5.5 "DENSITY" OF TRILATERALS IN CONFIGURATIONS

Another classical question is the following: Consider (combinatorial or geometric) 3-configurations for which the group of automorphisms acts transitively on the points and on the lines; for the purposes of this section (and only here) we shall follow the early writers and call such configurations **regular.** For a regular configuration we shall denote by $t = t(C)$ the number of trilaterals that meet one (hence every) point of a regular configuration C. The question of possible values of the "density" t is a topic raised by Schoenflies in [S1] and [S2], where he provided a partial answer. We shall present his results soon, but first a word of warning. These papers, like many in that period, are far from clear regarding exactly what are the configurations under discussion; quite a few of the assertions show a degree of naïveté that one does not expect from serious mathematicians. For example, Schoenflies states in [S1] that every regular configuration is self-dual; he repeats the assertion in [S2]. This error was noticed by Steinitz [S20]; he mentions that the smallest counterexample is a geometric (18_3) configuration; its underlying combinatorial version is denoted 18.7 in [B14, p. 337]. Also, although Schoenflies *seems to think* that he deals with geometric configurations (in the complex plane !) and therefore does not notice the Fano plane (7_3), Schoenflies does not address the question whether his configurations can actually be *geometrically* realized — even in the complex plane.

Acknowledging the work of Martinetti [M1] on configurations with no trilaterals (though giving an incorrect reference for it), Schoenflies asserts in [S1] that t must have one of the values 9, 6, 4, 3, or 2. (Note that the Fano plane has $t = 12$.) However, as noticed by Steinitz [S20], $t = 1$ is possible as well. The first result in [S1] is that $t(C) = 9$ happens if and only if $C = (8_3)$, the Mobius-Kantor configuration. (Unless one considers this as a combinatorial configuration, it can be "realized" only in the complex plane.) But Schoenflies here clearly missed the restriction to connected configurations — if C consists of several disconnected copies of (8_3) then $t(C) = 9$ as well; he corrected this error in [S2], as well as providing there a correct reference for [M1]. Similar shortcomings afflict some of his other assertions. For example, another statement is that there are three possibilities for $t = 6$: (i) The Pappus configuration (9_3); (ii) The Desargues configuration (10_3); and (iii) The cyclic configurations $\mathscr{C}_3(n)$ for $n \geq 9$, with lines $(j, j+1, j+3)$, (which we consider in Section 2.1 and 5.6). But as noted by Steinitz [S19, p. 488], t = 8 for the cyclic (9_3). The possibilities for t = 2, 3, or 4 are also discussed in detail in [S1] and in particular in [S2]; they consist of various families of mutually inscribed/circumscribed multilaterals. Steinitz [S19] also corrected Schoenflies' erroneous assertion that t(C) = 1 is not possible.

Steinitz [S20] and [S19, p. 489] also notes that all points of a 3-configuration may be in one orbit under automorphisms of the configuration, without the lines necessarily being in one orbit. Also, even if both points and lines are in single orbits, the configurations needs not be self-dual. On the other hand, all examples of this type are not relevant to cyclic configurations and their properties.

There is little motivation to discuss in more details of the results of Schoenflies and others and their proofs. However, several open questions deserve mention.

- Are there are any 3-configurations that are not regular but for which there exists a "density" of trilaterals — in other words, every point is incident with as many trilaterals as every other point.

Possibly some of the astral 3-configuration may provide examples, but the question seems not to have been investigated.

- Is there a reasonable classification of "regular" 4-configurations that have a "density" of trilaterals?

For example, the astral (24_4) configuration shown in Figure 3.6.2 has 16 trilaterals incident with every point.

- What about "densities" of quadrilaterals in "regular" 3- or 4-configurations? The same question may be asked for other multilaterals, including Hamiltonian multilaterals.

Questions in the same spirit have often been asked and investigated for graphs, and in some cases for combinatorial configurations or more general incidence structures. As this material has little connection to geometric configurations, we do not consider it here.

Exercises and problems 5.5

1. Verify the claim that the configuration in Figure 3.6.2 has 16 trilaterals incident with each point.

2. Find the number of trilaterals incident with each point of the 3-astral configuration (21_4) in Figure 3.7.1. How many quadrilaterals are incident with every point?

3. Are all points of the 2-astral configuration (48_4) shown in Figure 3.6.1 incident with the same number of trilaterals?

4. Investigate the astral 3-configurations (n_3) with $10 \leq n \leq 14$ concerning the numbers of trilaterals incident with each point. Quadrilateral? Any general hypotheses?