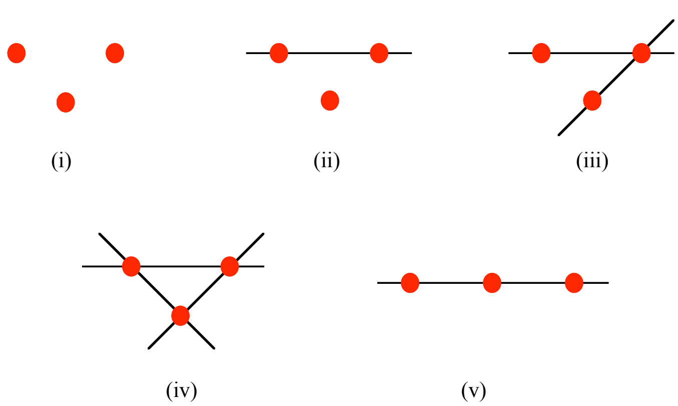


Figure 1.7.2. The possible "derived figures" for points of configurations (10_3).

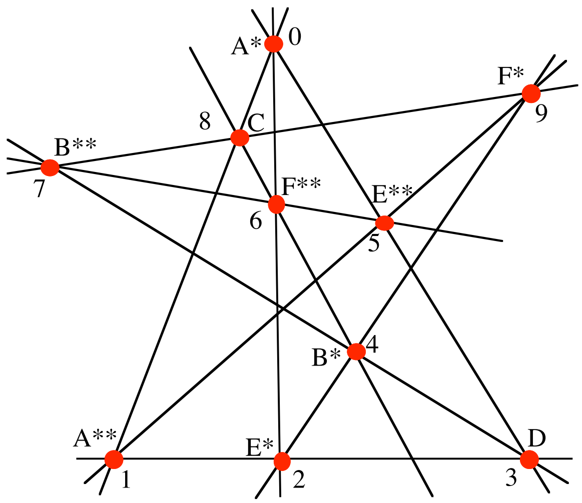


Figure 1.7.3. A revised labeling of the configuration $(10_3)_9$, making visible its automorphism group.

As an illustration of a second variant of the method of derived figures we investigate the four $(24_3, 18_4)$ configurations shown in Figure 1.7.4.

We shall call them C_1, C_2, C_3 and C_4. In this case we shall associate a "derived figure" with every line L of the configuration, by taking the 9 lines of the configuration that do not meet L in a point of the configuration, as well as the points of the configuration incident with these 9 lines. Since the configurations in Figure 1.7.4 have only two symmetry classes of lines, it is easy to determine all the derived figures; the isomorphism types that occur are **schematically** indicated in Figure 1.7.5. (Since we are interested only in isomorphisms, the relative positions of the points and lines are not relevant; only the incidences that occur in the configuration matter.) The results are:

C_1 has 6 derived figures of type (i), and 12 of type (ii), and so does C_4;

C_2 has 6 derived figures of type (iii), and 12 of type (iv);

C_3 has 18 derived figures of type (v); note that these are isomorphic to the ones of type (iii).

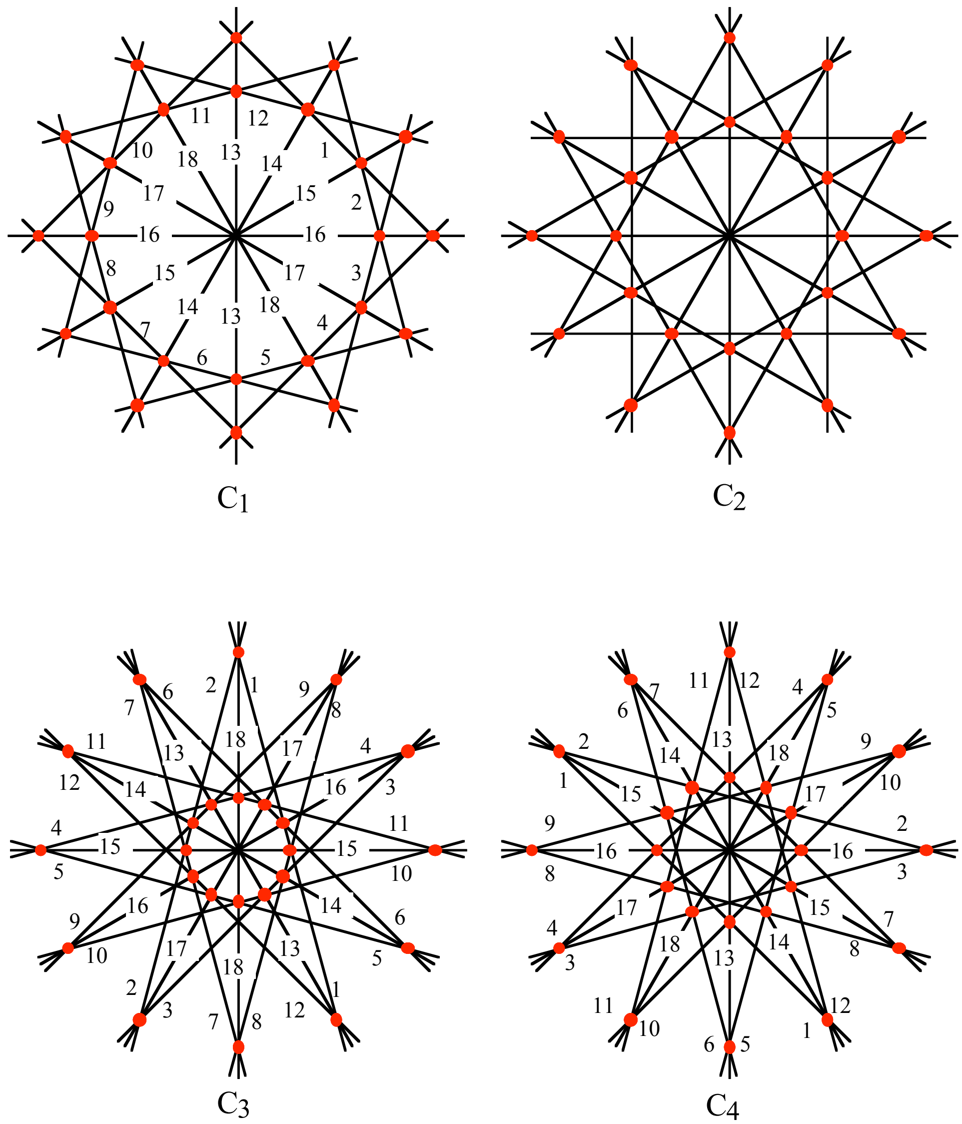


Figure 1.7.4. Four configurations $(24_3, 18_4)$.

It follows that the lines of C_1 form two transitivity classes, and so do the lines of C_2 and those of C_4. As far as the derived figures are concerned, the lines of C_3 could all belong to the same transitivity class. They, in fact, do so, as is shown, for example, by the permutation $t = (1)\ (2, 18)\ (3, 11, 9, 5)\ (4, 14, 10, 17)\ (6, 13)\ (7)\ (8, 15)\ (12, 16)$.

It also follows that C_2 and C_3 are isomorphic neither to each other nor to either of C_1 and C_4. However, as far as the derived figures are concerned, C_1 and C_4 could

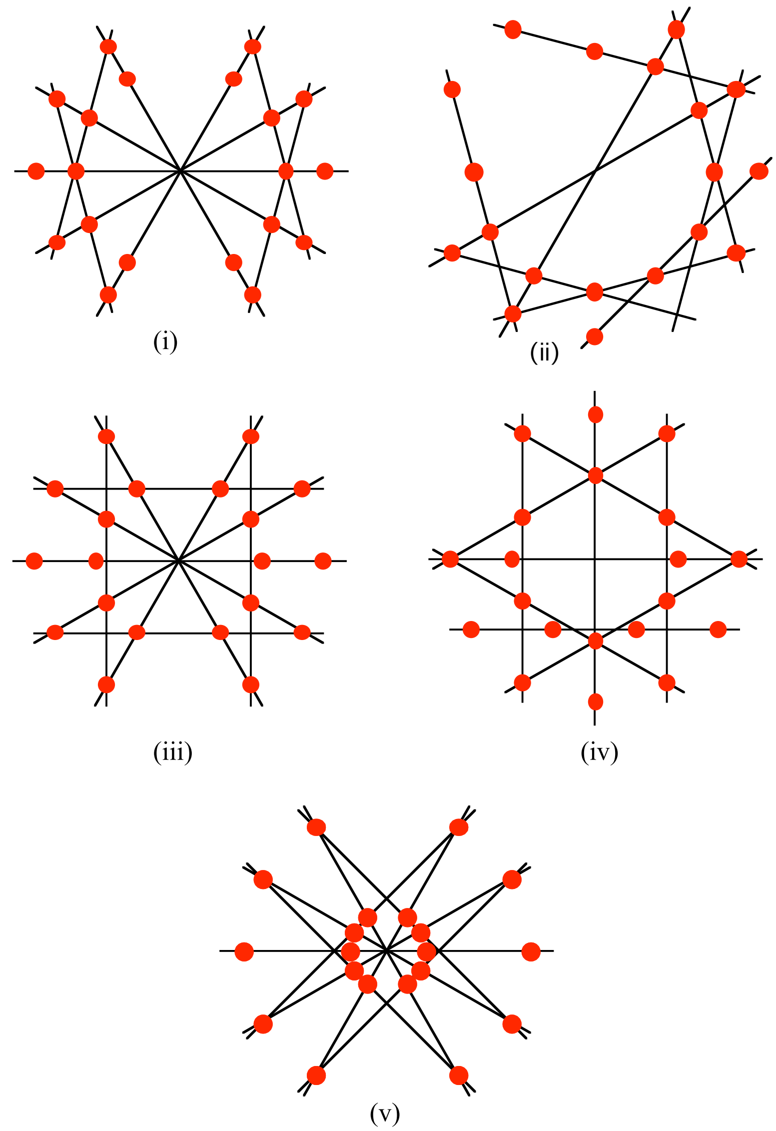


Figure 1.7.5. Derived figures that are possible for the configurations $(24_3, 18_4)$ in Figure 1.7.4.

be isomorphic. The labeling of these configurations in Figure 1.7.4, which was obtained with the help of the derived figures, is easily checked to represent an isomorphism between the two configurations.

As the next example we consider the three (9_3) configurations in Figure 1.1.6, and again find the derived figures for the various points. (We will consider these three configurations again in Section 2.2. There we will use a slightly different method.) It is clear that for all points, in all three configurations, the derived figure consists of two points. In $(9_3)_1$ none of these pairs is incident with a line of the configuration; in $(9_3)_2$ each such pair of points is incident with a line of the configuration, while in $(9_3)_3$ in six of the pairs the two points are incident with a line of the configuration and in the three remaining pairs this is not the case. Therefore no two of the configurations in Figure 1.1.6 are isomorphic. Moreover, the points in $(9_3)_3$ form (at least) two transitivity classes, while it is possible that in each of $(9_3)_1$ and $(9_3)_2$ they are in a single transitivity class. This is indeed the situation, as is easy to verify and as we will see in Section 2.2. (Note that $(9_3)_1$ is the Pappus configuration, which we encountered in Section 1.1.)

So far all the derived figures we considered were of a "local" type, related to features of the configuration that depended on the relations of each individual point with the other elements of the configuration. We shall now consider a "global" derived figure, which was used in several works of H. Gropp.

In Section 1.4 we briefly mentioned the Menger graph of a configuration. Its nodes are all the vertices of the configuration, and edges connect pairs of nodes that correspond to vertices incident with a line of the configuration. As easy to see, this graph is almost always rather unwieldy, and has had very little use. However, in many instances its **complement**, that is, the graph on the same nodes, but with edges connecting precisely those pairs which are not endpoints of an edge in the Menger graph. This graph has been called by Gropp the **configuration graph**; in Section 1.4 following [M5] we called it the **deficiency graph**. The deficiency graphs of the three (9_3) configurations in Figures 1.1.6 and 2.2.1 are shown in Figure 2.2.2 and used in Section 2.2 to distinguish between the three possible configurations (9_3).

The deficiency graph can be used to quickly decide whether the vertices of the $(12_4, 16_3)$ configuration in Figure 1.7.6 form one transitivity class. We consider its deficiency graph shown in Figure 1.7.7. The nodes P, Q, X are not in any 3-circuit, while the nine other nodes are in such circuits. Together with the obvious d_3 symmetry of the geometric realization in Figure 1.7.6, this means that P, Q, X are in one orbit, and that the other vertices form either one orbit, or they are in two orbits. It is easy to verify that the permutation (NMO)(RTV)((SWU)(P)(Q)(X), together with the geometric symmetries, establishes the single transitivity class of the nine points.

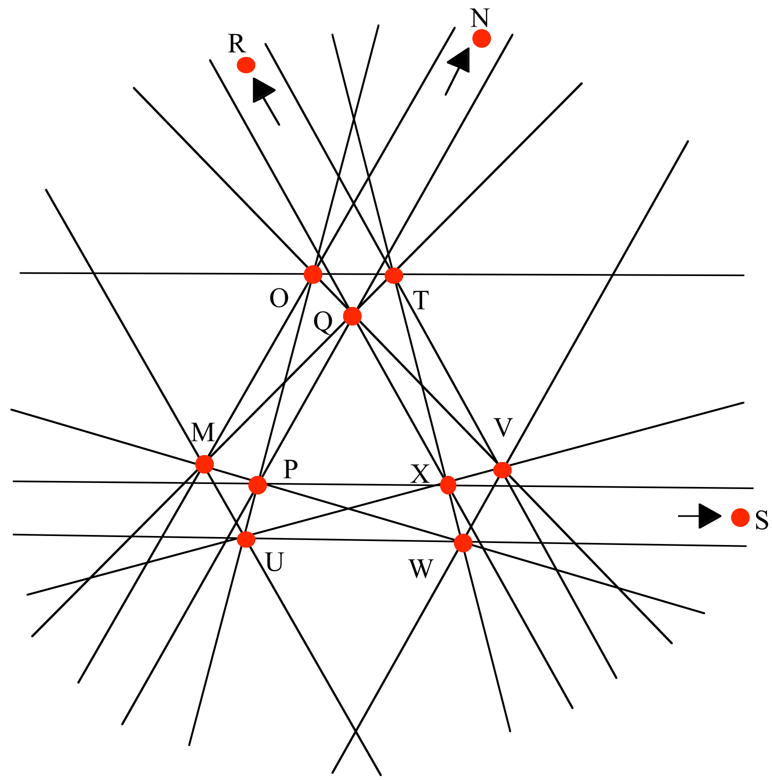


Figure 1.7.6. A $(12_4, 16_3)$ configuration.

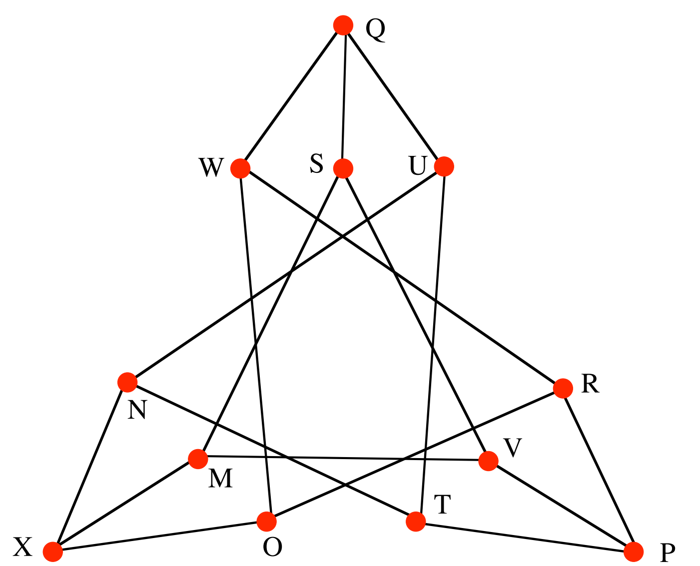


Figure 1.7.7. The deficiency graph of the $(12_4, 16_3)$ configuration in Figure 1.7.6.

Another example of the use of deficiency graphs is given in DiPaola-Gropp [D6], where combinatorial configurations (21_4) are studied. Using various combinatorial techniques they produce 200 non-isomorphic configurations of this kind, 12 of which are selfdual. They are recognized as non-isomorphic by the use of their deficiency graphs — except that one pair of non-isomorphic configurations has the same graph. They do not list the other configurations, but present configuration tables for these two (and their configuration graphs). They also do not make any statements regarding geometric realizability. I was curious whether the selfdual geometric configuration (21_4) from [G50] (shown in Figure 1.7.8) is isomorphic to one of these – and using the given configuration graph of the DiPaola-Gropp paper, it is easy to verify that our configuration is not isomorphic to either of these. In Table 1.7.1 we show a corrected copy of their tables. Since the vertices of our (21_4) form one orbit under automorphisms, in order to show that it is not isomorphic to the DiPaola-Gropp configurations it is enough to compare one of its remainder figures with any one of the latter. The remainder figure of the vertex A in Figure 1.7.8 is shown in Figure 1.7.10(a). The remainder figure of node 1 of the first graph in Figure 1.7.9 is shown in Figure 1.7.10(b); since it has a 5-valent vertex, the geometric configuration is not isomorphic to either of the two combinatorial ones.

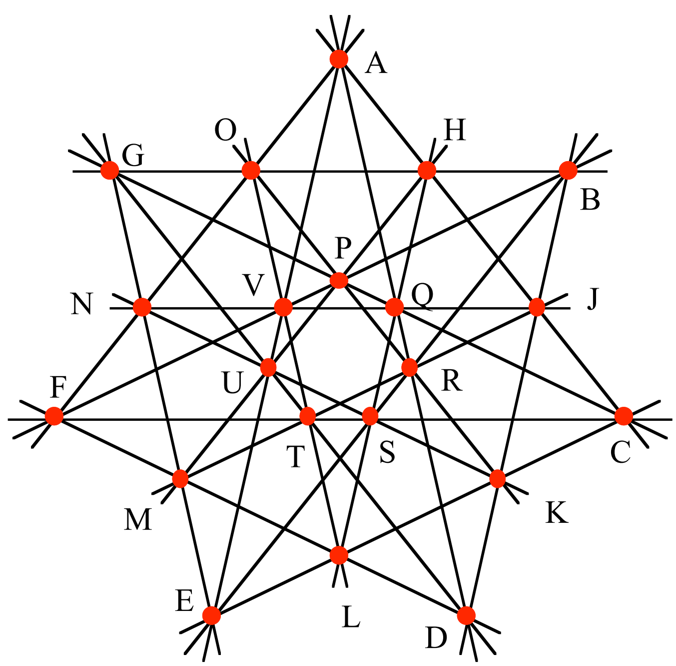
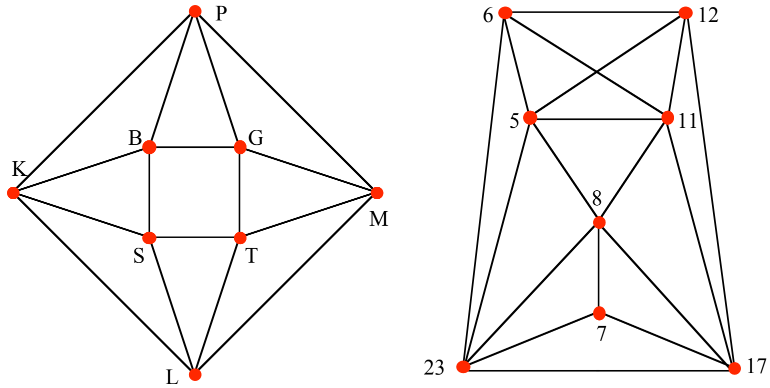


Figure 1.7.8. A geometric configuration (21_4) from [G50].

Configuration 749				Graph (749)			
1	2	3	22	(1,5)	(1,6)	(1,7)	(1,8)
1	4	13	24	(1,11)	(1,12)	(1,17)	(1,23)
1	9	16	18	(2,4)	(2,6)	(2,8)	(2,9)
1	10	14	15	(2,10)	(2,12)	(2,18)	(2,23)
2	5	14	24	(3,4)	(3,5)	(3,7)	(3,9)
2	7	16	17	(3,10)	(3,11)	(3,16)	(3,23)
2	11	13	15	(4,8)	(4,9)	(4,14)	(4,15)
3	6	15	24	(4,16)	(4,18)	(5,7)	(5,9)
3	8	17	18	(5,13)	(5,15)	(4,16)	(5,17)
3	12	13	14	(6,7)	(6,8)	(6,13)	(6,14)
4	5	6	23	(6,17)	(6,18)	(7,11)	(7,12)
4	7	10	22	(7,14)	(7,15)	(8,10)	(8,12)
4	11	12	17	(8,13)	(8,15)	(9,10)	(9,11)
5	8	11	22	(9,13)	(9,14)	(10,13)	(10,17)
5	10	12	18	(10,21)	(10,23)	(11,14)	(11,18)
6	9	12	22	(11,21)	(11,23)	(12,15)	(12,16)
6	10	11	16	(12,21)	(12,23)	(13,16)	(13,17)
7	8	9	24	(13,22)	(14,17)	(14,18)	(14,22)
7	13	18	23	(15,15)	(15,18)	(15,22)	(16,22)
8	14	16	23	(16,24)	(17,22)	(17,23)	(18,22)
9	15	17	23	(18,24)	(22,23)	(22,24)	(23,24)

Configuration 799				Graph (799)			
1	2	3	22	(1,5)	(1,6)	(1,8)	(1,9)
1	4	13	24	(1,10)	(1,12)	(1,16)	(1,23)
1	7	17	18	(2,4)	(2,6)	(2,7)	(2,9)
1	11	14	15	(2,10)	(2,11)	(2,17)	(2,23)
2	5	14	24	(3,4)	(3,5)	(3,7)	(3,8)
2	8	16	18	(3,11)	(3,12)	(3,18)	(3,23)
2	12	13	15	(4,8)	(4,9)	(4,14)	(4,15)
3	6	15	24	(4,17)	(4,18)	(5,7)	(5,9)
3	9	16	17	(5,13)	(5,15)	(5,16)	(5,18)
3	10	13	14	(6,7)	(6,8)	(6,13)	(6,14)
4	5	6	23	(6,16)	(6,17)	(7,11)	(7,12)
4	7	10	22	(7,13)	(7,14)	(8,10)	(8,12)
4	11	12	16	(8,14)	(8,15)	(9,10)	(9,11)
5	8	11	22	(9,13)	(9,15)	(10,15)	(10,16)
5	10	12	17	(10,23)	(10,24)	(11,13)	(11,17)
6	9	12	22	(11,23)	(11,24)	(12,14)	(12,18)
6	10	11	18	(12,23)	(12,24)	(13,16)	(13,18)
7	8	9	24	(13,22)	(14,16)	(14,17)	(14,22)
7	15	16	23	(15,17)	(15,18)	(15,22)	(16,22)
8	13	17	23	(16,24)	(17,22)	(17,24)	(18,22)
9	14	18	23	(18,24)	(22,23)	(22,24)	(23,24)

Table 1.7.1. The configuration tables of two non-isomorphic combinatorial configurations (21_4) and their isomorphic configuration graphs (from [D6]). The isomorphism is established by the permutation mapping Graph (749) onto Graph (799): (1,13,2,14,3,15)(4,8,6,7,5,9)(10,17,11,18,12,16)(22,23). The peculiar names of the marks (19,20,21 not used) are from [D6]. The two red entries are not correct in the original.



(a) (b)

Figure 1.7.9. (a) The remainder figure of vertex A in Figure 1.7.8. All other vertex figures are isomorphic to this. (b) The remainder figure of vertex 1 in the configuration (749) of Table 1.7.1.

Exercises.

1.7.1. For the $(16_3, 12_4)$ configuration of Figure 1.7.4(c) consider the derived figures of lines of each of the two symmetry classes. Find an automorphism that maps one line in one symmetry class to a line in the other class, and use that to show that all the lines belong to a single transitivity class. Show that all points belong to a single transitivity class, and that, in fact, all flags are in one transitivity class.

1.7.2. Use derived figures of lines to show that the two $(20_3, 15_4)$ configurations in Figures 4.3.3 and 4.3.4 are not isomorphic. Show also that in each of these configurations the lines (as well as the points) form two transitivity classes. How many transitivity classes of flags are there?

1.7.3 For the configuration we denote $(10_3)_3$ in Section 2.2, conduct an analysis of its automorphisms and symmetries analogous to the one we did above for $(10_3)_9$.

1.7.4. In continuation of our discussion concerning the configuration $(12_4, 16_3)$ shown in Figure 1.7.6, decide how many orbits of lines are there under the group of automorphisms.

1.7.5 Investigate the orbits, automorphisms and symmetries of the configuration $(12_4, 16_3)$ shown in Figure 4.3.7(b).

1.7.6 Are any of the three configurations in Figure 4.3.7 isomorphic?

1.7.7. Show that the vertices of the (21_4) configuration in Figure 1.7.8 are in a single orbit under automorphisms.

1.7.8 Find the automorphisms of the configuration denoted (749) in Table 1.7.1.