5.3 MULTILATERAL DECOMPOSITIONS

In 1828 Moebius [M20] pointed out that it is obvious that two trilaterals cannot be mutually inscribed/circumscribed, and proved the impossibility of two quadrilaterals be in such mutual relationship; he dealt here with the real Euclidean plane. (This seems to be the first paper dealing with this topic.) Moebius concludes his paper by saying: "I have not extended this investigation to multilaterals with more sides." [My translation] Therefore the statement in the Wikipedia [W4] that "Möbius (1828) asked whether there exists a pair of polygons with *p* sides each, having the property that the vertices of one polygon lie on the lines through the edges of the other polygon, and vice versa" has to be taken with a grain of salt. In fact, the answer to the question is affirmative, even if the author is lost in history. The first mention of three mutually inscribed/circumscribed trilaterals seems to be in Graves [G6*], where the Pappus configuration (9_3) is shown to have that property. Moreover, Graves shows that the Desargues configuration (10_3) can be presented as a pair of mutually inscribed/circumscribed pentalaterals. He illustrated this in [G6*] by a diagram (reproduced here as Figure 5.3.1) in which one of the pentalaterals in rendered in color (this was in 1839!). Without diagrams, Cayley [C2*] describes this and several other examples of mutually inscribed/circumscribed families of multilaterals.

For any multilateral decomposition of a 3-configuration by a family F of p-laterals P_i, $1 \le i \le r$, we shall say that it is an **inscribed/circumscribed family** (or decomposition)

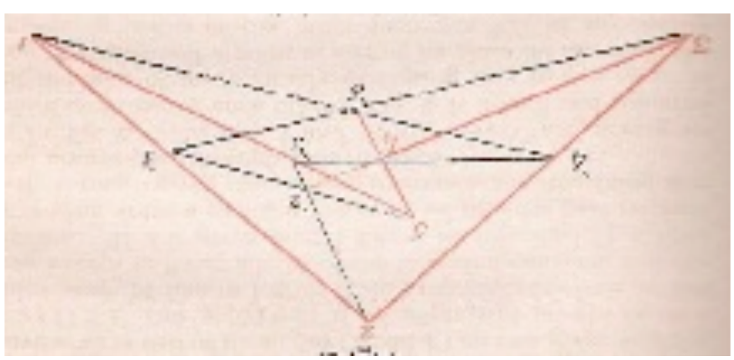


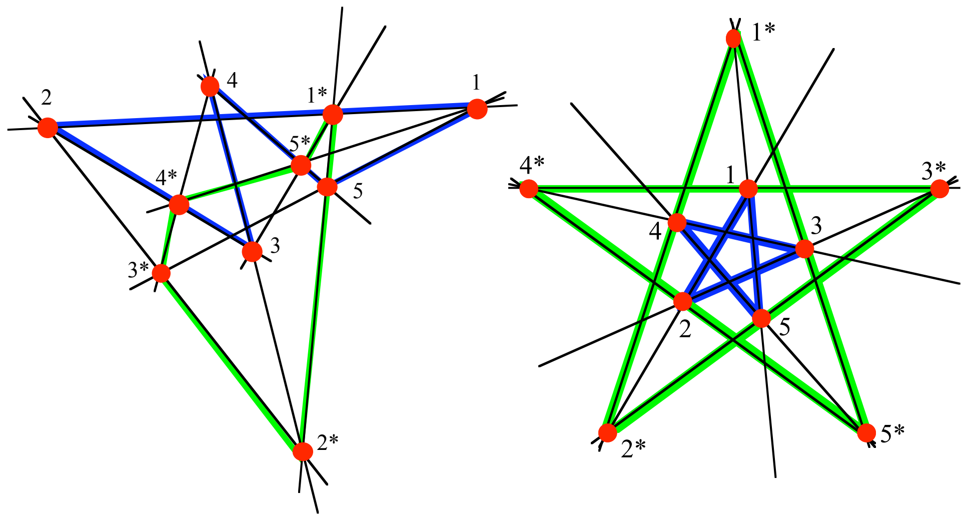
Figure 5.3.1. The Desargues configurations (10_3) presented as a pair (black, and color) of mutually inscribed/circumscribed pentalaterals. From Graves 1839 paper [G2*].

provided every line of P_i contain a vertex of P_{i+1}, for all i (subscripts taken mod r). This is illustrated (for r = 2) by the two examples in Figure 5.3.2. (The first is clearly the same as the Graves example in Figure 5.3.1.) As we discussed previously, Hamiltonian multilaterals of 3-configurations fit this description for r = 1. The astral 3-configurations considered in Section 2.7 show that for r = 2 there exist inscribed/circumscribed families of n-laterals for all $n \geq 5$. The case r = 2, n = 5 is illustrated in Figure 5.3.2(b). The k-astral 3-configurations discussed in Section 2.10 show that for every r, there exist inscribed/circumscribed families of n-laterals for all $n \geq 3$. However, in different forms, these results go much farther back.

The Desargues configuration (which we denoted $(10_3)_1$ in Section 2.2) can also illustrate a refinement of the definition. We say that a family F is an **orderly** inscribed/circumscribed family if consecutive vertices of P_{i+1} belong to consecutive lines of P_i. Although the example in Figure 5.3.2(a) shows a pair of inscribed/circumscribed pentalaterals, this family is not orderly — in contrast to the example in Figure 5.3.2(b). In fact, it can be proved that the Desargues configuration does not admit any orderly pair of pentalaterals. It is worth stressing (as Graves [G2*] did long ago) that the two pentalaterals in the Desargues configuration are combinatorially in a very symmetric reciprocal relationship although this finds no reflection in the geometric rendition.

Among other early publications are the papers [S1] and [S2] by Schoenflies. He was led to this topic by his investigation of the density of trilaterals which we shall discuss in Section 5.5. In these papers Schoenflies formally introduced the families of mutually inscribed/circumscribed multilaterals, and provided examples. (Minor errors of [S2] are corrected in [S3]. More serious shortcomings are pointed out by Steinitz in [S19, p. 488], [S20, p. 307].) Additional related investigations by Schoenflies are reported in [S3] and [S5]. Brunel [B30] considers the topic as well.

Other examples of orderly inscribed/circumscribed families are provided in Figure 5.3.3. A variety of examples is shown in [D10]. These can be generalized to all n-laterals with $n \geq 3$, and all $r \geq 3$.



(a) (b)

Figure 5.3.2. The configurations $(10_3)_1$ (in the labeling of Section 2.2, shown in (a)) and $(10_3)_{10}$ can both be interpreted as pairs of inscribed/circumscribed pentalaterals. However, the one in (b) has an orderly pair or inscribed/circumscribed pentalaterals: The line determined by the points i and i+1 of the red pentalateral contains the point i+1 of the green one. The Desargues configuration $(10_3)_1$ does not have such an orderly pair.

Like many other aspect of the theory of configurations, families of inscribed/circumscribed multilaterals have found a home of sorts in the theory of set configurations and more general combinatorial incidence systems. Recent publications one may wish to consult for these developments include [P2], [P9], [L6], and especially [V2]; additional references may be obtained through these, as well as from other publications of the authors. However, despite the language used in the publications of this trend (including words such as points, lines, polygons, collinearity, Pappus and Desargues configurations, and others), in most cases the meaning is completely divorced from the accepted geometric interpretation of the terms used. In some of the publications there are diagrams, but they are only schematic tools, not configurations in the geometric (or even topological) sense.

As far as I am aware, there has been no consideration given to any concepts analogous to inscribed/circumscribed families of multilaterals or multilateral decompositions in k-configurations, for k ≥ 4. In fact, it is easy to come up with several meaningful interpretations. In Figure 5.3.4 we show the possibly simplest of these. A family of multilaterals in a 4-configuration is an inscribed/circumscribed family if each line of a multilateral is incident with two vertices of another multilateral in a cyclical arrangement of the multilaterals. This is illustrated in Figure 5.3.4 in case of a 3-astral configuration (21_4); in both versions the red heptalateral is inscribed in the blue one, which is inscribed in the green one, which is inscribed in the original red heptalateral.

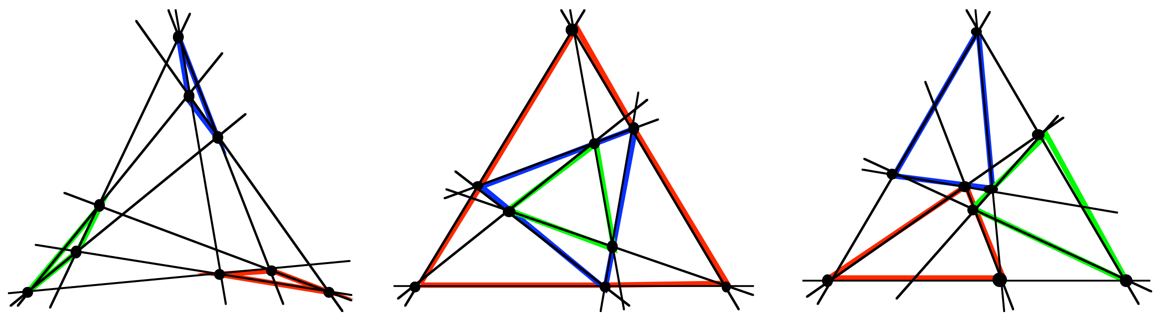


Figure 5.3.3. Examples of orderly inscribed/circumscribed families with r = 3 in rotationally symmetric realizations of the three configurations (9_3).

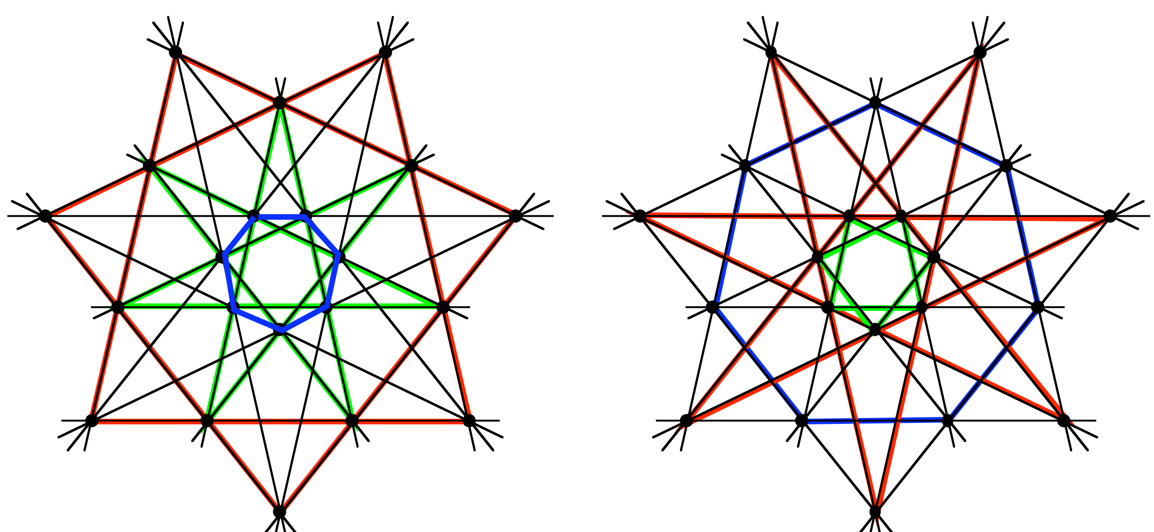


Figure 5.3.4. Two families of three mutually cyclically inscribed heptalaterals, that form a multilateral decomposition of the 3-astral configuration (21_4).

A different interpretation is illustrated in Figure 5.3.5. The multilateral decomposition of the astral configuration (24_4) consists of eight mutually inscribed/circumscribed trilaterals in a more complicated way. Each line contains points of three distinct trilaterals, and each point in incident with lines of three distinct trilaterals.

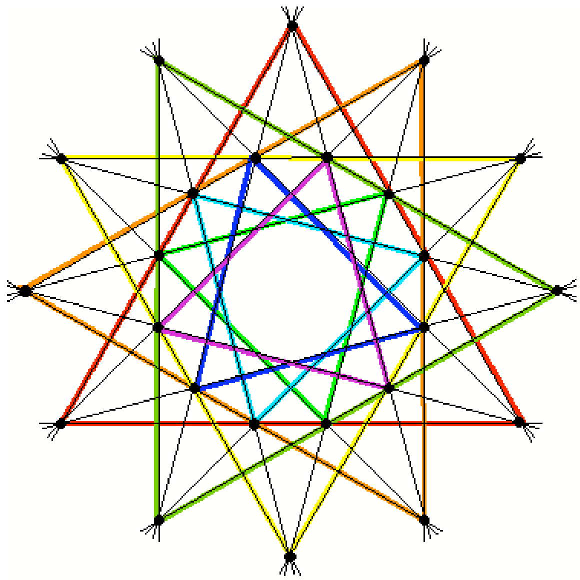


Figure 5.3.5. An illustration of the alternative interpretation of inscribed/circumscribed trilaterals in the astral configuration (21_4).

Exercises and problems 5.3

1. Prove that there is no *orderly* pair of pentalaterals in the Desargues configuration $(10_3)_1$.

2. Show that the cyclic configuration $\mathscr{C}_3(n)$ (see Section 2.1) contains an orderly family of three inscribed/circumscribed m-laterals whenever n = 3m. Does it contain a family of m trilaterals?

3. Investigate the general cyclic configurations $\mathscr{C}_3(n,1,k)$ (see Exercise 2 of Section 2.1) for the presence of orderly families of inscribed/circumscribed multilaterals.

4. Prove the result of Schoenflies [S5]: None of the three (9_3) configurations can be obtained by starting with nine points in general position in Euclidean 3-space, generating some of the lines and planes they determine, and intersecting these by a plane.

5. Can each of the (10_3) configurations be presented as an inscribed/circumscribed family of two pentalaterals?

6. Consider the astral configuration (60_4) shown in Figure 5.3.6. Find inscribed/circumscribed multilateral decomposition of this configuration into (i) twelve pentalaterals; (ii) four 15-laterals.

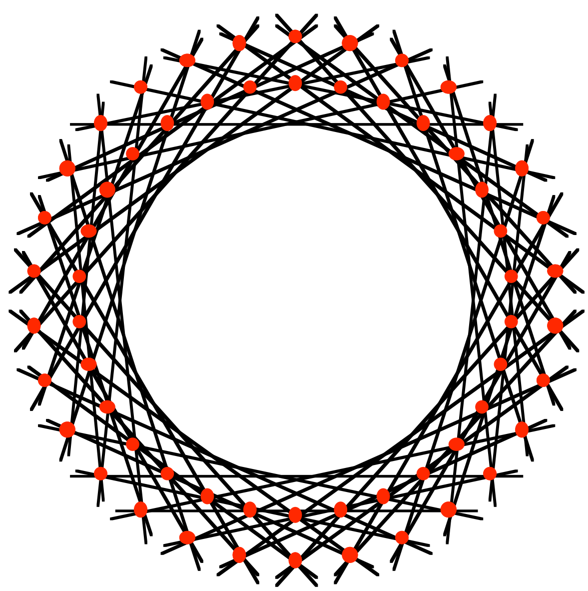


Figure 5.3.6. The astral configuration (60_4) used in Exercise 6.