5.6 THE DIMENSION OF A CONFIGURATION

In Section 1.3 we introduced the concept of "dimension of a configuration". For convenience, we repeat it here. If C is a configuration we say that C has dimension d if this is the largest integer for which G admits a geometric representation (by points and straight lines) in some Euclidean space, such that the affine hull of the imbedding has dimension d.

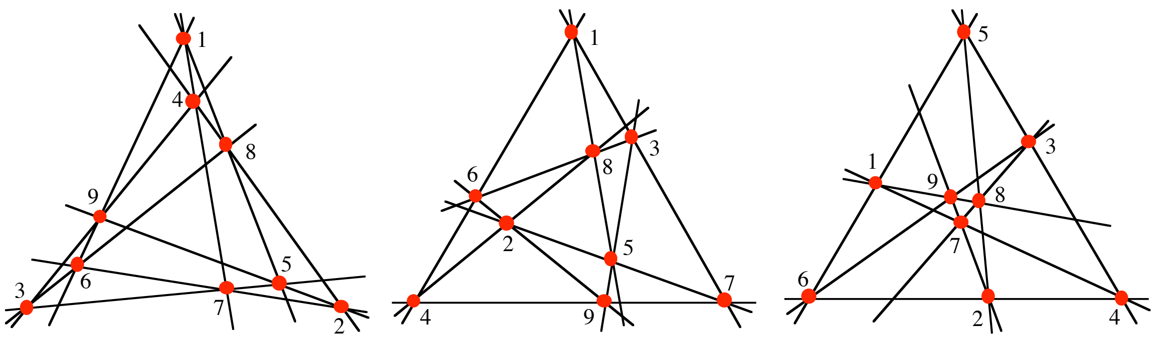
Among meaningful questions that one can ask is the determination of the dimension of a given configuration, the possible dimensions of k-configurations for a given k, what criteria can we find to determine whether a given configuration (or class of configurations) has this or that dimension, and so on. The material presented in this section has not been published before; it was developed in an ongoing collaboration with Tomaz Pisanski.

Here are some examples to help develop an understanding of the issues.

First, we consider the three smallest 3-configurations, the (9_3) configurations shown in Figure 5.6.1 (which we have seen earlier, as Figure 2.2.1). Each is (obviously) drawn in the plane — but could we somehow imbed it in 3-space so that it not be contained in a plane? The negative answer is easily established: Regardless of the dimension of the space, the plane determined by points 1, 5, 7 of $(9_3)_1$ necessarily contains also the points 3, 4, 8 and hence also 2, 6, 9, and thus the whole configuration. Similarly, the plane containing the points 1, 5, 7 of $(9_3)_2$ contains 2, 3, 8, hence 4, 6, 9; the plane containing 1, 5, 8 of $(9_3)_3$ contains also 2, 6, 9, and then 3, 4, 7 — in both cases the whole configuration.

A different situation prevails with respect to the Desargues configuration which we have denoted $(10_3)_1$; see Figure 5.6.2. Consider the four points 0, 1, 2, 3 imbedded in any Euclidean space of dimension at least 3, in such a way that no plane contains all four; this we occasionally call "general position". Then the points 4, 5, 6 determine another plane; we choose them so that the plane is not parallel to the plane of 1, 2, 3, and does not contain any of the four earlier points. These two planes determine (as their intersection) a line, which contains the points 7, 8, 9 determined, respectively, on that line by the planes of 1,3,6,4, 1, 2, 4, 5, and 2, 3, 6, 5. Hence all points (and all lines) of the configuration are in the 3-dimensional space affinely spanned by 0, 1, 2, 3. It follows that the dimension of the configuration $(10_3)_1$ is 3.

However, it would be wrong to conclude that an increase in the number of points implies an increase in the dimension. For example, the configuration $(10_3)_2$ shown in Figure 5.6.3 is 2-dimensional. Indeed, the plane containing points 1, 3, 5 contains also 2, 4, 9, hence 6, 7, 8, and 0, and thus the whole configuration.



$(9_3)_1$ $(9_3)_2$ $(9_3)_3$

Figure 5.6.1. The three configurations (9_3).

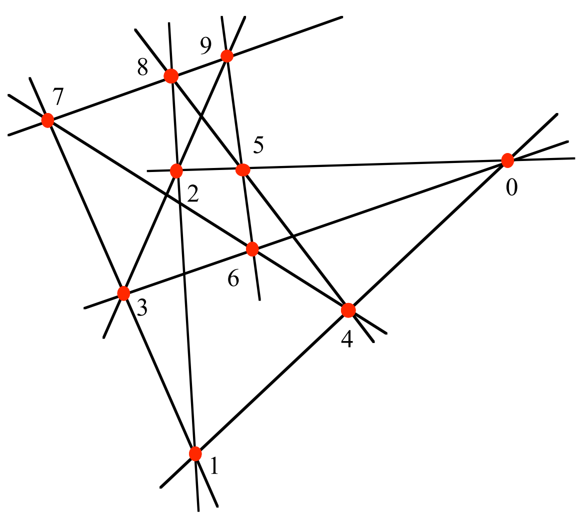


Figure 5.6.2. The configuration $(10_3)_1$ – the Desargues configuration — has dimension d = 3.

Theorem 5.6.1. There exist 3-configurations with arbitrarily large dimensions.

Proof. Start with any 3-configuration in the plane. Take three copies vertically above each other in 3-space, delete copies of the same line from each, and insert three vertical lines through the points on these lines. This raised the dimension by 1 (at least). Repeating the same procedure with three copies of this configuration, placed in suitable positions in parallel 3-spaces within a 4-dimensional space, deleting copies on one line from each and adding three transversals, raises the dimension of the resulting configuration to 4 (at least). Obviously we can continue indefinitely by the same method. ◊

The configurations constructed in the proof of Theorem 5.6.1 are quite large. It may be of interest to find smaller examples, at least for small dimensions d. We have already seen such configurations for $d = 2$ and 3. For $d = 4$ we can use the Cremona-Richmond configuration shown in Figure 5.6.4; we encountered this configuration earlier, in Sections 1.1 and 5.4. Indeed, consider the 15 points in the 4-dimensional Euclidean space E^4, listed in Table 5.6.1 by their labels in Figure 5.6.4. Then it is easily checked that all 15 triplets that are supposed to be collinear indeed are, while it is obvious that the affine hull of the set is 4-dimensional.

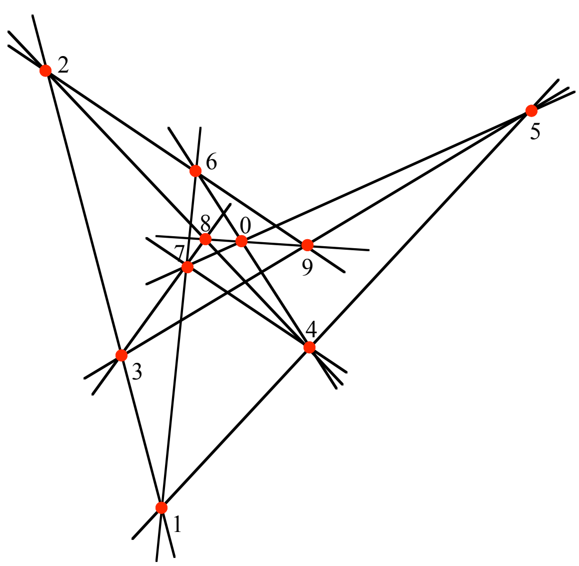


Figure 5.6.3. The configuration $(10_3)_2$ has dimension $d = 2$.

As reported in Section 2.3, there is no information available on the family of geometric configurations (n_3) for n = 13 or 14 (beyond some examples). Hence it is not possible to definitely assert that the Cremona-Richmond configuration is the smallest 3-configuration of dimension d = 4. We venture:

Conjecture 5.6.1. All configurations (n_3) with $n \leq 14$ have dimensions 2 or 3.

A question that arises quite naturally is whether dual configurations have the same dimension. A negative answer is obvious from the example in Figure 5.6.5: The configuration $(6_2, 4_3)$ is clearly contained in the plane determined by any two of its lines, while the dual configuration $(4_3, 6_2)$ spans the 3-dimensional space if the four points are chosen in affinely independent positions. While it is possible to generalize this example, the situation concerning connected but not 2-connected configurations discussed in Section 5.1 makes it plausible that balanced configurations may behave differently from unbalanced ones.

Conjecture 5.6.2. If C is a balanced configuration then the dimensions of C and its dual C* are the same.

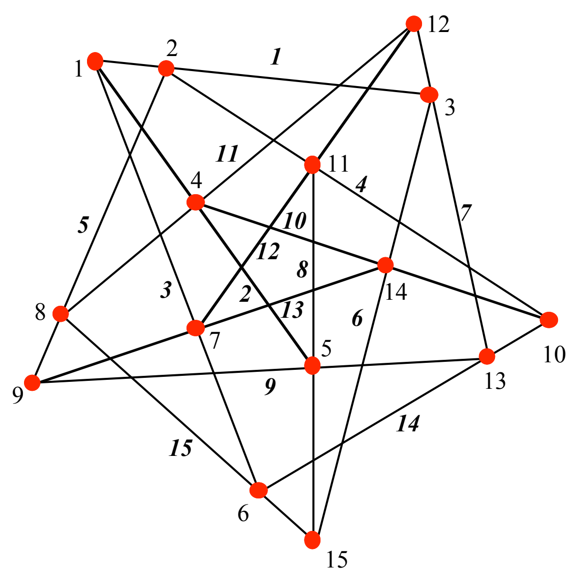
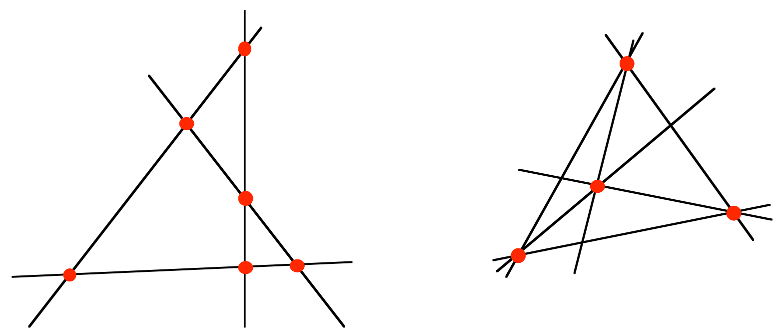


Figure 5.6.4. The Cremona-Richmond configuration (15_3) is 4-dimensional.

1 = (0,0,0,0), 2 = (1,0,0,0), 3 = (-1,0,0,0), 4 = (0,1,0,0), 5 = (0,-1,0,0),
6 = (0,0,1,0), 7 = (0,0,-1,0), 8 = (0,0,0,1), 9 = (2,0,0,2), 10 = (1,1,1,1),
11 = (0,2,2,2), 12 = (0,2,0,2), 13 = (2,2,0,2), 14 = (2,0,2,2), 15 = (0,0,2,2).

Table 5.6.1. Coordinates for the points of a realization of the Cremona-Richmond configuration (15_3) in the 4-dimensional Euclidean space. The names of the points refer to the labels in Figure 5.6.4.



(a) (b)

Figure 5.6.5. (a) The configuration $(6_2, 4_3)$ known as the *complete quadrilateral* is 2-dimensional. (b) Its dual $(4_3, 6_2)$, the *complete quadrangle*, is 3-dimensional.

It is easy to show that the cyclic configuration $\mathscr{C}_3(n)$ is 2-dimensional in all cases in which it is realizable by a geometric configuration, namely $n \geq 9$. Indeed, consider the typical Levi diagram of $\mathscr{C}_3(n)$, shown in Figure 5.6.6. The plane that contains the points P_0, P_1, P_2 contains the lines L_0, L_1, and hence the points P_3, P_4; then the lines L_2 and L_3 are in this plane, therefore the points P_5 and P_6 as well. Since this pattern continues indefinitely, the whole configuration is in one plane.

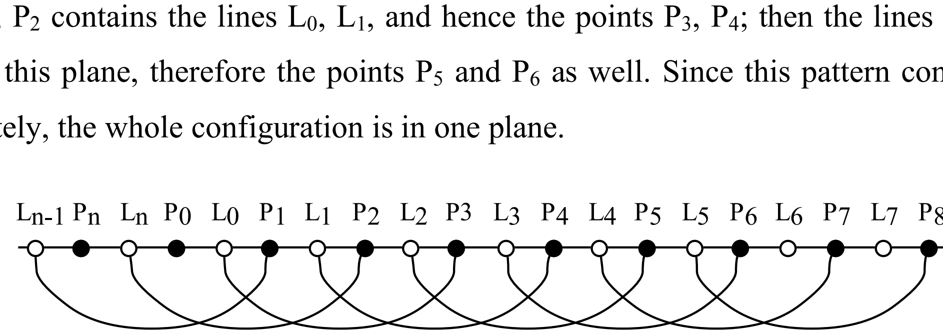


Figure 5.6.6. A stretch of the Levi graph of the cyclic configuration $\mathscr{C}_3(n)$ used to show that that the configuration is 2-dimensional for all $n \geq 9$.

So far we have dealt mainly with the dimension of 3-configurations. What is known about 4-configurations? Very little seems to be known at present. It is easy to verify that the astral configuration (24_4) shown in Figure 3.6.5 is 2-dimensional, as is the 3-astral configuration (21_4) shown in Figure 3.7.1. In case of the six astral configurations (36_4) shown in Figure 3.6.6 the proof that all are 2-dimensional is only slightly more involved. Experimental evidence on k-astral configurations has not turned up any that are demonstrably d-dimensional with $d \geq 3$. However, it is well possible that for reasonably large n some k-astral (n_4) configurations are not 2-dimensional; it would be interesting to decide this question at least for astral 4-configurations, or for 3-astral ones.

On the other hand, the (41_4) configuration in Figure 3.3.16 is easily seen to be 3-dimensional. The two parts that are joined at the four collinear points by the four concurrent lines show how to "bend" the configuration into 3-dimensional space.

A challenging task — that may be impossible to fulfill — is finding combinatorial criteria for the dimension of a configuration.

Exercises and problems 5.6

1. Determine the dimensions of the remaining seven configurations (10_3), shown in Figures 2.2.3 and 2.2.5.

2. Does the analogy with the results of Section 5.1 for unbalanced configurations extend to the dimensions? Specifically, do there exist [q,k]-configurations (with $3 \leq q \neq k \geq 3$) such that the dimensions of C and its dual C* are different?

3. How large can be the difference between the dimensions of a dual pair of configurations is Exercise 2 ?

4. Recall from Section 2.1 that a *general cyclic configuration* $\mathscr{C}_3(n,a,b)$ consists of triples {j, a+j, b+j}, for given a, b with $0 < a < b < n$ and for $1 \leq j \leq n$, all entries taken mod n. Determine the dimension of the various configurations $\mathscr{C}_3(n,1,b)$, and possibly of the general $\mathscr{C}_3(n,a,b)$.