5.4 MULTILATERAL-FREE CONFIGURATIONS

We turn now to one of questions concerning trilaterals (and multilaterals) in configurations, that go back to the classical period of configurations in the last quarter of the 19th century. It has seen new life in the recent decades, mostly without any acknowledged relation to the earlier results.

The first question that will occupy us asks for configurations that contain no trilaterals. Here is what is known.

Theorem 5.4.1. For every $k \geq 2$ there exist geometric k-configurations that are trilateral-free.

The proof is immediate on recalling the configurations LC(k) described in Section 1.1 as well as in [P5], and utilized in Section 5.1. The only drawback of this answer is the rather large size of these configurations. The resulting trilateral-free geometric configurations are (n_k) with $n = k^k$. ♦

We shall see below how smaller trilateral-free geometric configurations can be found in some cases. For some general estimates see Lazebnik *et al.* [L2].

Another general result gives a lower bound on the size of trilateral-free configurations.

Theorem 5.4.2. If an (n_k) configuration with $k \geq 2$ is trilateral-free then $n \geq k(k-1)^2 + k$.

The proof is straightforward on considering the situation schematically presented in Figure 5.4.1, assuming k = 4. Any one line (represented by the horizontal one) carries k points; through each of the points go k–1 other lines of the configuration, each carrying k–1 additional points. The only remark that needs to be made is that these points must all be distinct, since otherwise there would be a trilateral present in the configuration. This argumentation (or something similar) was shown to me by J. Bokowski. Notice that the argument does not use any geometry, and hence the result holds for combinatorial configurations as well. ♦

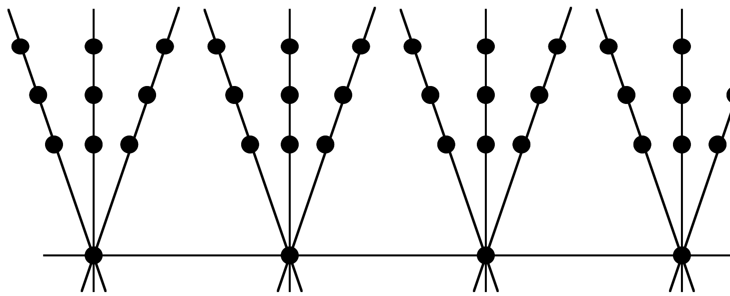


Figure 5.4.1. Schematic representation of the proof of Theorem 5.4.2.

For k = 3 the result was known to Martinetti [M1] in 1886.

The cubic bound in Theorem 5.4.2 is in contrast to the exponentially large examples in Theorem 5.4.1. We shall next show that we can do much better than the exponential example for k = 3, and slightly better for k = 4.

As a consequence of Theorem 5.4.2 we see that for k = 3 any trilateral-free k-configuration must have at least $n \geq 15$ points. Martinetti [M1] seems to be the first to have raised in 1886 the question of trilateral-free 3-configurations. He proved that such configurations have at least 15 points, and provided a combinatorial description of the unique (15_3) that is trilateral free. It needs to be stressed that, from all that we can read in his publications, Martinetti thought at that time (as well as later) that combinatorial 3-configurations are all geometrically realizable. In fact, Martinetti's trilateral-free (15_3) configuration is, indeed, geometrically realizable, see Figure 5.4.2. Moreover, in the prehistory of configurations, traces of this (15_3) geometric configuration can be found in considerations of families of straight lines on cubic surfaces, by Schläfli in 1858 and Cremona in 1868. The configuration itself is frequently called the Cremona-Richmond configuration; see [C6], [W2]. More detailed historical explanations and references can be found in [B19].

The Cremona-Richmond configuration is shown in Figure 5.4.2, and a Levi graph (see [W7]) based on a Hamiltonian multilateral, is shown in Figure 5.4.3. Since the configuration is trilateral-free, its Levi graph has no circuits of size smaller than 8; in other words, its *girth* is 8. In fact, it is the smallest 3-valent graph of girth 8, and it is famous as

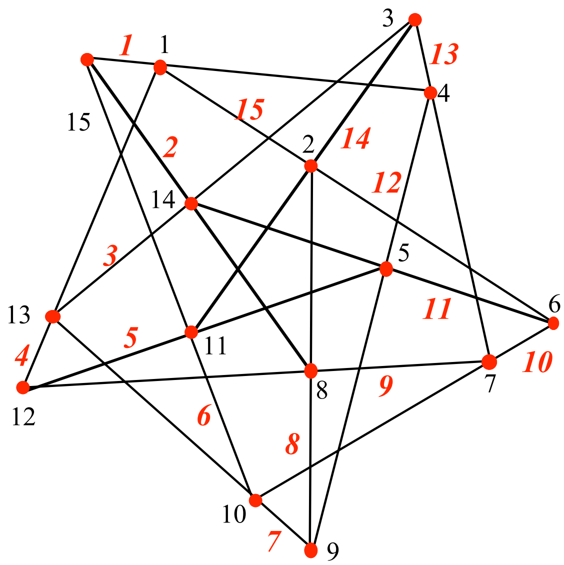


Figure 5.4.2. The Cremona-Richmond trilateral-free configuration (15_3). Point labels are in plain font, line labels are in bold italics. Same digits establish a duality correspondence between points and lines.

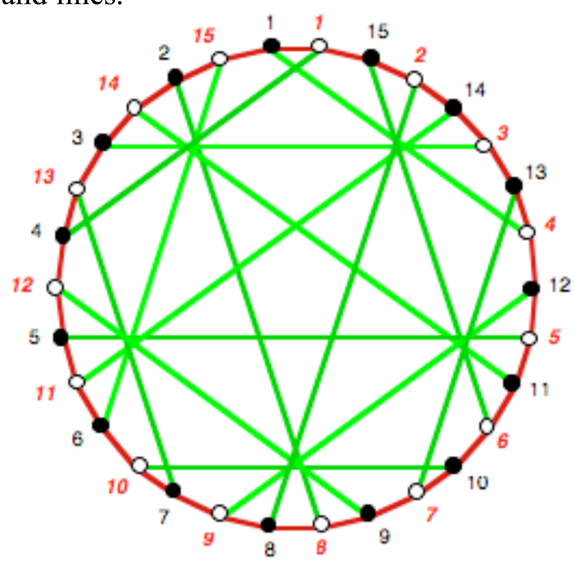


Figure 5.4.3. A Levi graph of the Cremona-Richmond configuration. It is the "Tutte 8-cage", the smallest 3-valent graph with girth 8. Similar presentations appear in [C6] and many other places.

Tutte's (3,8)-cage, or more simply, *Tutte's 8-cage.* Coxeter [C6] provides an ingenious labeling of its vertices, and calls it "the most regular of all graphs". Figure 5.4.3 also shows that this graph has a color-reversing symmetry, hence the Cremona-Richmond configuration is selfdual; this result can also be deduced from the fact that there is only one type of trilateral-free (15_3) configuration. On the other hand, it should be noted that there are infinitely many *projectively* inequivalent geometric realizations of this configuration. This is most easily seen by manipulations in some software such as Geometer's Sketchpad™.

Before continuing our description of the other results about trilateral-free configurations, we need to present some of the more recent definitions and results that deal with the same topic in a different language.

A **(k,g)-cage** is a graph with all vertices of **valence** k and of **girth** g, having the smallest possible number of vertices. For this definition, and most of the known results concerning cages, see [G19], [W7] and the references given there. For attractive illustrations of some of the cages see [P7].

The Levi graph of a (combinatorial or geometric) (n_k) configuration has, as we mentioned in Section 1.5, girth $g \geq 6$; since, as we have seen in Section 2.1, the Fano configuration (7_3) has the smallest number of vertices, its Levi graph with 14 vertices is a (3,6)-cage — in fact, the only (3,6)-cage.

Trilateral-free 3-configurations have girth at least 8; hence the Levi graph of the smallest such configuration — the Cremona-Richmond (15_3) — is the (3,8)-cage, with 30 vertices. Since the Cremona-Richmond (15_3) is the unique trilateral-free (15_3) configuration, this cage is also unique: it is the Tutte (3,8)-cage, mentioned earlier.

Another related concept is that of "generalized quadrangles". A **generalized quadrangle** is an incidence structure in which each pair of distinct points determines at most one line, and for each non-incident pair consisting of a point P and a line L, there is precisely one line L* that is incident with P and with a point P* of L. A (finite) generalized quadrangle is of **order** (s, t) if every line contains precisely s+1 points and every point is incident with precisely t+1 lines. The terminology is often justified by the fact that an ordinary quadrangle can be interpreted as a generalized quadrangle of order (1,1). Obviously, each generalized quadrangle of order (2,2) is a combinatorial 3-configuration.

The smallest generalized quadrangle of order (2,2) has 15 points; hence it is a (15_3) configuration, for which the definition of generalized quadrangles implies that it is trilateral-free. It follows that it is isomorphic with the Cremona-Richmond configuration. Polster [P7] shows several diagrams of this generalized quadrangle; two are particularly interesting. The first, which he attributes to Stanley Paine, is shown in Figure 5.4.4; the labels establish its isomorphism with the configuration in Figure 5.4.2.

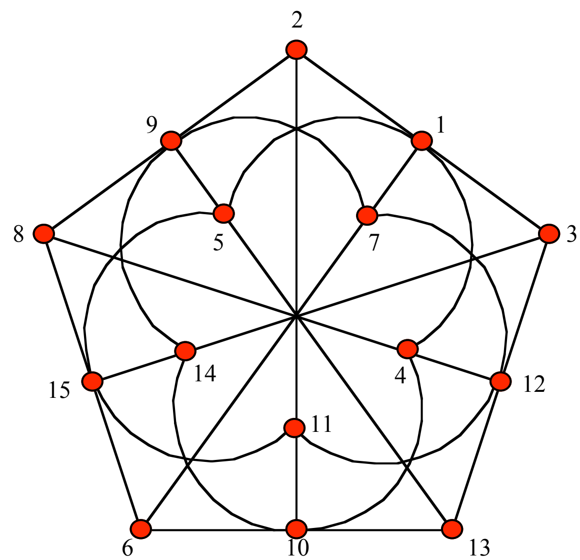


Figure 5.4.4. The "doily" of S. Payne: a geometric model of the 15-point generalized quadrangle of order (2,2) — also known as the Cremona-Richmond configuration.

The second interesting model shown by Polster [P7] is by lines (actually, line segments) in 3-dimensional space; see Figure 5.4.5. The model is best understood as being spanned by a regular tetrahedron; the tetrahedron's edges are indicated by the dashed lines and are not part of the configuration. Naturally, any appropriate projection of this model into the plane provides a planar realization of the Cremona-Richmond configuration. A figure resembling such a projection illustrates the Cremona-Richmond configuration in Wells' "Dictionary" [W2, p. 40].

Returning now to the configuration language, here are some of the additional results on trilateral-free 3-configurations, all established by Martinetti [M1]:

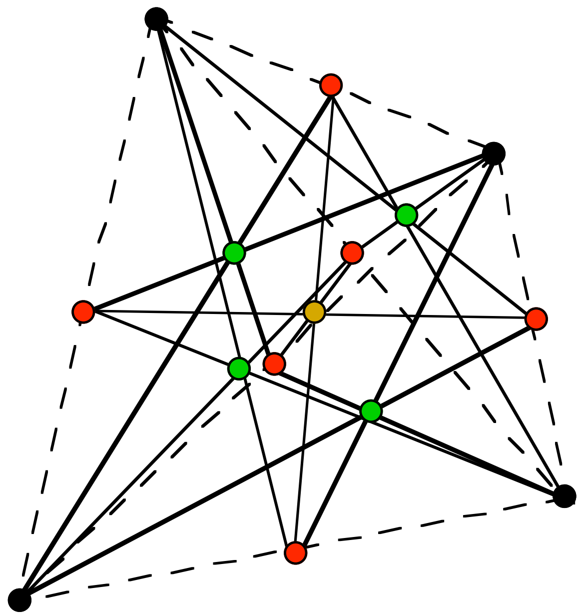


Figure 5.4.5. A realization of the 15-points generalized quadrangle of order (2,2), alias Cremona-Richmond configuration, supported in 3-space by a regular tetrahedron. (Adapted from [P7].)

- There are no trilateral-free configurations (16_3). This is not hard to show, starting with the arrangement shown in Figure 5.4.1, and noting that for a sixteenth point there are only relatively few possibilities of collinearities with the other points — none leading to a configuration, even in the combinatorial sense.
- There is a single trilateral-free configuration (17_3); again, the uniqueness implies that it is selfdual. It is interesting because of its very low symmetry. Under its group of automorphisms it has four point orbits: two of size 6 each, one of size 3, and one of size 2. It is geometrically realizable, but with no symmetry. Details (such as configuration table, geometric realization, Levi graph, automorphism group, orbits) can be found in [B19].
- There are precisely four trilateral-free configurations (18_3). Two are dual to each other, and each of the other two is self-dual. Data on all four, with geometric realizations, are given in [B19]. One of the selfdual configurations (denoted 18-D in [M1] and [B19]) is interesting because of its symmetry; it admits a selfpolar realization as an astral configuration 9#(4,2;3) in the notation of Section 2.7, and is shown in Figure 5.4.6.

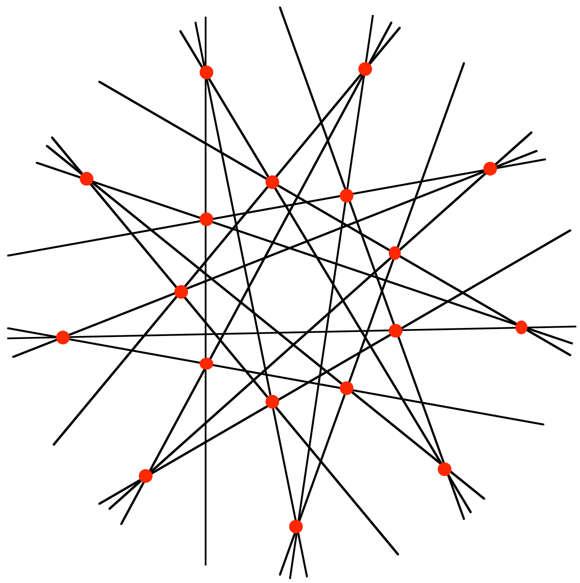


Figure 5.4.6. A realization of the trilateral-free selfpolar configuration (18_3) denoted 18-D in [B19]; it is astral with symbol 9#(4,2;3), and is the first of an infinite series of trilateral-free selfpolar configurations.

In considering these results of Martinetti [M1], one should bear in mind that although he uses geometrical language there is no diagram presenting these configurations, nor is there any hint how the corresponding geometric configurations should be constructed. The first geometric realizations seem to be the ones in [B19].

According to the data in [B14] (reproduced in [B19]) there are 19 combinatorial trilateral free configurations (19_3), 162 such configurations (20_3), and 4713 (21_3). It is not known how many are geometrically realizable.

On the other hand, we have the following:

Theorem 5.4.3. For every $n \geq 15$ except $n = 16$ and possibly $n = 23$ and 27, there are trilateral-free geometric configurations (n_3).

Proof. For n = 15, 17, 18, 19, 20, 21 trilateral-free geometric configurations are shown in [B19]. It is easy to verify that all astral configurations $m\#(4,2;3)$ for $m \geq 9$ and $m \neq 12$ are trilateral free; this shows that for all even $n \geq 18$, $n \neq 24$, there are trilateral-free configurations (n_3). The (18_3) and (20_3) configurations mentioned above are of

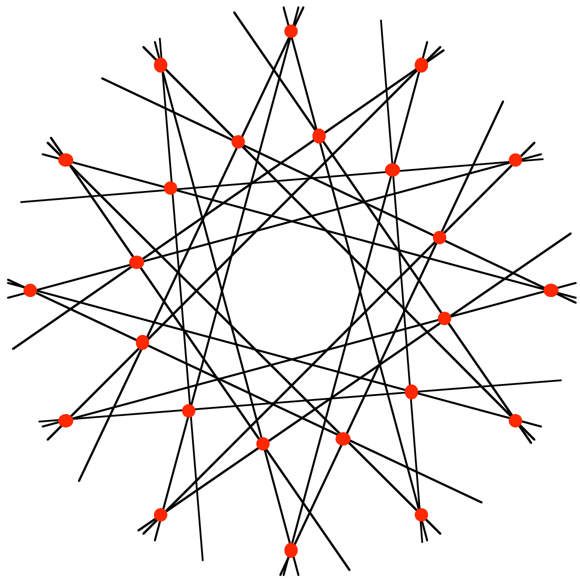


Figure 5.4.7. A trilateral-free configuration (24_3); it is astral with symbol 12#(5,3;4).

this type. For the exceptional value n = 24 a trilateral-free geometric configuration is shown in Figure 5.4.7.

The construction of the appropriate configurations for odd n is slightly more complicated. In almost all cases, the following construction works. Starting with trilateral-free geometric configurations (p_3) and (q_3), we delete one line in each and connect the three pairs of orphan points with an additional, new point. (The required alignment can always be obtained through suitable projective transformations.) This yields a trilateral-free geometric configuration (n_3) with $n = p + q + 1$. Starting from the trilateral-free configurations we already constructed, this yields the required geometric configurations for all odd $n \geq 31 = 15 + 15 + 1$. An alternative construction works for all $n \geq 29$: In analogy to the "deleted union" construction (DU-1) described in Section 3.3, we delete a line from a trilateral-free geometric configuration (p_3) and delete a point from a trilateral-free geometric configuration (q_3); by placing appropriate copies of the two configurations so that the lines of the latter (which are missing a point) pass through the points of the former (that are missing a line), we obtain a trilateral-free geometric configuration of $n = p + q - 1$ points. For $p = q = 15$ this yields $n = 29$.

The case (25_3) is particularly interesting. Visconti [V4] gives configuration tables for two distinct trilateral-free combinatorial configurations (25_3), each consisting of a family of five mutually inscribed/circumscribed pentalaterals. These are reproduced, in Visconti's notation, in Tables 5.4.1 an 5.4.2. In the somewhat analogous case of trilateral-free configuration (20_3) consisting of four mutually inscribed/circumscribed pentalaterals, Visconti provides a graphical representation, that seems to be the first *symmetric* rendition of any multiastral configuration. However, contrasting this is the fact that there is no indication in [V4] whether the (25_3) configurations described are geometrically realizable. We have verified that at least one of these can be drawn, but not in a polycyclic manner; see Figure 5.4.8. Just as in the case of the (17_3) and (19_3) configurations investigated in [B19], the configuration is asymmetric, and was constructed by successive approximations. It is very likely that the same situation exists for Visconti's other (25_3). ♦

Visconti [V4] and Martinetti [M3] provide additional examples of trilateral-free combinatorial 3-configurations consisting of mutually inscribed/circumscribed pentalaterals, and some other multilaterals as well. It may be conjectured that these are geometrically realizable as well.

It is worth noting that most astral configurations m#(b,c;d) are trilateral-free for sufficiently large m. Exceptions (such as the n = 24 case mentioned above) are usually easy to spot, but there seem to be some subtler issues that have not been tackled so far. An example of such a situation is given in Figure 5.4.9.

We are turning now to quadrilateral-free configurations; this term is somewhat of a misnomer — at least in the sense we shall use it. By **quadrilateral-free** we shall refer to configurations that have neither a trilateral nor a quadrilateral. We have no example of a configuration that has no quadrilateral but does have trilaterals; it appears to be an open question whether such configurations exist. This leads to our terminology that simplifies the locutions.

In discussing quadrilateral-free configurations, we consider only 3-configurations, since nothing on the topic of quadrilateral-free k-configurations seems to be known for $k \geq 4$.

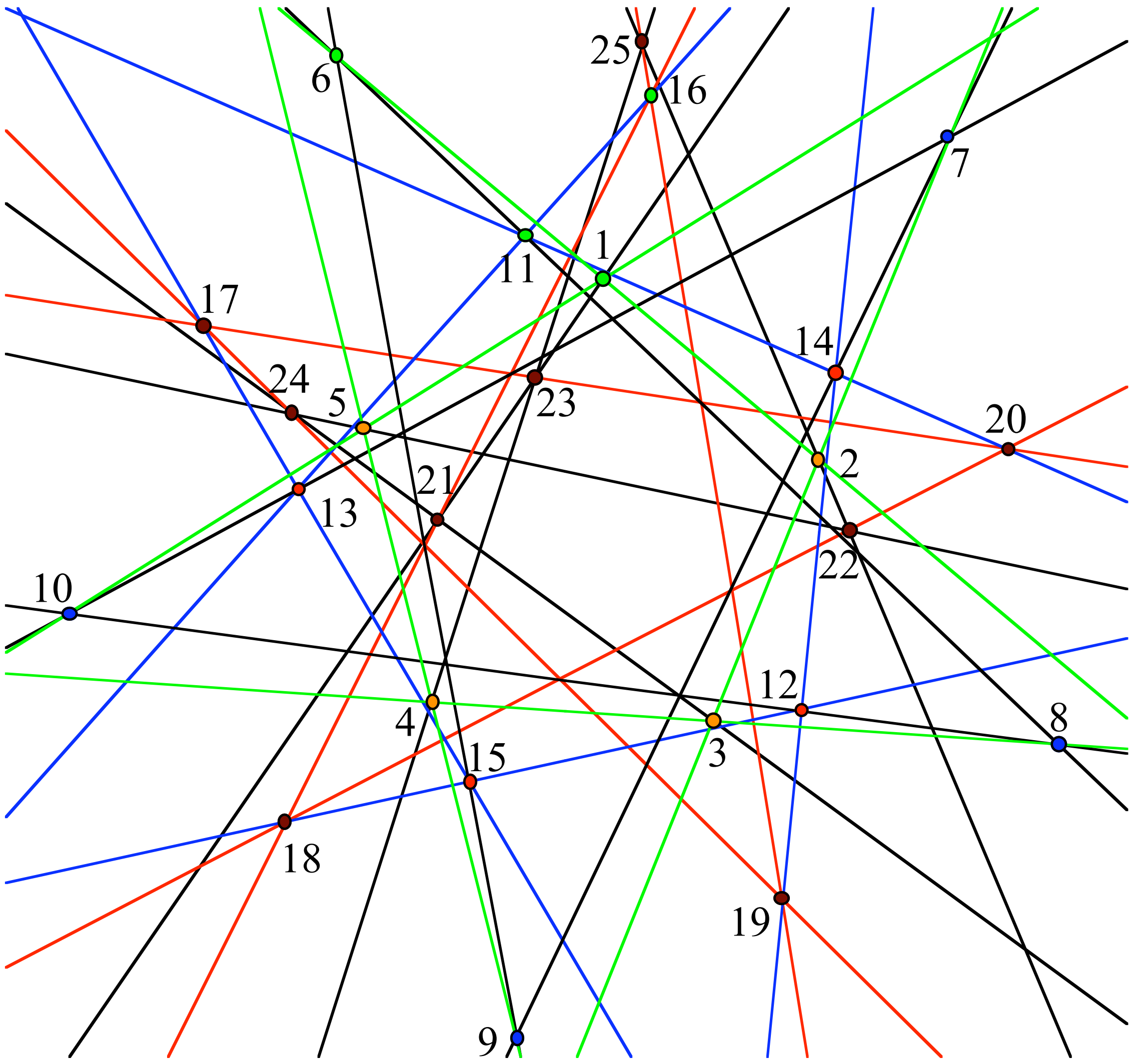


Figure 5.4.8. A geometric realization of one of the trilateral-free configurations (25_3) given by configuration tables in [V4] and Table 5.4.2 below.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
2	3	4	5	1	8	9	10	6	7	13	14	15	11	12	18	19	20	16	17	23	24	25	21	22
6	7	8	9	10	11	14	12	15	13	16	19	17	20	18	21	24	22	25	23	1	2	3	4	5

Table 5.4.1. The configuration table of one of the trilateral-free configurations (25_3) found by Visconti [V4].

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
2	3	4	5	1	8	9	10	6	7	13	14	15	11	12	18	19	20	16	17	23	24	25	21	22
6	7	8	9	10	11	14	12	15	13	16	19	17	20	18	21	24	22	25	23	1	5	4	3	2

Table 5.4.2. The configuration table of the other trilateral-free configuration (25_3) found by Visconti [V4]. A geometric realization of this configuration is given in Figure 5.4.8.

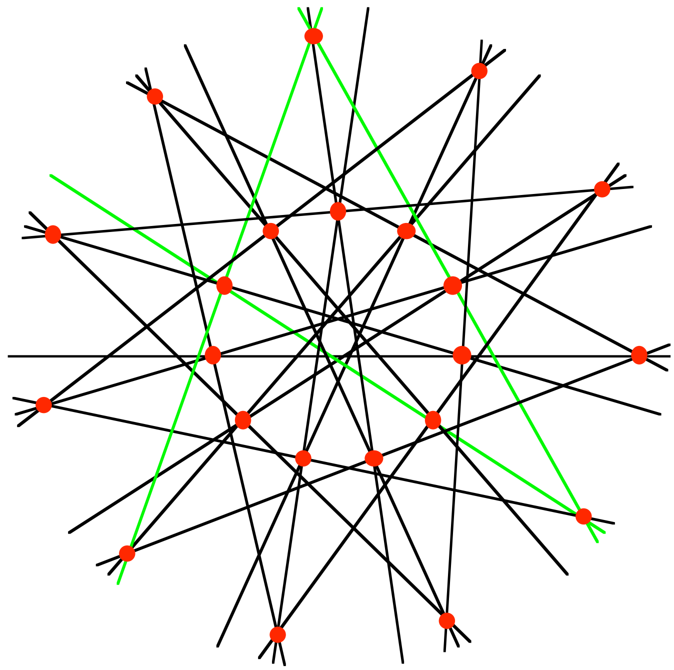


Figure 5.4.9. The astral configuration 11#(5,4;2) is not trilateral-free; one trilateral is shown by green lines. For m such that $13 \leq m \neq 15$, the configuration m#(5,4;2) is trilateral-free.

The only published work I am aware of that deals with quadrilateral-free geometric 3-configurations is [P6]; the configurations are studied using their Levi graphs.

Through the Levi graphs, the question of quadrilateral-free configurations is related to 3-valent bipartite graphs of girth at least 10. Such graphs have been extensively investigated; a large quantity of relevant literature can be found in [W6] and [W7].

The result of these graph-theoretic studies that is most relevant to our topic is that there exist exactly three 10-cages (also called (3,10)-cages), that is, 3-valent graphs of girth 10 with the smallest number of vertices, namely 70; all three are bipartite. The first one was found by Balaban [B1], the other two by O'Keefe and Wong [O2]; Wong [W6] proved that these three are the only ones. Balaban's 10-cage has a color-interchange automorphism, the other two do not have any such automorphism.

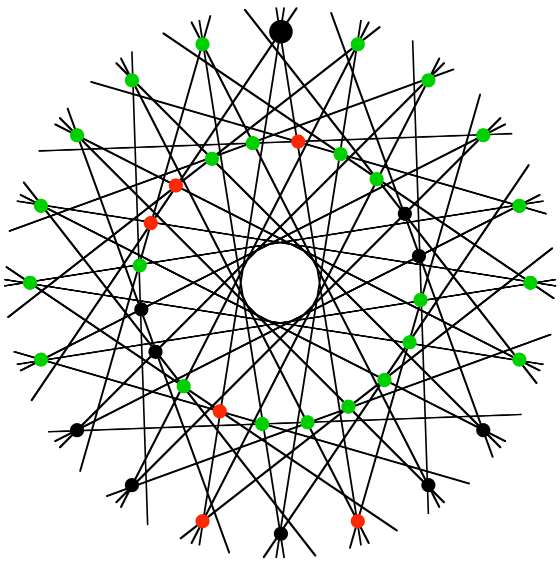
In [P6], Pisanski *et al.* describe in detail these three 10-cages, and the resulting five quadrilateral-free configurations (35_3). Since Balaban's 10-cage has a color-reversing symmetry, the corresponding configuration is selfdual. The other two 10-cages yield a pair of dual configurations each. It is clear that the three 10-cages can be interpreted as Levi graphs of quadrilateral-free *combinatorial* 3-configuration (35_3); however, Pisanski *et al.* prove that they admit *geometric* realizations, and provide in [P6] diagrams for three of the five. These three admit polycyclic representations which the last two do not have; for them there is in [P6] a description of the method of proof (following [B25]), and a reference to the full set of coordinates listed at a website. In [P6] there is also described a construction of quadrilateral-free geometric configurations (n_3) for an infinite sequence of values of n.

An improvement of this last result is the following:

Theorem 5.4.4. For every $n = 4m \geq 40$, there exists a quadrilateral-free geometric configuration (n_3). There exists an n_0 such that for every $n \geq n_0$ there is a quadrilateral-free geometric configuration (n_3). The available estimate is $n_0 \leq 320$.

Proof. It is easily verified that for $m \geq 10$ the astral configuration (2m)#(m-1,1;4) is quadrilateral-free. I am indebted to T. Pisanski for showing me one of the two smallest of these configurations, 20#(9,1;4), see Figure 5.4.10(a). The next members of the sequence, the pair of dual 22#(10,1;4) are shown in Figure 5.4.11. The proof of the fact that the (2m)#(m-1,1;4) configurations are quadrilateral-free is easy by generalizing the argument indicated by the coloring of the points in Figures 5.4.10 and 5.4.11. Since the configuration is astral, and any quadrilateral would have to contain a point of the outer ring, it is enough to show that the point marked by the large black dot is not part of any quadrilateral. The only six points at (graph-)distance 1 are the six red points, and those at distance 2 are the 24 green ones. The presence of any quadrilateral would imply that two of the green points coincide — which does not happen since this would imply that there are at most 23 green points.

In order to prove the existence of n_0 we may use the same construction as in the proof of theorem 5.4.3 for odd n. We take two quadrilateral-free configurations (p_3) and (q_3), and using convenient representatives delete one line from each; an additional point and three lines through it and the points on the two deleted lines form a quadrilateral-free configuration (n_3) with $n = p + q + 1$. Repeating the construction r times leads to configurations with r points more than the sum of the numbers of points of the configurations used. This yields the bound $n_0 \leq 320$. ♦



(a) (b)

Figure 5.4.10. The two dual astral configurations 20#(9,1;4). Both (40_3) configurations are quadrilateral-free.

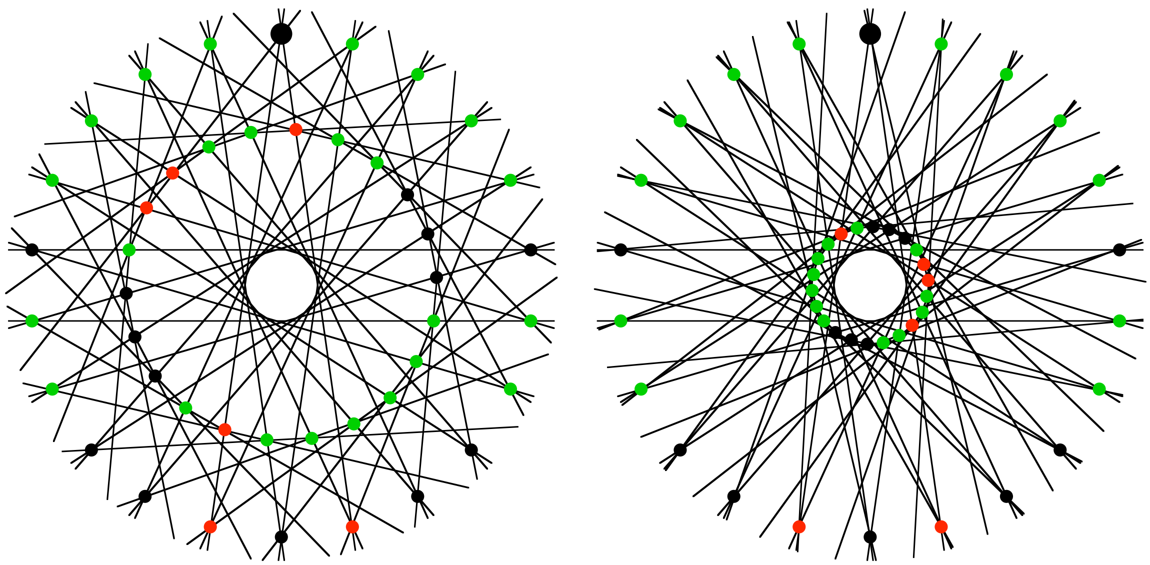


Figure 5.4.11. A dual pair of astral, quadrilateral-free (44_3) configurations 22#(10,1;4).

In analogy to our convention concerning quadrilaterals, we say that a configuration is **pentalateral-free** if it contains no t-laterals for t = 3, 4, 5. The information available is exceedingly meager. A (3,12)-cage has 126 points and happens to be bipartite; hence it can be interpreted as the Levi graph of a pair of dual (63_3) *combinatorial* configurations. Schroth [S9] found graphic representations of these two configurations, which we reproduce in Figures 5.4.12 and 5.4.13. (The title of Schroth's paper refers to the "generalized hexagons" of order (2,2); the uniqueness of the dual pair was established in [C3].)

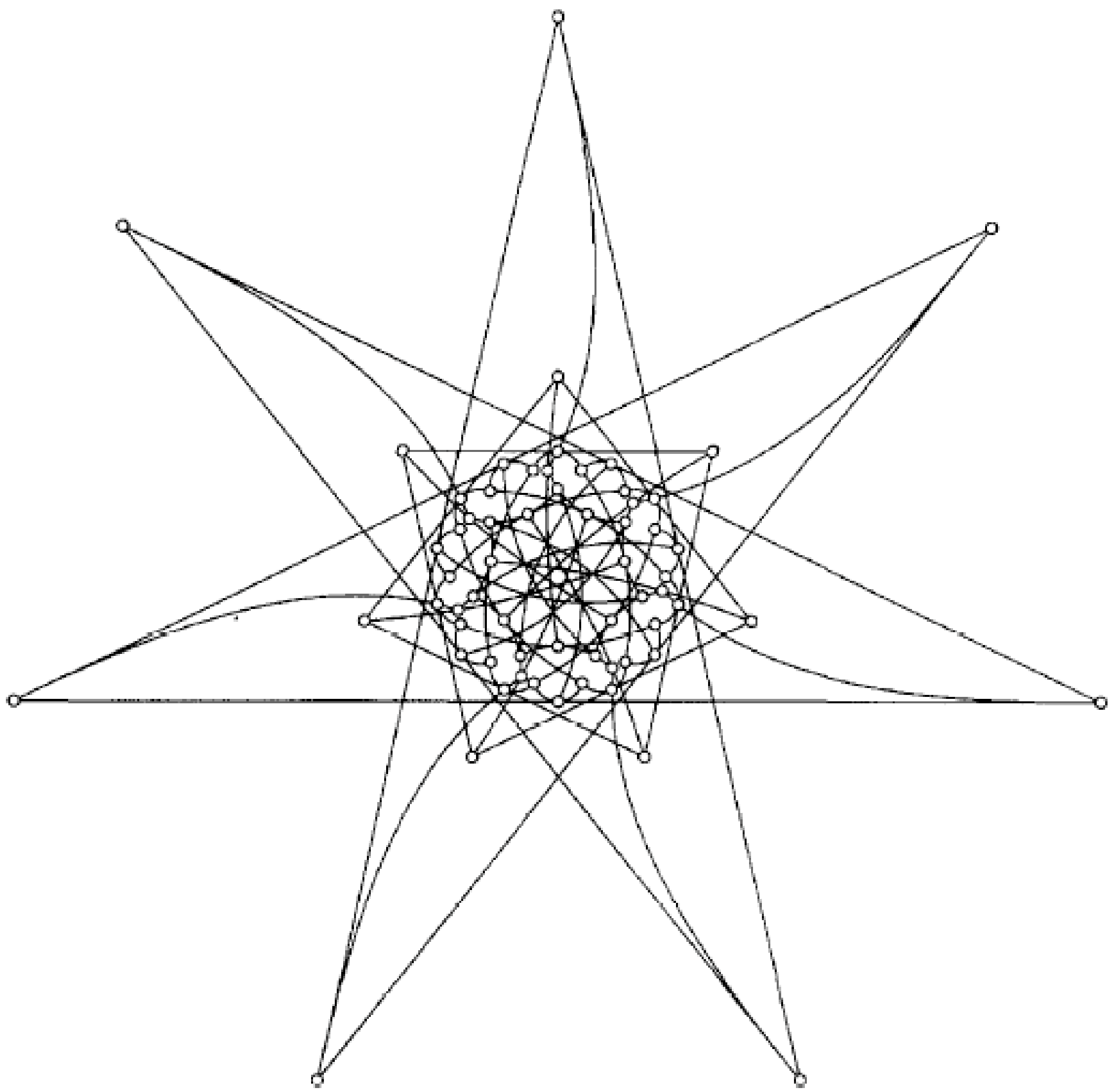


Figure 5.4.12. One of the "generalized hexagons of order (2,2)" shown by Schroth [S9]. Its points and arcs are a graphic rendition of a (63_3) configuration.

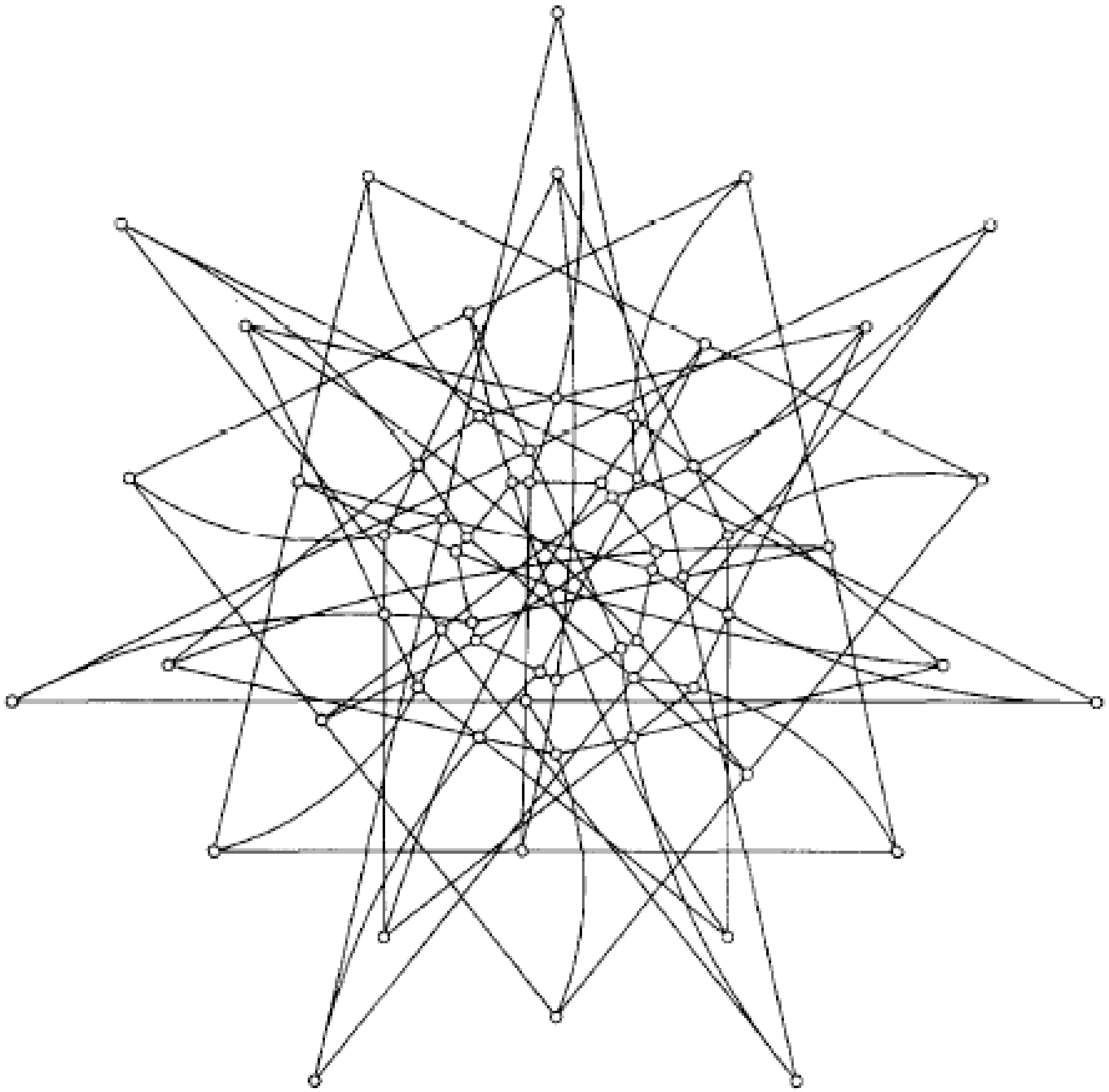


Figure 5.4.13. The other (63_3) configuration from Schroth [S9].

The diagrams in these two figures naturally lead to the question whether the two pentalateral-free configurations are geometrically realizable. The affirmative answer was provided by M. Boben and T. Pisanski, soon after the publication of [S9], but never published. With their permission, one of the two geometric pentalateral-free configurations (63_4) is reproduced in Figure 5.4.14, by a diagram they supplied. In Figure 5.4.15 we show the reduced Levi diagram of this configuration, as kindly provided by Boben and Pisanski.

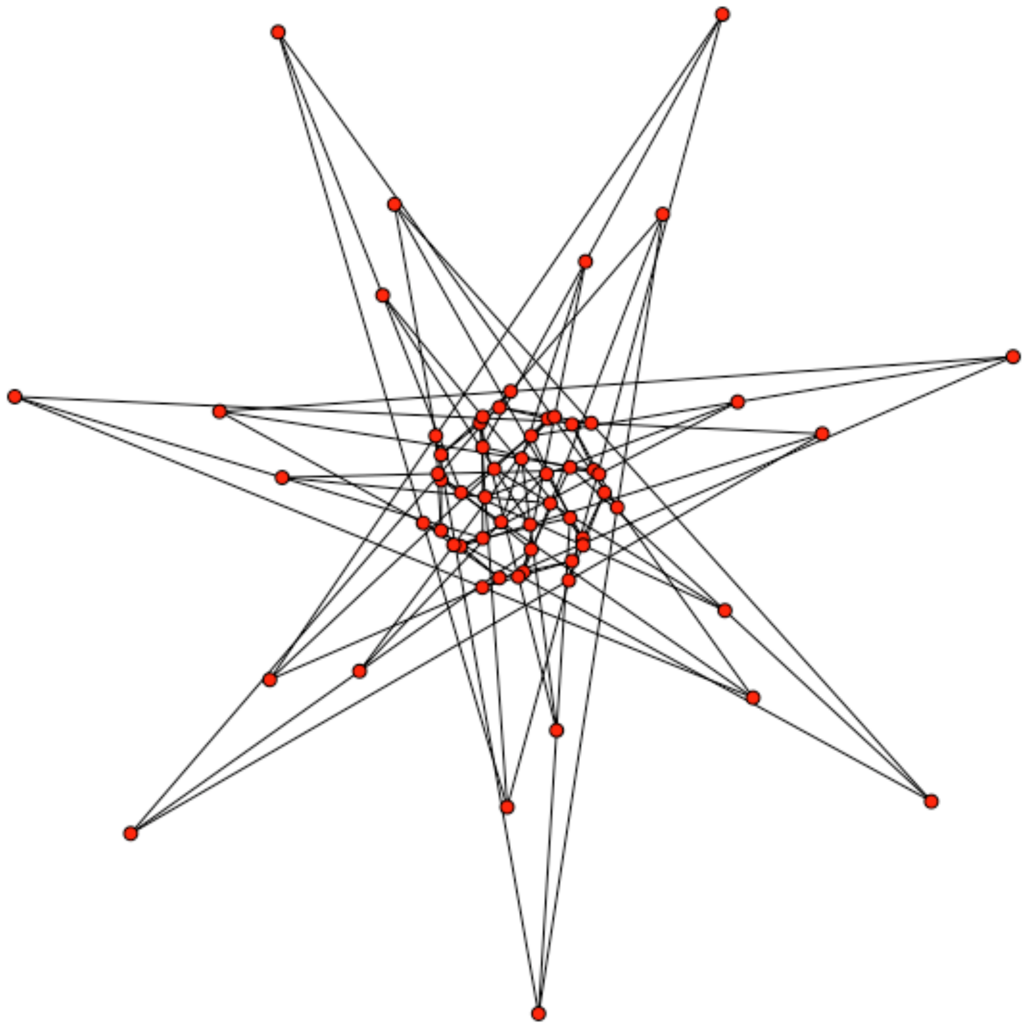


Figure 5.4.14. A geometric realization of a pentalateral-free configuration (63_3). (Courtesy of M. Boben and T. Pisanski, from unpublished work.)

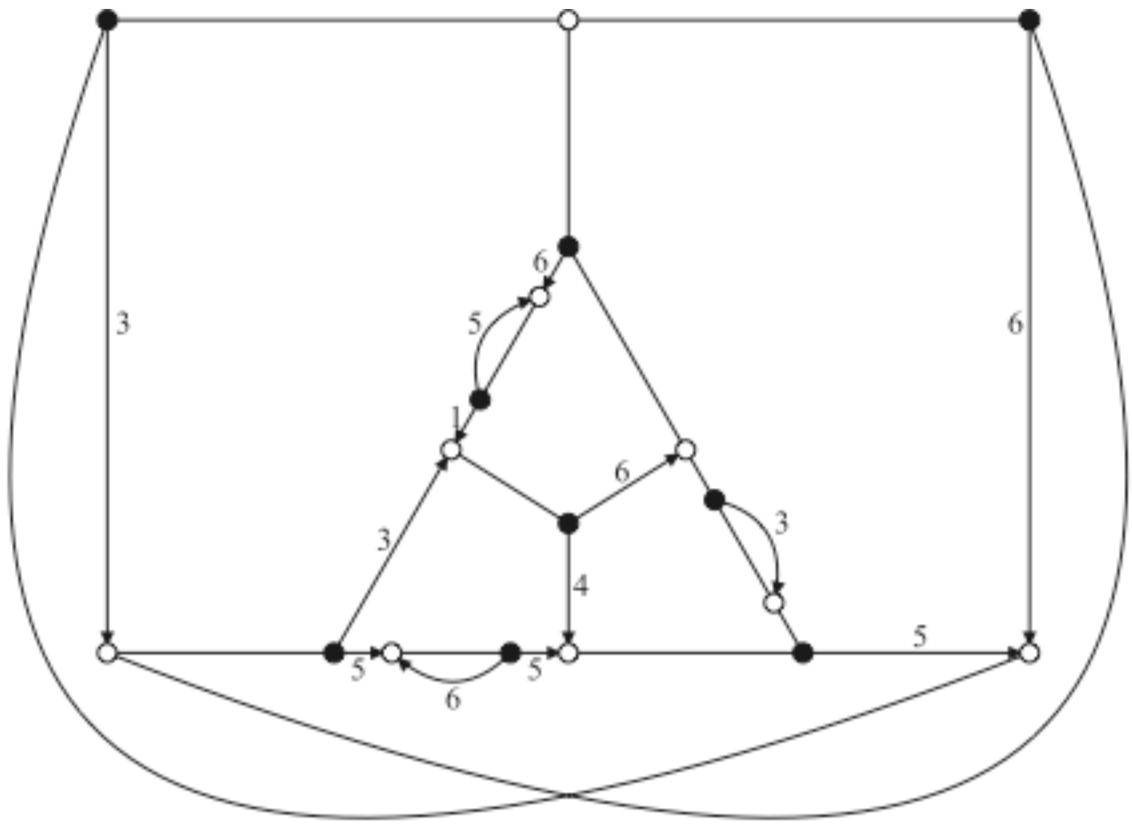


Figure 5.4.15. The reduced Levi graph (in their notation) of the pentalateral-free configuration in Figure 5.4.14. (Courtesy of M. Boben and T. Pisanski, from unpublished work.)

The Boben and Pisanski construction of the pentalateral-free (63_3) configuration is a piece of supporting evidence for Conjecture 2.6.1, according to which all 3-connected combinatorial 3-configurations can be realized by points and (straight) lines. (The 3-connectedness of the 12-cage can be directly established, but it also follows from the more general result of Fu *et al.* [F3] that all (3,g)-cages are 3-connected, or the more general result of Daven and Rodger [D4] that for $k \geq 3$ all (k,g)-cages are 3-connected; there is a conjecture in [F3] that all (k,g)-cages are k-connected.)

* * * * * *

Turning next to the case of trilateral-free 4-configurations, there is much less information available. A trilateral-free *combinatorial* configuration (40_4) was found recently by Hendrik van Maldeghem. Van Maldeghem's example attains the bound of Theorem 5.4.2 for $k = 4$; hence it corresponds to a (4,6)-cage. The construction of this example is described by Bokowski in [B21, pp. 263–265], where an incidence matrix is also shown (in two forms). However, no information seems available concerning the possibility of realizing this configuration geometrically, or even just topologically.

As already mentioned, the configuration LC(4) provides an example of a trilateral-free geometric configuration (n_4) with $n = 4^4 = 256$. Smaller trilateral-free configurations (120_4) are the astral configurations 60#(22,21,2,9) and 60#(27,26,3,14) shown in Figures 5.4.16 and 5.4.17, and their duals. By using the (3m+) and (DU-1) constructions described in Section 3.3, from the (120_4) configurations we can construct infinite families of geometric trilateral-free configurations.

However, much a better example of a trilateral-free 4-configuration is a (60_4) found very recently by M. Boben. It and its polar are shown in Figure 5.4.18.

The procedures analogous to the one describe earlier, applied to the configurations LC(k) for $k \geq 5$, show that in all these cases there are infinite families of trilateral-free geometric configurations. Unfortunately, they are all far too large for intelligible graphics ...

It is not known whether for $k \geq 5$ there exist trilateral-free *combinatorial* configurations (n_k) with $n = k(k-1)^2 + k$.

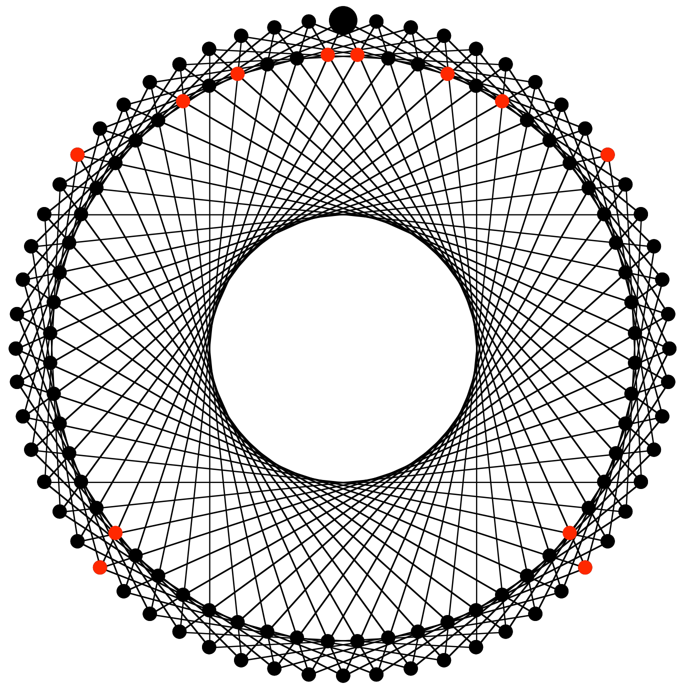


Figure 5.4.16. A trilateral-free geometric configuration (120_4). It is a sporadic astral configuration, with symbol 60#(22,21,2,9).

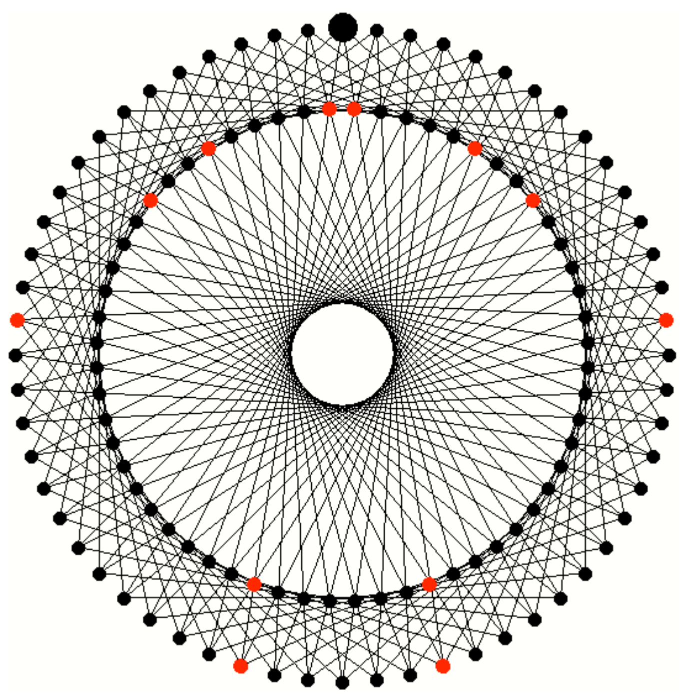


Figure 5.4.17. 60#(27,26,3,14)

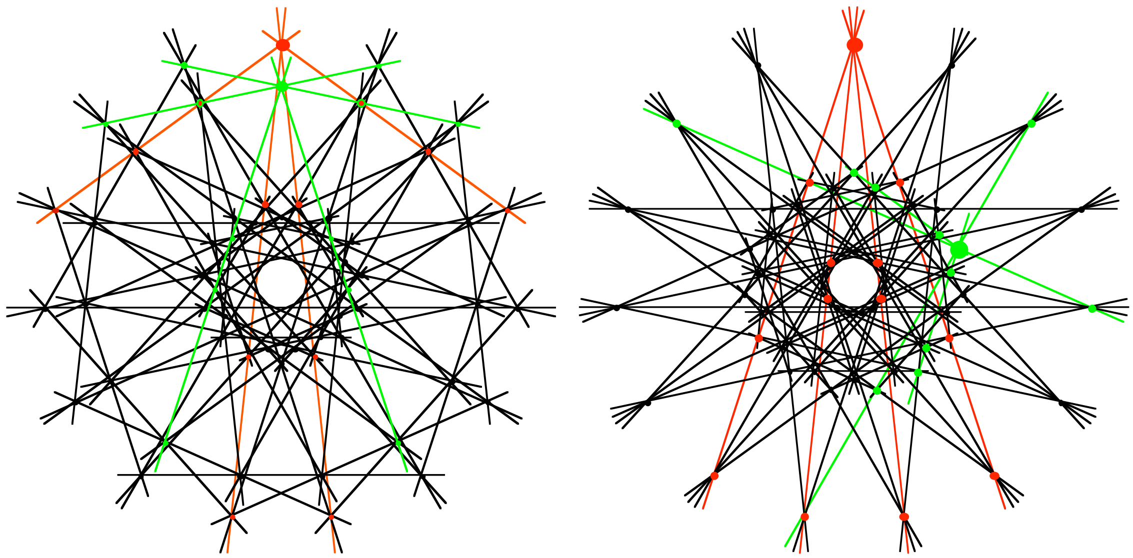


Figure 5.4.18. The trilateral-free (60_4) 4-astral configuration 15#(1,3;7,6;4,3;2,6) found by M. Boben, and its polar. (Courtesy of M. Boben.)

Exercises and problems 5.4

1. Find other astral families of quadrilateral-free 3-configurations.

2. Is there a quadrilateral-free configuration (n_3) for every $n \geq 35$? Or for all but a very small number of values of n?

3. Decide whether it is possible for a 3-configuration to have no quadrilaterals but contain some trilaterals.