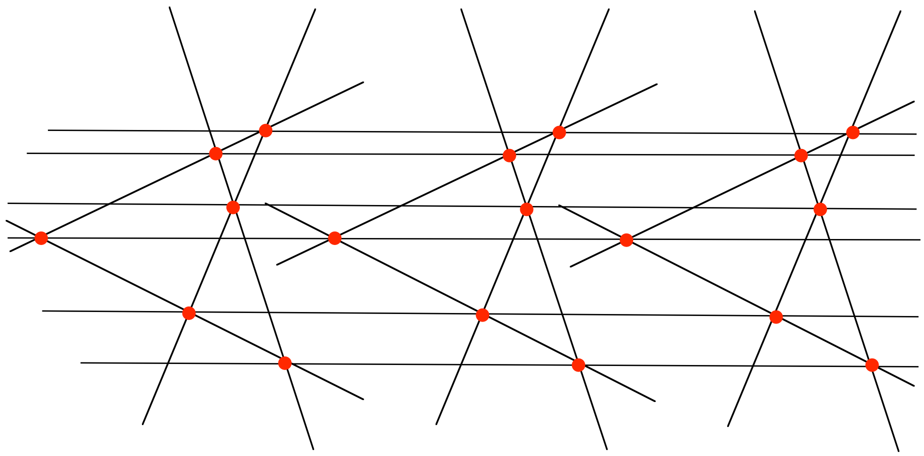
5.8 AUTOMORPHISMS AND DUALITY

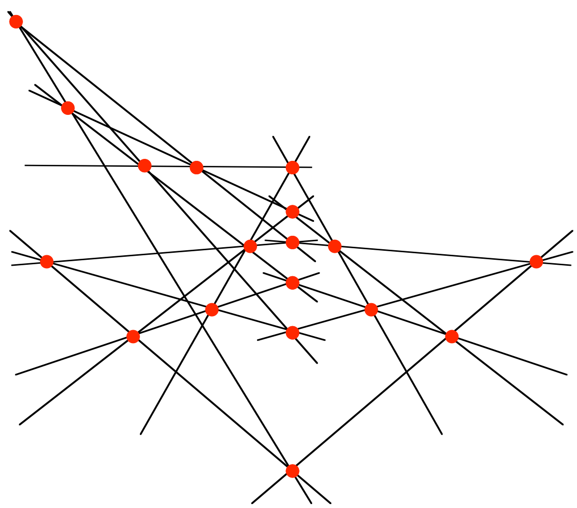
In Section 1.5 we introduced the combinatorial concepts of automorphism and duality in a superficial way. In Section 2.10 we presented some results on the duality of 3-configurations, in particular, the astral ones. Here we will discuss such topics in more detail, with particular attention to the various ways in which such combinatorial properties and relations interact with geometric symmetries. The main reason for doing so is the absence of a coherent account of such interdependence of combinatorics and geometry; the examples we shall present are only brief glimpses of these relations. They lead to a large number of essentially unexplored questions.

We recall that an incidence-preserving map between the elements (points and lines) of two configurations is an **isomorphism** if points are mapped to points and lines to lines, and it is a **duality** if points are mapped to lines and vice versa. An isomorphism of a configuration with itself is an **automorphism**, while a duality of a configuration with itself is a **selfduality**. An automorphism of a configuration that is induced by an isometry of the plane is a **symmetry** of the configurations, while a selfduality that is induced by reciprocation in a suitable circle is a **selfpolarity**. Similarly, a pair of dual configurations may be **polar** to each other; this means that when situated appropriately, the reciprocation in a circle maps one onto the other. On the other hand, as we shall discuss later, the situation may be much more complicated.

We shall first illustrate the possibilities of the duality relations on a variety of examples, introducing additional classes and concepts as we are led by these experiences. In Figure 5.8.1 we show a pair of dual configurations (18_3). It is an easy exercise to find a duality mapping between the two configurations. There is no polarity between the two realizations of the dual pair. In contrast, in Figures 5.8.2, 5.8.3, and 5.8.4 we show the (24_4) configuration 12#(5,4;1,4) with three distinct labelings of its points and edges. The first shows that this configuration is selfdual, under the mapping that interchanges the green and blue labels. Since the configuration is very symmetric, it is reasonable to inquire about polarity. The correspondence between the labels actually establishes a more particular kind of selfduality. Although the polar of the configuration does not coincide with the configuration itself, it coincides with a copy reflected in a suitable mirror. In such a situation we shall say that the configuration is **oppositely selfpolar**. The necessity of using



(a)



(b)

Figure 5.8.1. A pair of dual configurations (18_3). The configuration in (a) consists of three $(6_2, 4_3)$ subconfigurations (also known as "complete quadrilaterals") connected by six "parallel" lines. The dual configuration in (b) consists of three $(4_3, 6_2)$ subconfigurations (known as "complete quadrangles") and six points at which the three subconfigurations are joined.

a mirror is obvious from the opposite orientations of the red and blue labels. The impossibility of finding a different selfduality that would not require a mirror follows from the observation that the outer ring of points gets mapped into the inner family of lines – but these are not aligned in the required way. Similar arguments will be applicable in several cases to be mentioned later.

The same configuration (24_4) exhibits **orbit transitivity** (or "ring transitivity"). This means that there is an automorphism that interchanges the (geometric) symmetry orbits. For a proof of this assertion it is enough to consider Figure 5.8.3, in which the labels indicate an isomorphism with Figure 5.8.2.

We shall see similar behavior with other configurations. However, the (24_4) configuration exhibits a more rare kind of symmetry: It is not just point-transitive and line-transitive under automorphisms, but it is **flag-transitive.** (By this is meant that all **flags**, each consisting of an incident pair point-line, are equivalent under automorphisms of the configuration.) This is made visible by considering another labeling of the configuration, shown in Figure 5.8.4. With this labeling (and the geometric symmetries of the configuration) it is easy to show that all flags for a single orbit under automorphisms. Obviously, this implies point-transitivity and line-transitivity.

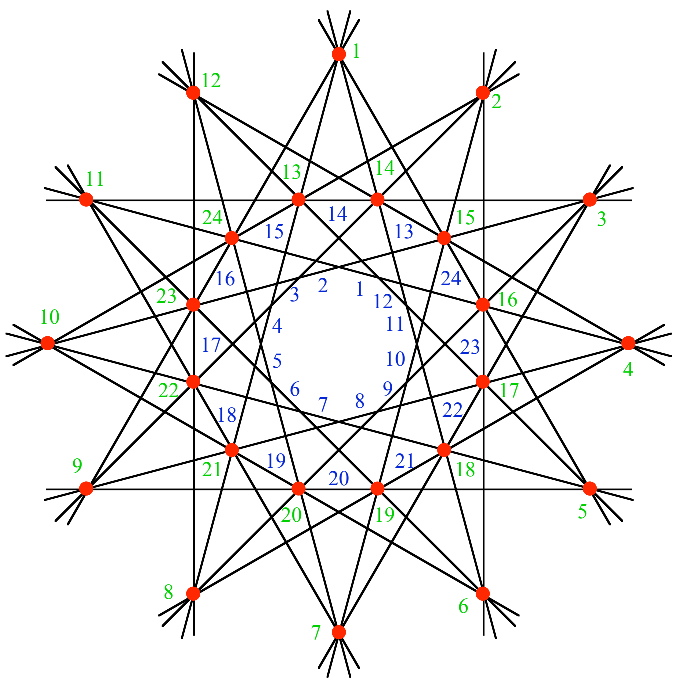


Figure 5.8.2. The (24_4) astral configuration 12#(5,4;1,4) is oppositely selfpolar; its polar needs to be reflected in order to coincide with the original.

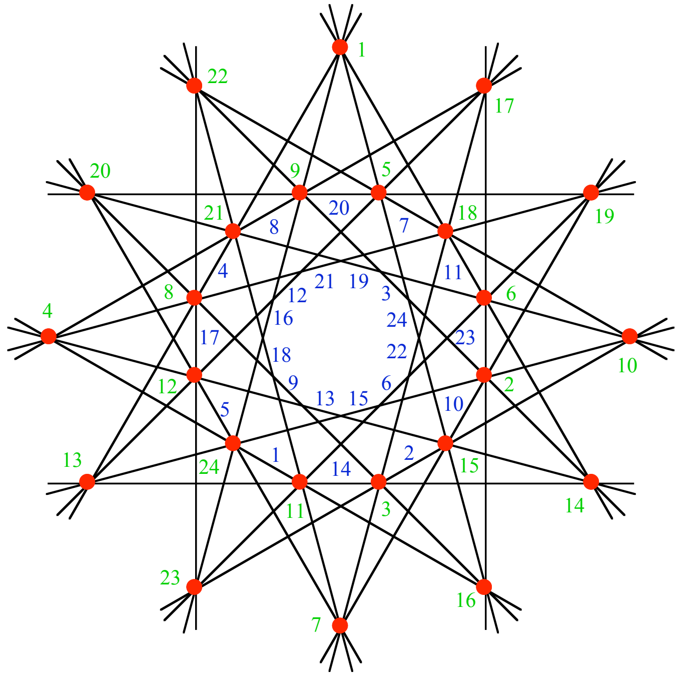


Figure 5.8.3. A different labeling of the (24_4) configuration from Figure 5.8.2. The identity map of the labels establishes that there is an automorphism of the configuration that maps the inner orbit of points onto the outer one, and similarly for lines.

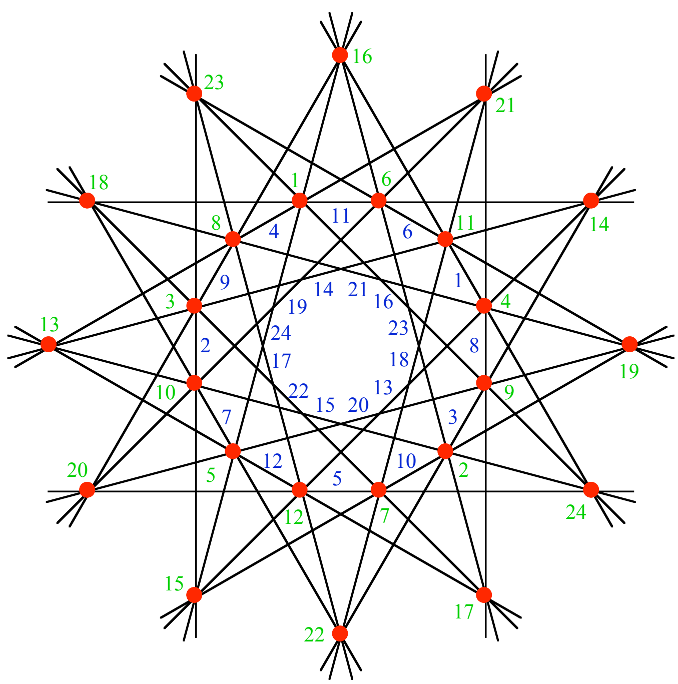


Figure 5.8.4. A labeling of the (24_4) configuration that can be used to establish the flag-transitivity of the configuration.

In Section 3.6 we mentioned that there are six different astral configurations (36_4), and we presented them in Figure 3.6.3. The labels on three of these show how they are isomorphic. Each can be shown to be polar to the one near it, hence these are all isomorphic as well. In contrast to the configuration (24_4), none of these six configurations is selfpolar. Moreover, there are two orbits of points, and two of lines, in each of these configurations; hence there is no transitivity of anything.

As mentioned in Section 3.6, the astral configurations (48_4) belong to four cohorts: 24#{{11,1},{10,8}}, 24#{{9,3},{8,6}}, 24#{{8,2},{7,5}}, 24#{{10,2},{8,8}}. This gives rise to seven distinct configurations, with symbols 24#(8,7;2,5), 24#(8,5;2,7), 24#(11,10;1,8), 24#(11,8;1,10), 24#(9,8;3,6), 24#(9,6;3,8), and 24#(10,8;2,8). By the general results from Section 3.5, the last one is disconnected and consists of two copies of 12#(5,4;1,4); hence we shall not be concerned with it here. Each of the other six is selfdual, but this does not translate into any geometric selfpolarity. In fact, the polars of 24#(8,7;2,5), 24#(11,10;1,8), 24#(9,8;3,6) are 24#(8,5;2,7), 24#(11,8;1,10), and 24#(9,6;3,8), , respectively. Moreover, 24#(8,7;2,5), 24#(8,5;2,7), 24#(11,10;1,8), 24#(11,8;1,10), are all isomorphic, and 24#(9,8;3,6) and 24#(9,6;3,8) are isomorphic. These combinatorial symmetries are far from obvious. In Figure 5.8.5 and 5.8.5 we show the first two of these configurations, with labels that indicate the selfduality of each as well as the polarity between them. By switching the label colors, it also indicates isomorphism. In Figure 5.8.7 we show the configuration 24#(11,10;1,8), with only the points labeled to show its isomorphism with the other two. (See exercise 5.) In Figure 5.8.8 and 5.8.9 we show the configurations 24#(9,8;3,6) and 24#(9,6;3,8), labeled to show their selfpolarity and the polarity and isomorphism between them.

There is no additional information available at present concerning the combinatorial properties of other 2-astral 4-configurations. The large number of such configurations — already for (60_4) there are 15 configurations — make "experimental" progress unlikely without some new theoretical insights.

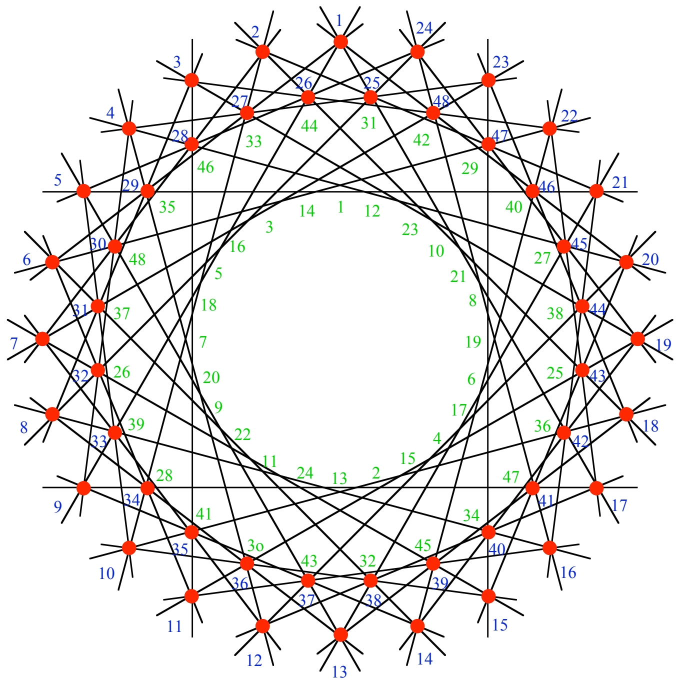


Figure 5.8.5. The configuration 24#(8,7;2,5) labeled to show its selfduality.

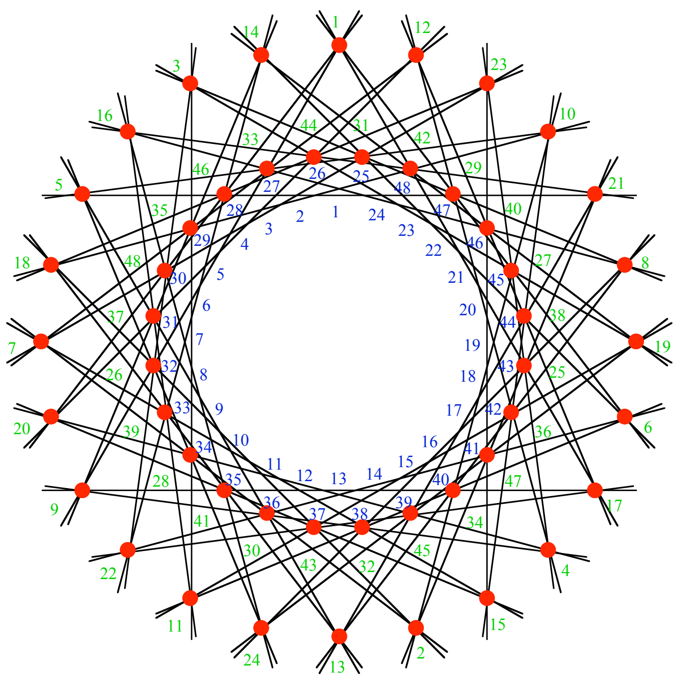


Figure 5.8.6. The configuration 24#(8,5;2,7) labeled to show its selfduality, and its polarity and isomorphism with 24#(8,7;2,5), the last by switching colors on labels.

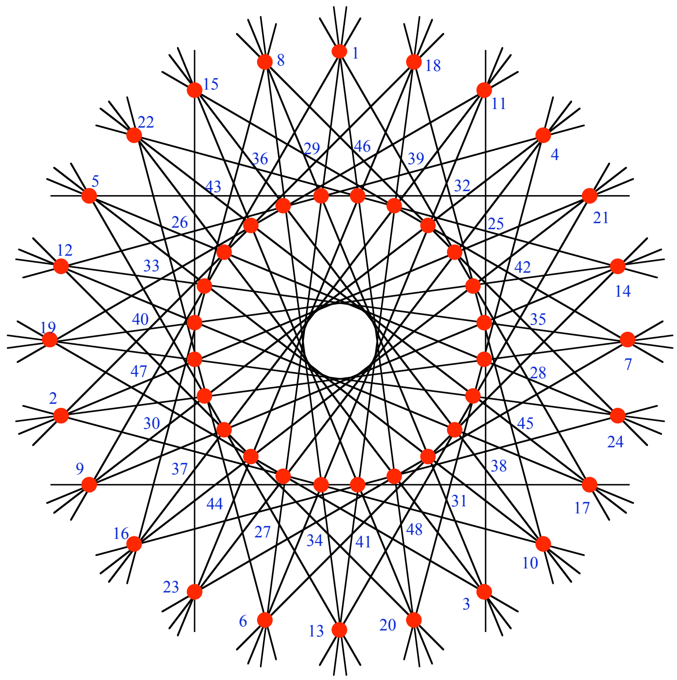


Figure 5.8.7. Configuration 24#(11,10;1,8) with points labeled to show isomorphism with 24#(8,7;2,5), and hence with 24#(8,5;2,7).

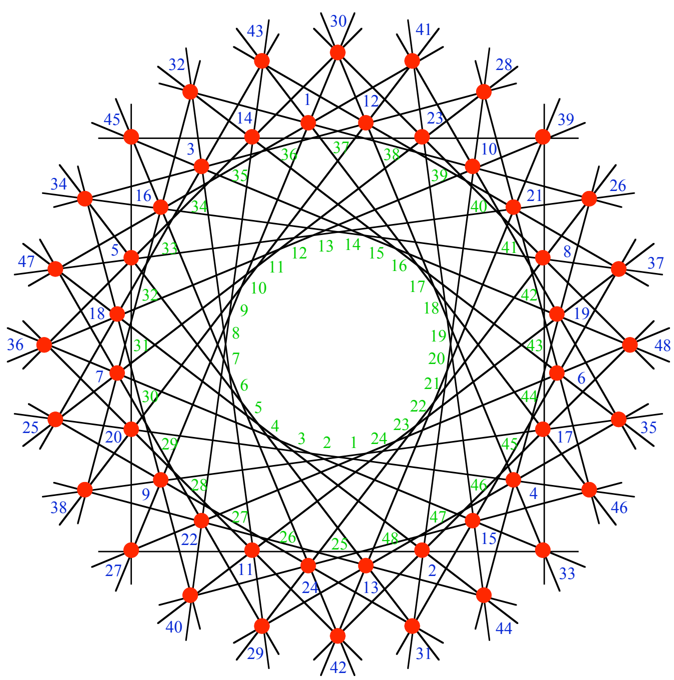


Figure 5.8.8. The configuration 24#(9,8;3,6) with labels that show the selfduality.

On the other hand, there is some knowledge of the situation concerning 3-astral 4-configurations. To begin with, by the general results of Sections 3.5, and 3.7, all such configurations are selfdual. Already in [G50] it was noted that the smallest 3-astral configuration (21_4) (shown in Figure 3.7.1, as well as in Figure 5.8.10) is selfpolar, orbit-transitive, and flag-transitive. We show these relations in the two parts of Figure 5.8.10.

This highly symmetric behavior does not extend to the next larger 3-astral configuration. In Figure 5.8.11 we show the (24_4) configuration 8#(3,2;1,3;2,1), with labels that show it is selfpolar. However, a count of symmetric trilaterals shows that there are three distinct orbits of points, hence also of lines.

The 3-astral configurations (27_4) show an interesting variety. There are six such configurations. Three of the trivial ones are isomorphic, as shown in Figure 5.8.12; one of these is selfpolar, while the other two are selfpolar*. We use the term **selfpolar*** to indicate that the polar configuration needs to be reflected in the origin (that is, turned 180°) in order to coincide with the original configurations. In each there are three orbits of points

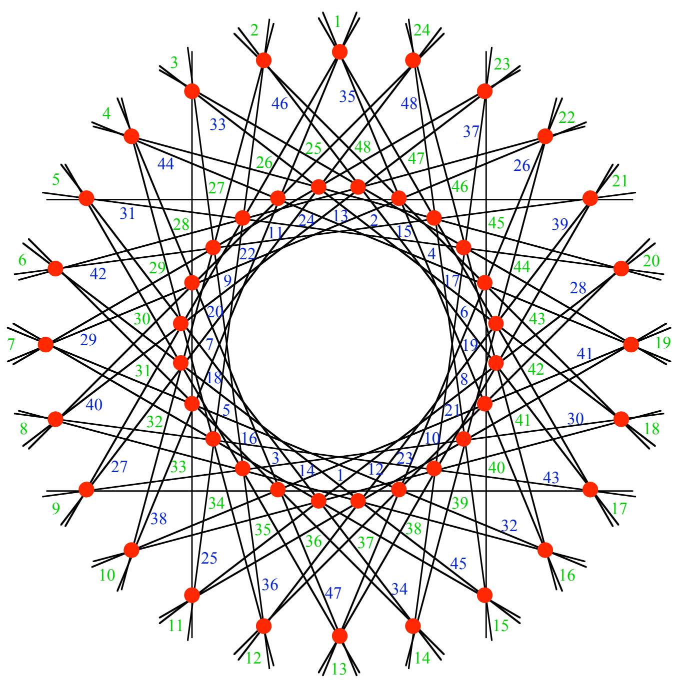
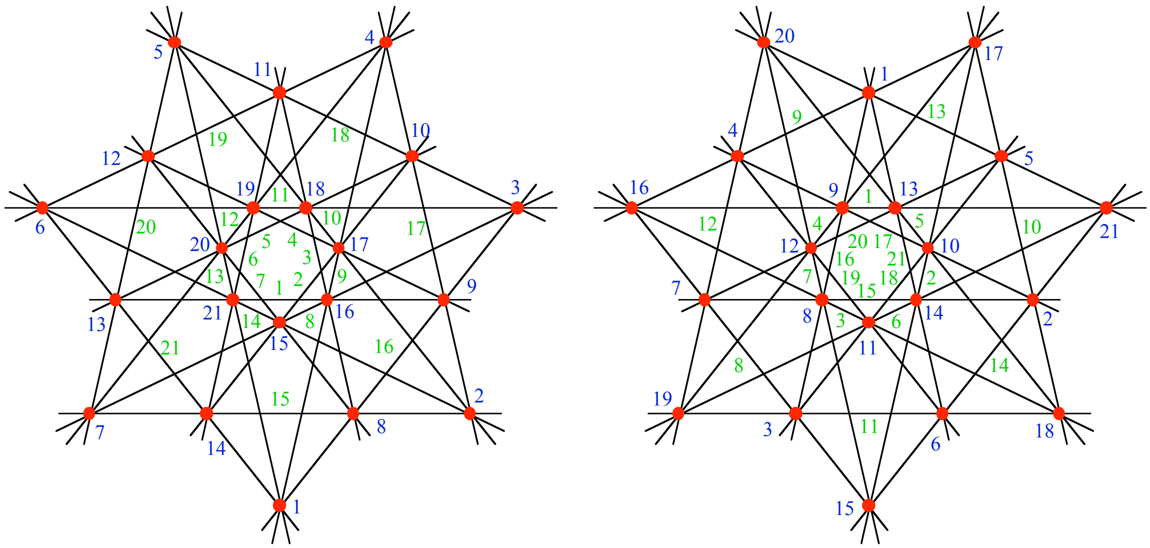


Figure 5.8.9. The configuration 24#(9,6;3,8) with labels that show its selfduality, and its polarity with 24#(9,8;3,6), as well as it isomorphism with that configuration (on interchanging the colors of the labels.

and three orbits of lines. There is one other trivial configuration, 9#(4,2;1,4;2,1), which is also selfpolar* but is not isomorphic to any of the other (27_4) 3-astral configurations. It is



(a) (b)

Figure 5.8.9. The 3-astral (21_4) configuration 7(3,2;1,3;2,1). (a) The labels indicate a selfpolarity. (b) Together with part (a), the labels indicate orbit-transitivity as well as flag-transitivity. (A more elegant labeling to exhibit these relations is used in [G50].)

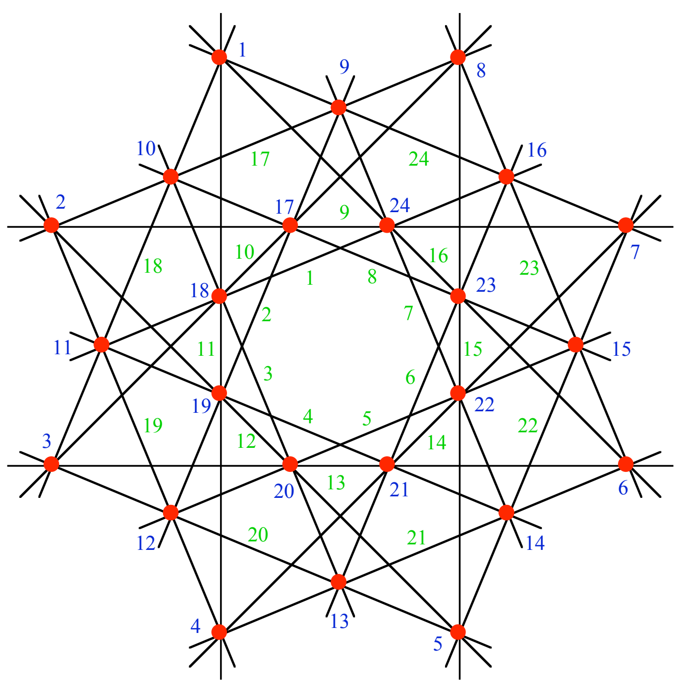


Figure 5.8.11. The 3-astral configuration (24_4), with symbol 8#(3,2;1,3;2,1) is selfpolar, but has three orbits of points and three of lines.

shown in Figure 5.8.13, with labeling that enables a check of the orbit-transitivity of the configuration. The last two configurations of this family, 9#(4,3;2,3;1,3) and 9#(4,3;1,3;2,3), are the first non-trivial 3-astral 4-configurations we encountered so far. In

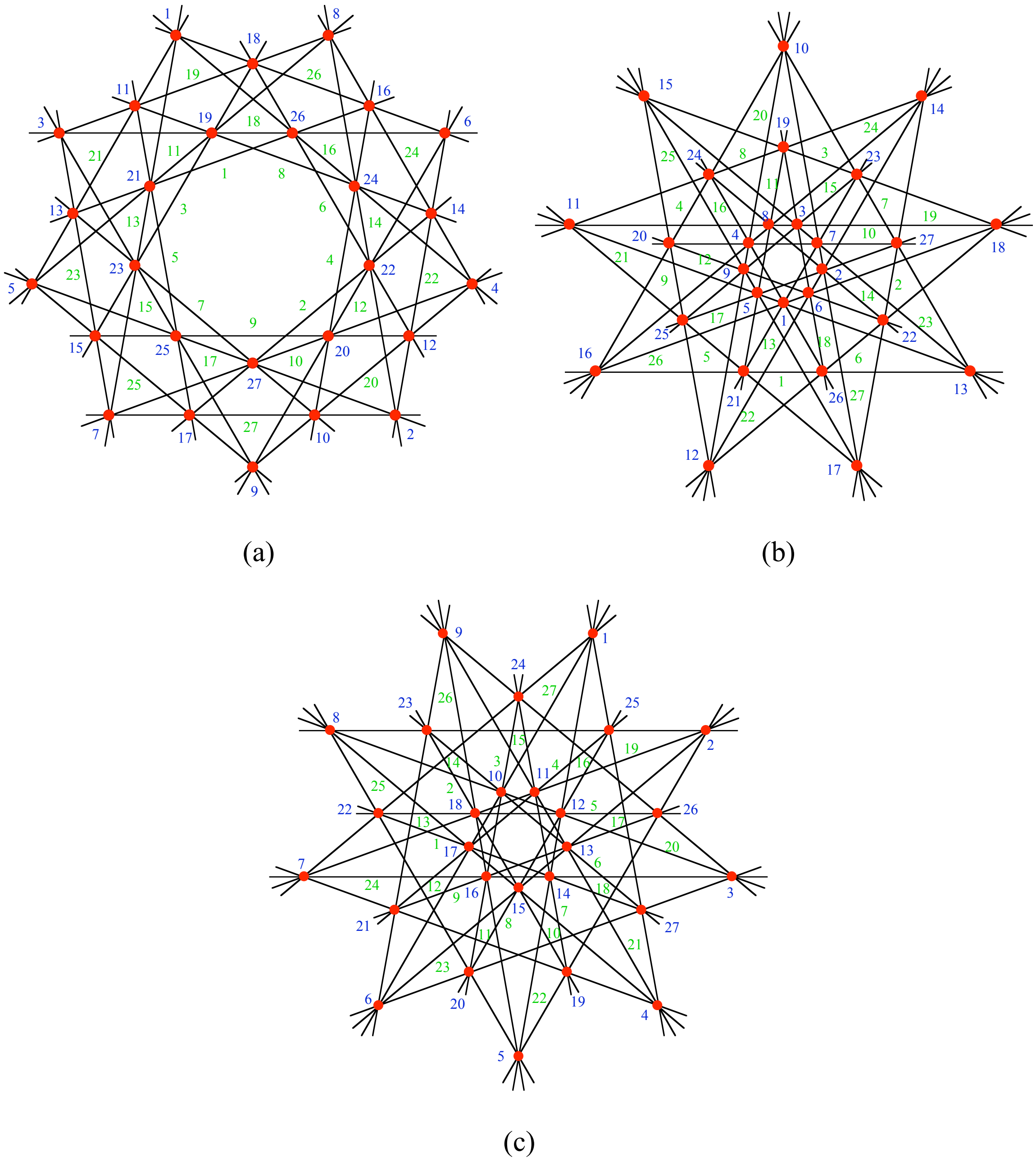
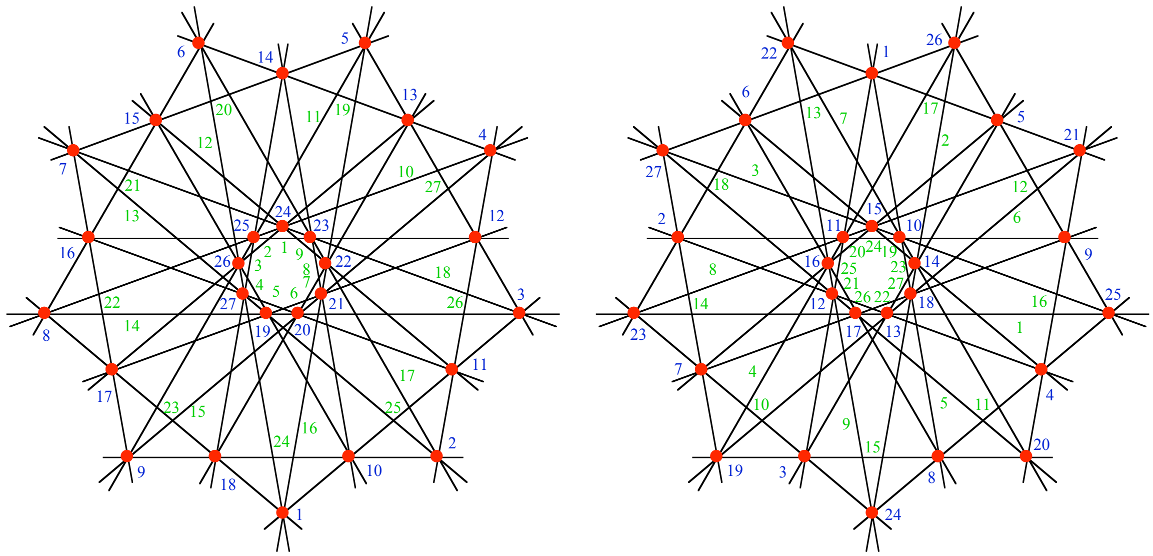


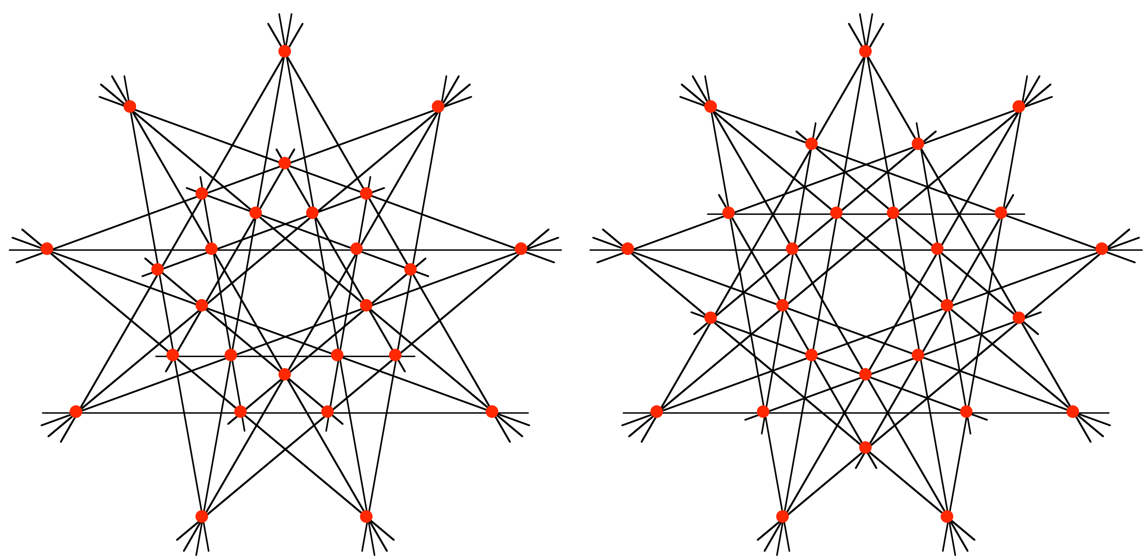
Figure 5.8.12. Three 3-astral configurations (27_4) that are isomorphic, with appropriate labels. (a) 9#(3,2;1,3;2,1) and (b) 9#(4,3;1,4;3,1) are selfpolar, while (c) 9#(4,3;2,4;3,2) is selfpolar*.

the terminology we used in Section 3.7, they belong to the family (2) with m divisible by 3. The two configurations are polars of each other, and are point- and line-transitive.



(a) (b)

Figure 5.8.13. (a) The 3-astral (27_4) configuration 9#(4,2;1,4;2,1) is selfpolar*, as shown by the labels. (b) Comparison with (a) shows that the three point-orbits are equivalent, hence there is point-transitivity and line-transitivity.



(a) (b)

Figure 5.8.14. The last two 3-astral configuration (27_4); they are polar to each other, but there are no transitivity properties beyond that. (a) 9#(4,3;2,3;1,3). (b) 9#(4,3;1,3;2,3).

There are seven 3-astral configurations (30_4). Four are trivial, and three are systematic. Among the trivial ones, 10#(3,2;1,3;2,1) and 10#(4,3;1,4;3,1) are selfpolar, the other two are oppositely selfpolar. In the notation of Section 3.7 the nontrivial configurations are in the cohort of the family (1) with q = 5, p = 1, and r = 2. The configuration 10#(4,3;1,2;1,3) is selfpolar, the other two in this cohort are polars of each other. The details of their isomorphisms and transitivities have not been investigated, nor is any non-obvious information available concerning the ten 3-astral configurations (33_4) or any (n_4) with n > 11.

D. Marusic and T. Pisanski [M3*] investigated configurations in which the group of automorphisms does not act transitively on flags, but automorphisms together with dualities do act transitively on flags. They call such configurations **weakly flag-transitive**, and prove that there are no weakly flag-transitive k-configurations for k = 2 and any odd k. To complement this they construct several weakly flag-transitive 4-configurations; they show the smallest known such configuration, a (27_4). In retrospect, it is easy to check that this is a 3-astral configuration from the trivial family, namely 9#(2,1;4,2;1,4). Similarly, their smallest known with an even number of vertices is the 3-astral configuration (42_4), namely 14#(3,1;5,3;1,5). Their final example is a trilateral-free 4-configuration (68_4), the sporadic 17#(3,1;6,2;5,4;7,8).

There is also a complete absence of information concerning the isomorphism and duality properties of k-astral configurations with k ≥ 4. There is a whole family of unexplored problems, waiting for new initiatives and insights.

* * * * * *

Returning now to the more complicated situations possible for more general configurations, we first note that is a configuration C is selfdual under a selfduality δ, then obviously the inverse map δ^{-1} is also a selfduality. In most situations it is also true that the automorphism δ∘δ is the identity map ι on C. However, this does not happen in all cases, a larger number of repetitions of the map δ needs to be applied to reach the identity automorphism ι. We call the smallest such number the **rank** r = r(δ) of δ. In Figure

5.8.15 we show a (5_2) configuration, with two selfdualities; one has rank 2, the other rank 10. The minimum of the ranks of all selfdualities of a selfdual configuration C is called the rank r(C). Obviously, the rank of the configuration in Figure 5.8.14 is 2. In contrast, the rank of the selfduality τ of the octalateral (8_2) in Figure 5.8.15 is 16; however, although there are several other selfdualities of this configuration, they all have rank 16, hence this is the rank of the configuration.

Several facts concerning rank can be established easily. First, the rank of any configuration C is a power of 2, that is, $r(C) = 2^k$ for some $k \geq 1$. It is also obvious that the case of the configuration in Figure 5.8.16 illustrates the general fact that the rank of any (2^k)-lateral is 2^{k+1}. Finally, if r(C) = 2, then one may use the same labels for the points and the lines of C. Conversely it is obvious that a selfduality that preserves the labels is of rank 2.

Regarding configurations (n_3) it is known (and easily verified) that for $n \leq 10$ all are selfdual of rank 2. According to Betten *et al.* [B14] among the 31 configurations (11_3) there are 25 selfdual ones of rank 2, and three pairs of mutually dual configurations — however, there is no identification of the dual pairs. In the same paper it is stated that among the 229 configurations (12_3) there are 95 that are selfdual of rank 2, the other forming dual pairs. Among the 2036 configurations (13_3) there are reported to be 366 selfdual ones, all but one of which have rank 2. There is no indication in [B14] what is the rank of that configuration, but it is possible to determine that the rank is 4. The authors also report in [B14] that among the 21399 configurations (14_3) there is one selfdual of rank greater than 2, and among the 245342 configurations (15_3) there are three of rank greater than 2. For neither of these is the rank indicated, but their configuration tables are given; this makes it possible to find out their rank, if enough effort is put into it. One may conjecture that they all have rank 4.

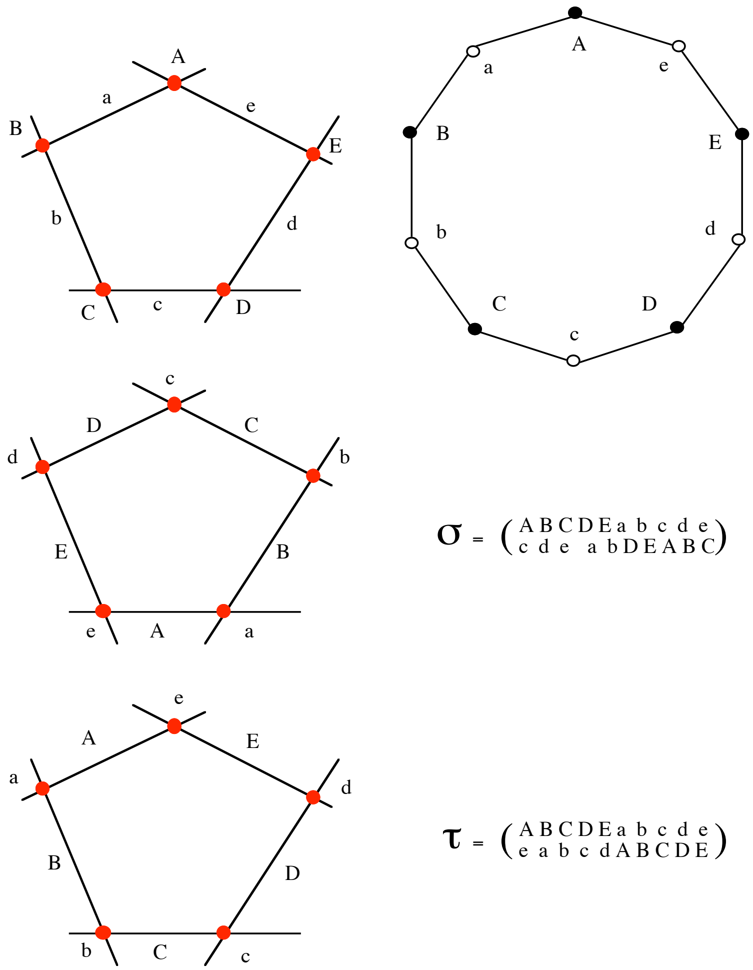


Figure 5.8.14. A configuration (5_2) (that is, a pentalateral) and two of its selfdualities. Either from the permutation representation, or by using the Levi graph shown, it is easy to verify that the rank of the selfduality σ is 2, and the rank of τ is 10.

In another direction of investigation of configurations, in [A3] Ashley *et al.* constructed for ever k, selfdual configurations (n_3) with rank 2^k. Moreover, one can find such configurations with $n \leq 2^{k+5}$. The construction is quite complicated, and we shall not reproduce it here.

There seems to be no information available concerning the rank of 4-configurations. However, we venture:

Conjecture 5.8.1. Every selfdual geometric configuration of rank 2 has a realization such that the polar (in a suitable circle) is congruent to the original configuration.

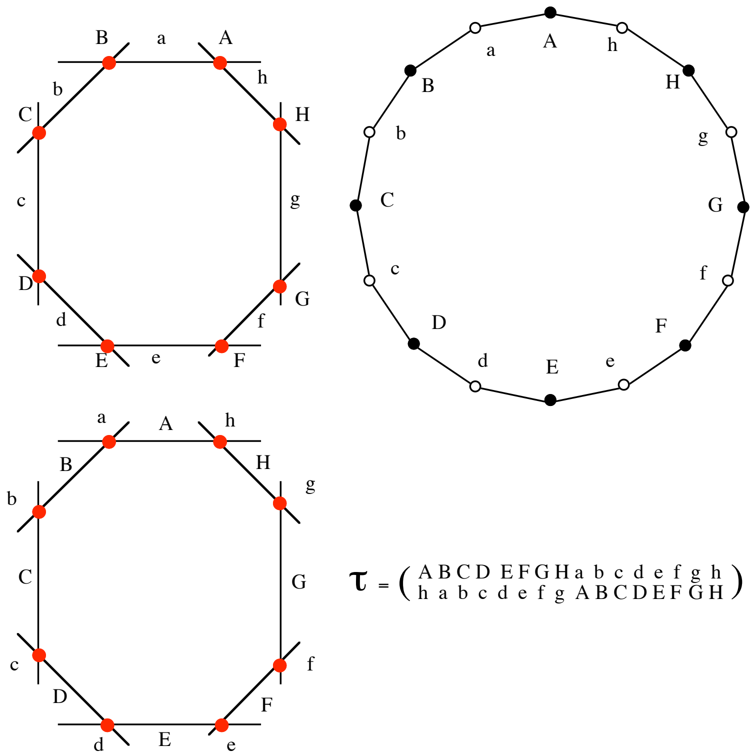


Figure 5.8.15. An octalateral (8_2) and one of its selfdualities. This selfduality, as well as all other selfdualities of the octalateral, have rank 16; this is therefore the rank of the octalateral as well.

Conjecture 5.8.2. For every q ≥ 1 and every k ≥ 1 there exist selfdual geometric q-configurations of rank 2^k.

Exercises and problems 5.8.

1. Show that the astral topological configuration 11#(5,4;1,4) shown in Figure 5.8.3 is selfpolar*. Investigate its transitivity properties (that is, orbit-, point-, line-, flag-transitivity).

2. Prove the assertions made above concerning the astral configurations (36_4).

3. The connectedness of a configuration has nothing to do with such properties as selfduality. Prove that the configuration 24#(10,8;2,8) is selfpolar and flag transitive.

4. Provide arguments that prove that the configuration 24#(8,7;2,5) is not selfpolar in any sense.

5. Draw the configuration 24#(11,8;1,10) and show its isomorphism with 24#(11,10;1,8).

6. Verify the claim that the 3-astral configuration (24_4), with symbol 8#(3,2;1,3;2,1) has three orbits of points.

7. Find a selfduality of rank 2 of the topological configuration $(10_3)_4$.