5.7 MOVABLE CONFIGURATIONS

In this section we shall investigate to possibilities of changes of shape among configurations of a fixed isomorphism type. Applying any affine or projective transformation is most likely to produce a different configuration – but we shall consider such differences trivial and endeavor to find and describe more substantial modifications. In other words, we are considering equivalence classes of configuration, where members of each class are projectively equivalent to each other.

For example, all geometric realizations of (3_2) configurations are in one equivalence class, as are those of (4_2). On the other hand, configurations (5_2) have infinitely many projectively distinct forms; in fact, since any four points in general position can be projectively mapped on any other such quadruple of points, the projective equivalence class is determined by any fifth point. It follows that the (5_2) configurations form a variety of dimension 2.

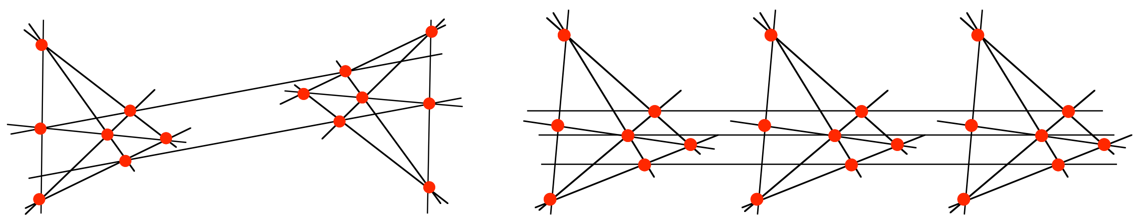
We shall say that a configuration is **rigid** if its geometric realizations form a single class under projective transformations. Both the theorem of Steinitz (as presented in Section 2.6) and practical experience suggest:

Conjecture 5.7.1. There are no rigid 3-configurations.

In view of the more stringent constraints that k-configurations with $k \geq 4$ have to satisfy, it may be tempting to believe that at least some of them are rigid. This may well be the case — however, none has been found that is demonstrably rigid. Hence we venture:

Conjecture 5.7.2. There are no rigid k-configurations for any $k \geq 3$.

A configuration that is not rigid shall be called **movable**. The motion itself can happen in a variety of ways. For example, with 3-configurations, in all cases that have been investigated, after having fixed a sufficient number of points and lines to eliminate projective maps it is at least possible to move either a point arbitrarily on a line, or pivot a line about a point (or both). This is illustrated by four of the (10_3) configurations illustrated in Figure 2.2.5; after arbitrarily choosing four points, the only remaining choice is that of a point on one of the already determined lines. In many 3-configurations (such as the ones illustrated in Figure 5.7.1) it is easy to see that they are movable even if keeping considerable parts of the configuration unchanged. However, there are analogous 3-configurations, such as the one in Figure 5.7.2, in which some of the parts have to be modified ; similar examples can easily be multiplied. There are other examples in which the connecting lines between parts are neither parallel nor concurrent, and others in which the connection between parts are through points rather than lines.



(a) (b)

Figure 5.7.1. Two 3-configurations in which solid parts may be simply pulled apart. (a) A 2-connected (14_3). (b) A 3-connected (21_3).

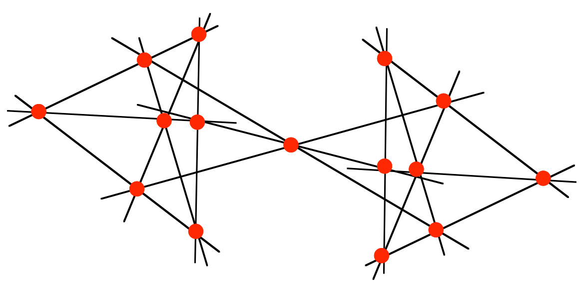


Figure 5.7.2. A 3-connected configuration (14_3) with a half-turn symmetry, in which solid parts may be separated by a greater or smaller distance.

In all the movable situations described so far there is essentially no symmetry except possibly by reflection in a mirror or by a halfturn, and in some special positions. Much more interesting are movable configurations in which the configuration retains some non-trivial cyclic or dihedral symmetry throughout the motion. We have encountered such configurations in Section 2.9, when discussing dihedral astral 3-configurations.

However, it seemed rather unlikely that analogous movable k-configurations, with considerable symmetry, can exist for $k \geq 4$. But exactly this type of configurations was discovered by L. Berman in the summer of 2006, and was first published in [B7]. We mention in passing that another kind of movable 4-configurations are the "floral configuration", the first of which was found by J. Bokowski somewhat later in 2006. We presented the relevant results in Section 4.7, and shall not dwell upon them here.

The simplest of Berman's methods of generating such configurations can be described as follows.

Starting with two 4-configurations, in one of them we omit one half of the lines of one orbit, and in the other one half of the points in one orbit. If the configurations and the orbits have been chosen appropriately, it is possible to locate the deficient configurations in such a way that the points that were incident with the deleted lines slide on the lines from which a point was deleted, thus supplying the correct numbers of incidences. The new configuration has four fewer points and lines than the original ones had jointly. Naturally, the choices of the points and lines to be omitted have to be made carefully, subject to some very stringent conditions. These restrictions are made explicit in [B7], and the complete characterization and proofs are given there. They are far too detailed and delicate to be included here, and the interested reader is advised to consult the original paper (which is easily accessible). A glimpse of the result of Berman's construction can be seen in Figure 5.7.3, which shows the smallest movable configuration obtainable by this method. Berman's paper [B7] contains some additional constructions and developments as well.

A new paper by Berman [B10] (private communication) presents additional constructions of movable configurations that retain cyclic symmetry during motion. It is more parsimonious, but the construction steps depend on the parity of the starting regular polygon. All configurations in this class have five point orbits of equal size. The smallest movable configuration that can be obtained is a (30_4). Several positions of this configuration are shown in Figure 5.7.4, adapted from [B10].

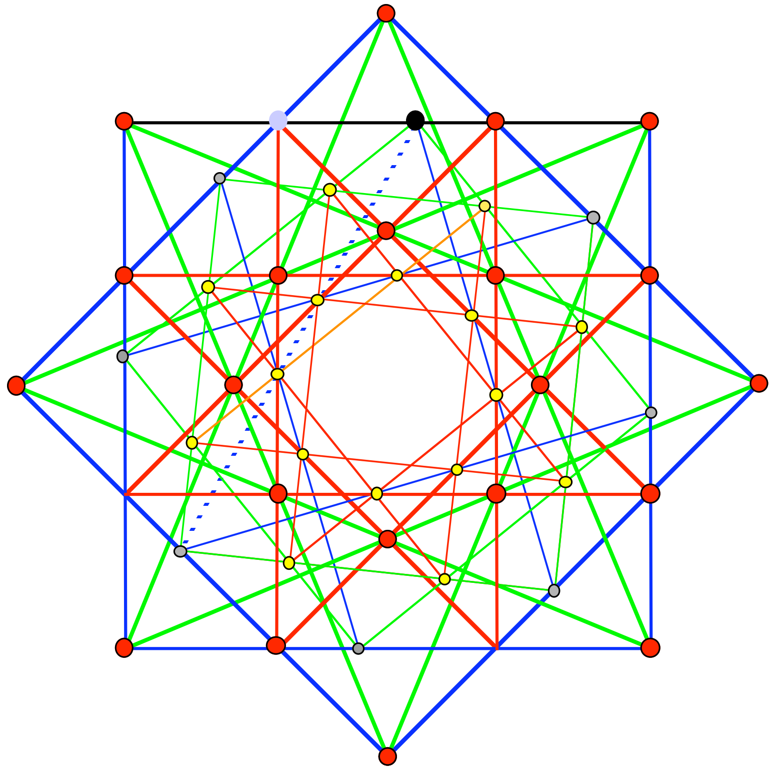


Figure 5.7.3. A movable (44_4) configuration, adapted from [B7], Figure 7. It is constructed from two copies of the 3-cyclic configuration (24_4) with symbol 8#(2,1;3,2;1,3]; one is shown in heavy lines and large red dots, the other with thin lines and smaller yellow dots. From the former the grey dot and the three analogous points (not shown) have been omitted, while from the latter the dotted line and its three analogs are deleted. The missing incidences are replaced by placing the black dot on the black line, and the corresponding points of its orbit on the corresponding lines; because of the choice of the parameters and orbits, the black point is freely movable on the black line, provided the sizes of the configurations are adjusted appropriately.

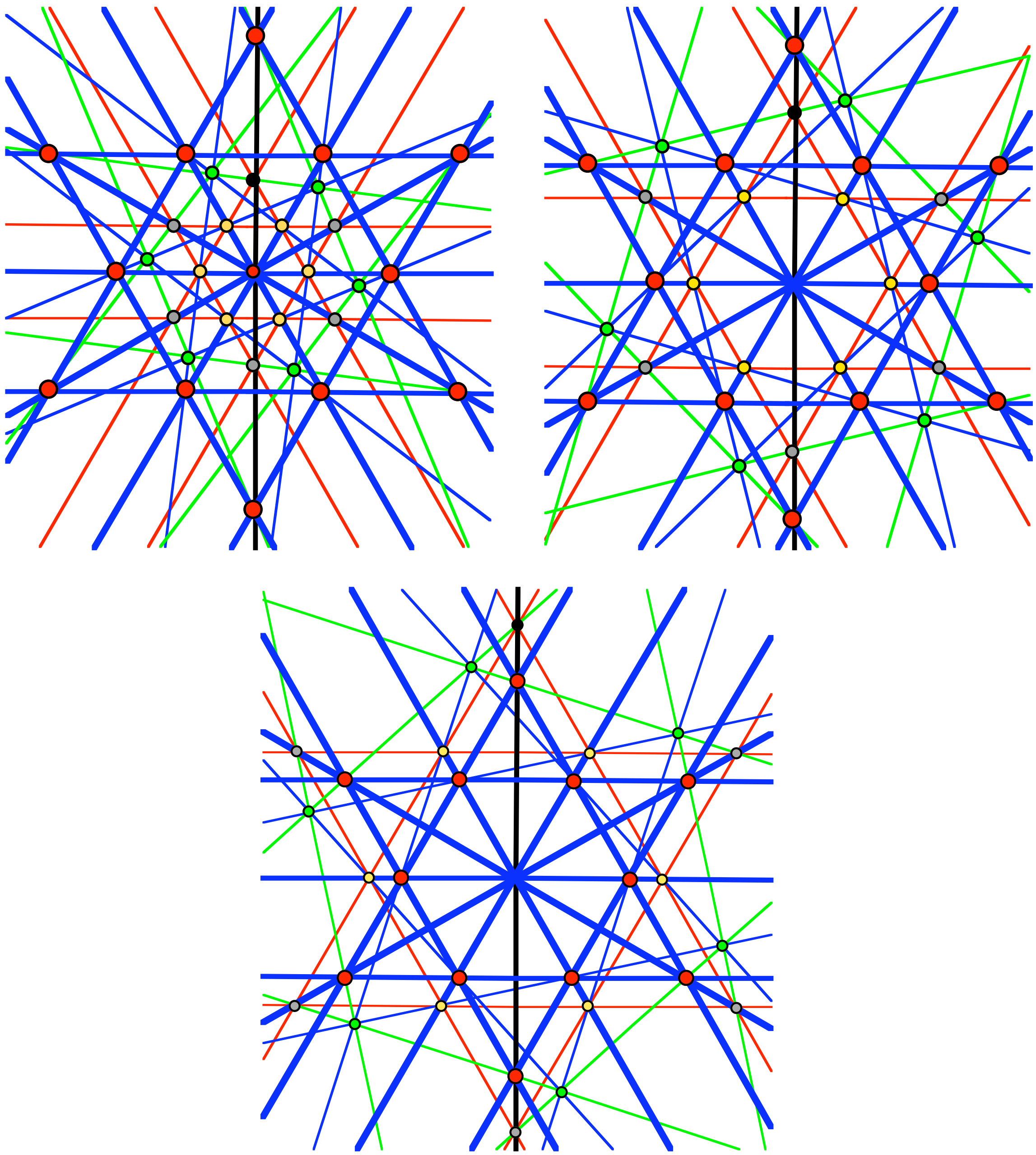
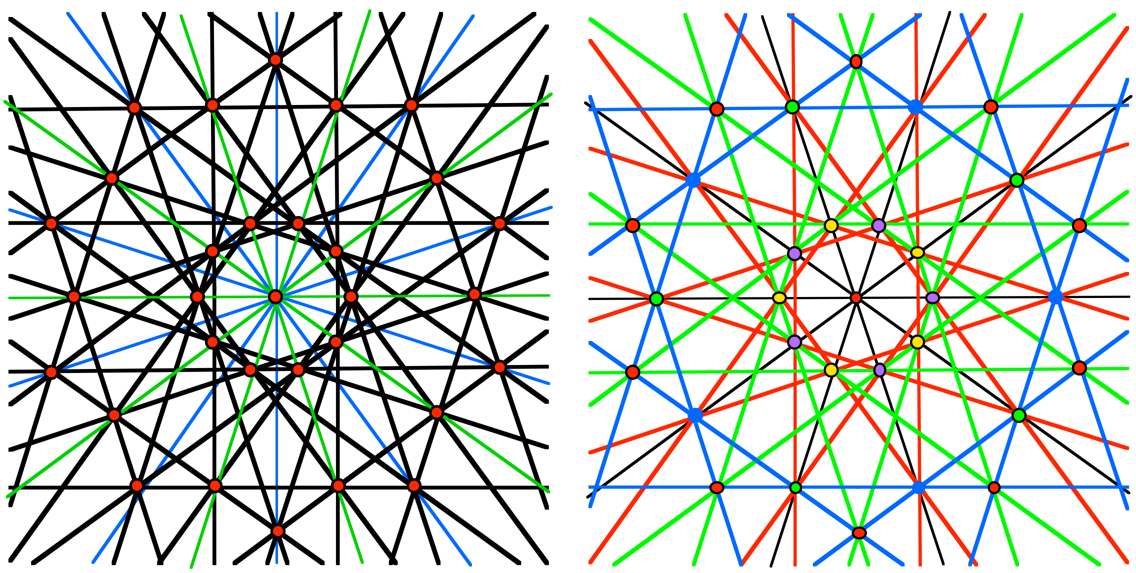


Figure 5.7.4. The smallest movable configuration from Berman's [B10]. It is a 5-orbits configuration (30_4). The black point can move freely on the black line as illustrated in the three parts.

Berman's construction in [B10] is considerably simpler than the ones in [B7], since it does not require deletion and pasting. However, it seems that the construction we shall consider next has certain advantages; it is presented here for the first time. It has been discovered a very short time before this book was going to print, and I have not had the time to figure out all conditions for the applicability of the method.

Consider the following example, which is the smallest to which the construction is applicable but is typical in all other respects. We start with the tricyclic configuration 10#(2,1;4,2;1,4); it has symmetry group d_{10}. However, we wish to consider it with only one of its d_5 subgroups. The situation is illustrated in Figure 5.7.5. Without the constraints imposed by the deleted mirrors and the accompanying rotations, the configuration is movable! In Figure 5.7.6 are shown several stages of the motion. The images were created with the Geometer's Sketchpad® software, the top green point being freely movable on the blue line with positive slope incident with it. An interesting and useful observation is that the points on the disregarded mirrors remain collinear throughout the motion.



(a) (b)

Figure 5.7.5. (a) The tricyclic configuration (30_4) with symbol 10#(2,1;4,2;1,4), shown with its ten mirrors; its symmetry group is d_{10}. (b) The same configuration, but equipped with only five of the mirrors. The three orbits of lines and the five orbits of points under the symmetry group d_5 are distinguished by their colors.

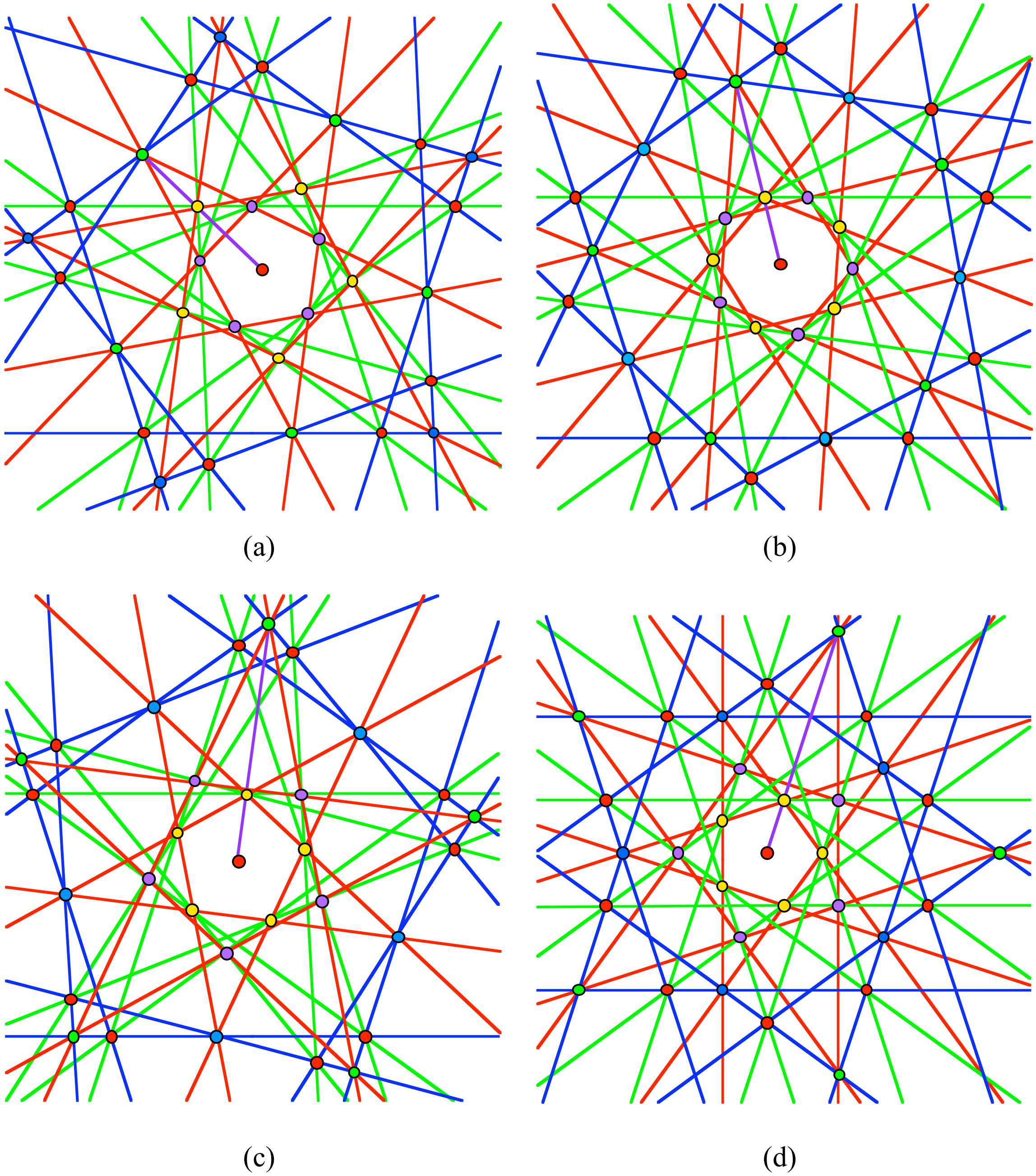
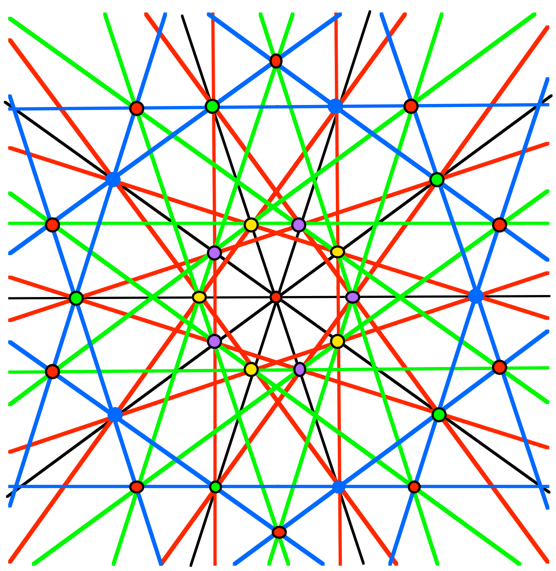
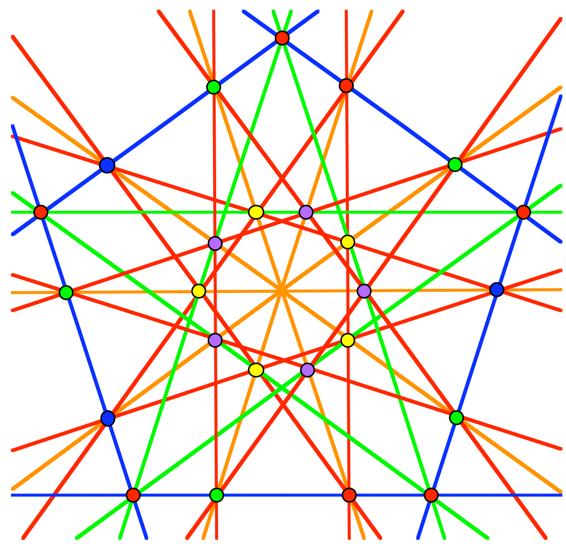


Figure 5.7.6. Four snapshots of different stages in the motion of the (30_4) configuration from Figure 5.7.5(b). The purple segment indicates one of the mirrors.

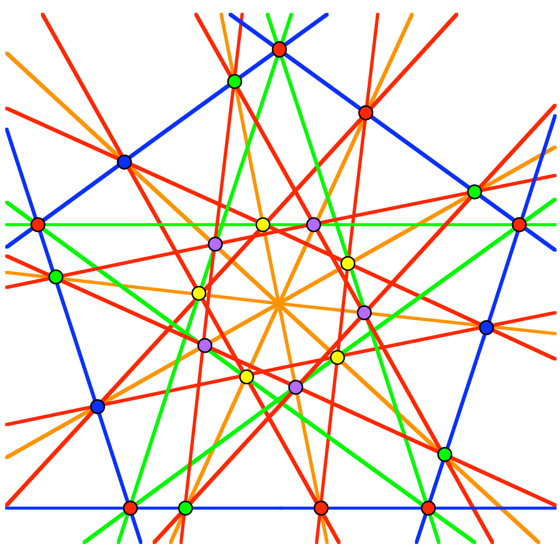
This circumstance can be used to construct another family of movable configurations, using the (5/6m) construction from Section 3.3. Applied to the (30_4) configuration in Figure 5.7.5(b) it yields the smallest known movable 4-configuration, a (25_4), illustrated with several snapshots of its motion in Figure 5.7.7.



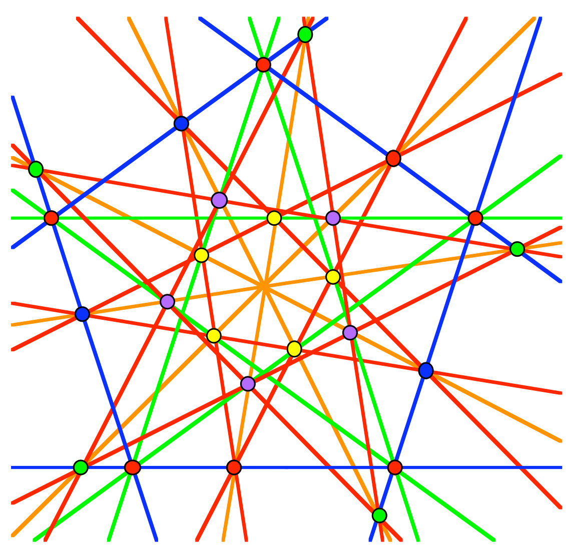
(a)



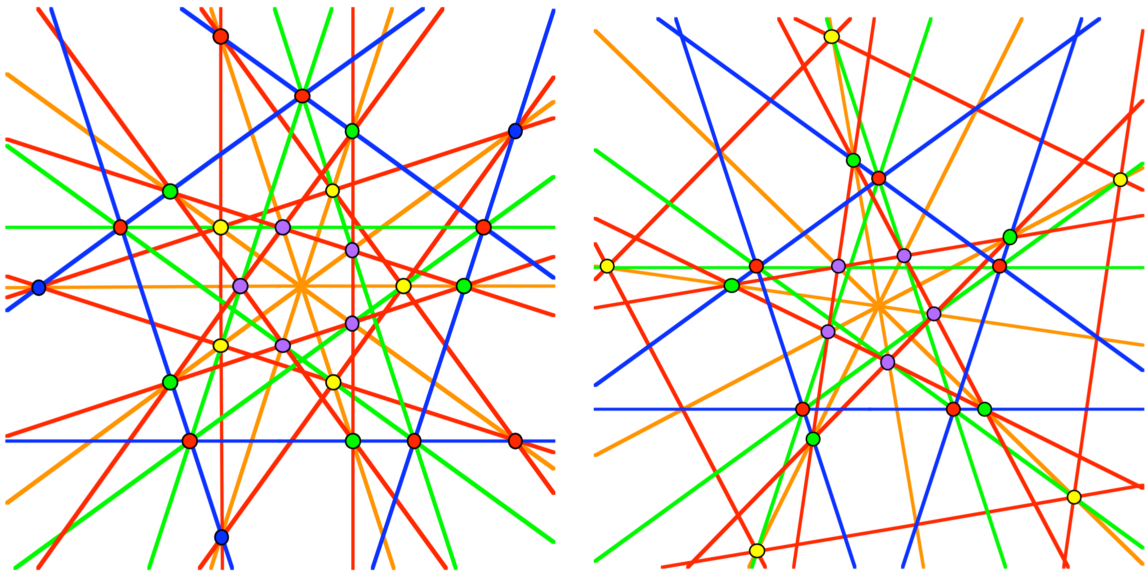
(b)



(c)



(d)



(e) (f)

Figure 5.7.7. (a) The (30_4) configuration with symmetry group d_5, from Figure 5.7.5(b). (b) A (25_4) configuration obtained from the configuration in (a) by omitting the points in the orbit of the lowest red point, and the lines incident with these points, and by adding the orange diametral lines (that go along the mirrors of the starting (30_4) configuration). This is the construction we introduced in Section 3.3 under the designation (5/6m); this particular instance is the same as the first part of Figure 3.3.13. (c) to (f) Four snapshots of different stages in the motion of the (25_4) configuration from part (b).

Exercises and problems 5.7

1. Investigate the possible motions of the astral configurations such as (10_3), allowing for departure from astrality.

2. Describe movable examples of 4-configurations that can be obtained by using copies of [3,4]-configurations. What are the smallest configurations of this kind? Apply the same construction to the superfiguration (9_3) shown in Figure 1.3.4.

3. Investigate to which configurations are the methods we used in Figures 4.7.6 and 5.7.7 applicable.