5.2 HAMILTONIAN MULTILATERALS

As another example of the utility of Levi graphs, we consider in this section and the next some of the known results on multilaterals in configurations. We start by expanding the appropriate definitions from Section 1.3.

We call **multilateral** (or, if appropriate, **r-lateral**) any sequence of points P_i and lines L_i of a configuration that can be written as $P_0, L_0, P_1, L_1, \dots, P_{r-1}, L_{r-1}, P_r = P_0$, with each L_i incident with P_i and P_{i+1} (all subscripts understood mod r). Thus, an r-lateral in a configuration C corresponds to a (2r)-circuit in the Levi graph L(C). A **multilateral path** satisfies the same conditions except the coincidence of the first and last elements. Instead of 3-lateral we shall say **trilateral**, and analogously **pentalateral**, etc.[1] A **circuit decomposition** of a graph is any family of disjoint simple circuits that together include all vertices of the graph. Clearly, not every graph has a circuit decomposition, but as we have seen in Section 2.5, the Levi graph of every connected k-configuration, $k \geq 2$, has such a decomposition. The corresponding multilaterals of the configuration are said to be a **multilateral decomposition** of the configuration. A multilateral decomposition consisting of a single multilateral is a **Hamiltonian multilateral** of the configuration. In other words, a Hamiltonian multilateral of a configuration passes through all its points and uses all its lines, each precisely once.

The Hamiltonian circuit of the Levi graph in Figure 5.1.1 corresponds to a Hamiltonian multilateral of the configuration. On the other hand, from the Levi graph of the same configuration shown in Figure 5.1.2 we easily see the possibility of a circuit decomposition of the Levi graph into four 6-circuits, hence yielding a decomposition of the configuration (12_3) in Figure 5.1.1 into four trilaterals (that form a cycle of mutually inscribed / circumscribed trilaterals). It is also easy to observe a decomposition of this configuration into three quadrilaterals, such as la8g6b71, 3d10i2c9h, 5f12k4e11j.

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[1] The terms "trilateral", "pentalateral" and others were used by Martinetti [M1] in 1886, but in a slightly different meaning.

As another example, in Figure 5.2.1 we show a Hamiltonian multilateral of a configuration (10_3), as well as a decomposition of the same configuration into two pentalaterals.

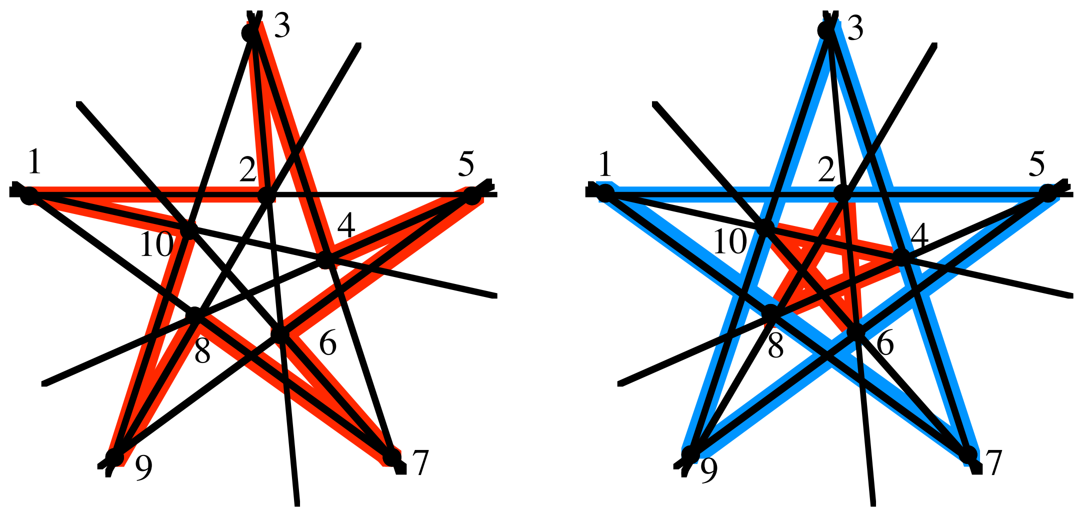

Figure 5.2.1. A configuration (10_3) with one Hamiltonian multilateral, and one decomposition into two pentalaterals that are mutually inscribed/circumscribed.

The remaining part of this section is devoted to a survey of results known about Hamiltonian multilaterals. The great majority of these results deal with 3-configurations.

To begin with, here are some historical notes; in all of these papers the *circuit* terminology has been used, but we present them in our terms. Kantor in 1881 [K4] states *as a theorem* that every (n_3) configuration has a Hamiltonian multilateral. Martinetti [M2] in 1887, Schoenflies [S2] in 1888 and Brunel [B30] in 1895 consider Hamiltonian multilaterals as self-inscribed/circumscribed polygons. Schröter in 1889 [S8] states that he confirmed the existence of Hamiltonian multilaterals in all (10_3) configurations, and Steinitz in 1897 [S18] does the same for all (11_3). Steinitz also observed that connectedness is a necessary condition, a fact not mentioned by earlier writers. He also provided a first example of a connected configuration that does not admit a Hamiltonian multilateral. The smallest configuration that would fit his description is (28_3). In 1990, Gropp [G9] announced that all connected configurations (n_3) with $n \leq 14$ have Hamiltonian multilaterals. The statement (a1) in the paper [K9, p. 128] by Kelmans, to the effect that every 3-valent, 3-connected bipartite graph with at most 30 vertices has a Hamiltonian circuit, implies that all connected (n_3) configurations with $n \leq 15$ have Hamiltonian multilaterals.

The example of Steinitz mentioned above was improved in [D10] by the construction of the configuration (22_3) shown in Figure 5.2.2. It is not known whether every connected configuration (n_3) with $16 \leq n \leq 21$ admits a Hamiltonian multilateral.

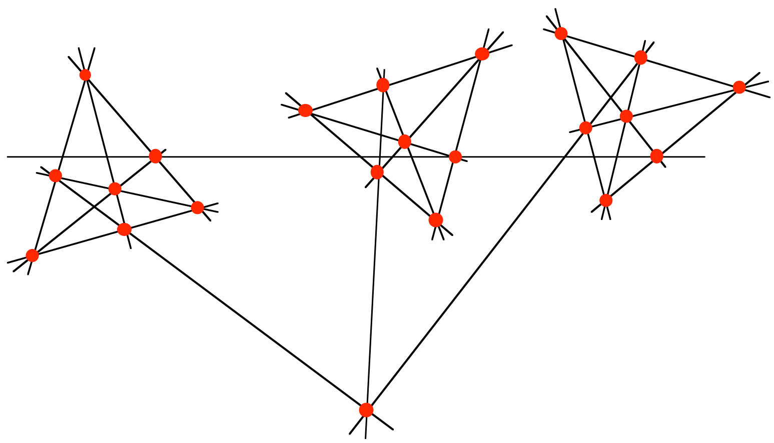

Figure 5.2.2. A (22_3) configuration that does not admit any Hamiltonian multilateral. It is obvious that such configurations (n_3) can be constructed for every $n \geq 22$.

The fact that all connected configuration (n_3) without a Hamiltonian multilateral known at the time were only 2-connected led the author to conjecture in 2002:

Conjecture 5.2.1. All 3-connected (n_3) configurations admit Hamiltonian multilaterals.

However, this conjecture was disproved in [G45]:

Theorem 5.2.1. There exists a 3-connected geometric configuration (25_3) that does not admit a Hamiltonian multilateral.

We shall prove this by a construction, fashioned after the arguments presented in [G45].

Our construction starts with the smaller graph shown in Figure 5.2.3, devised by M. N. Ellingham and J. D. Horton in [E1]. This graph has no Hamiltonian circuit that uses both heavily drawn edges. The *proof* of this assertion is simply a follow-up of a few alternatives — the non-trivial, clever part is the *discovery* of the graph. For the next step we insert two additional vertices in each of the heavily drawn edges of Figure 5.2.3, resulting in the graph shown in Figure 5.2.4. From this the graph of Figure 5.2.5 was constructed by Georges [G1]. It is a non-Hamiltonian, 3-connected, bipartite graph. Again, the proof of non-Hamiltonicity consists of the examination of several possibilities, and showing that neither leads to a Hamiltonian circuit. After the graph is constructed, this is a matter of routine checking, as are the other relevant properties. At this step, as in the earlier, it is the construction of the graph that is ingenious. The main result of the construction that is relevant to the present aim is the fact that the Georges graph has girth 6. Since it is bipartite, it follows that it is the *Levi graph* of a combinatorial configuration (25_3). This is its relevance for the present goal.

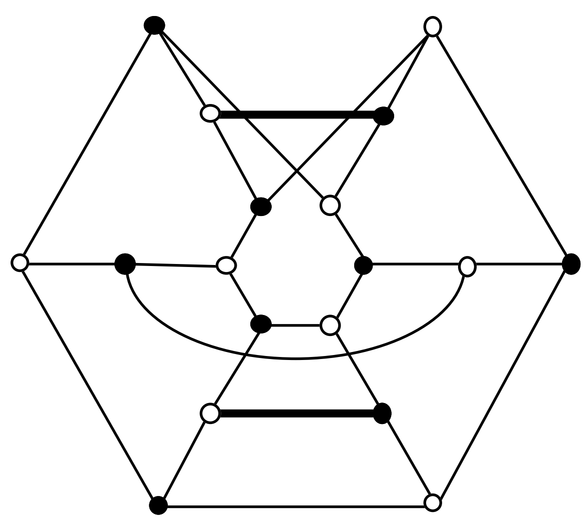

Figure 5.2.3. This bipartite graph found by Ellingham and Horton [E1] does not admit a Hamiltonian circuit that uses both heavily drawn edges.

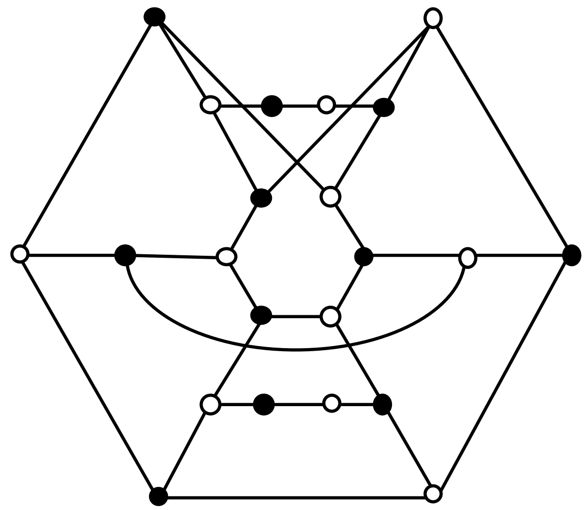

Figure 5.2.4. A modification of the graph in Figure 5.2.3.

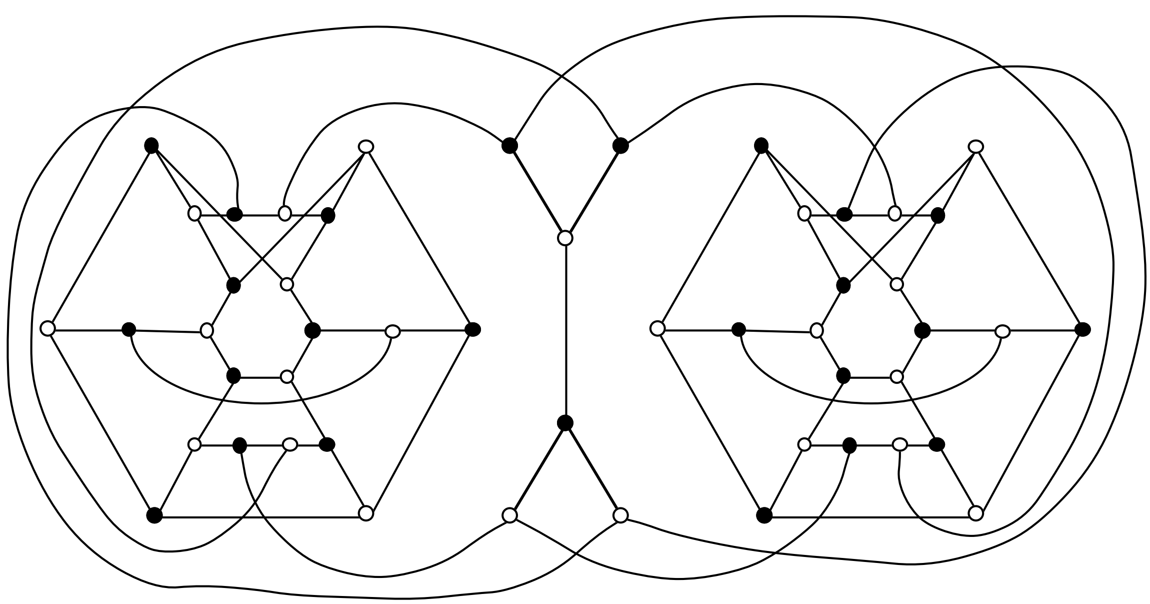

Figure 5.2.5. The Georges graph, resulting from a combination of two copies of the graph in Figure 5.2.4. The graph is bipartite, 3-connected, non-Hamiltonian, and has girth 6.

We shall now apply the method of Steinitz described in Section 2.6 to obtain first a realization of this combinatorial configuration (25_3) as a prefiguration, and then apply a continuity argument to establish the possibility of its realization by a geometric configuration. To begin with we label the Georges graph in a more-or-less random way, as in Figure 5.2.6. This labeling leads to the configuration table, shown in Table 5.2.1. This is turned into an orderly configuration table, shown in Table 5.2.2, as required for the application of Steinitz's construction. (This table was not constructed by following the rather cumbersome Steinitz algorithm, but by a straightforward "greedy" algorithm: Taking the first available choice at each step. It worked very well in the present case.)

Table 5.2.3 shows the Georges configuration with columns (that is, lines) permuted so that a decomposition of the configuration into "multilaterals" becomes obvious. This is accomplished by making the second entry in a column equal to the third entry in the preceding column, the first column being chosen arbitrarily. The exception is the last column of a multilateral, in which the last entry is the same as the second entry of the

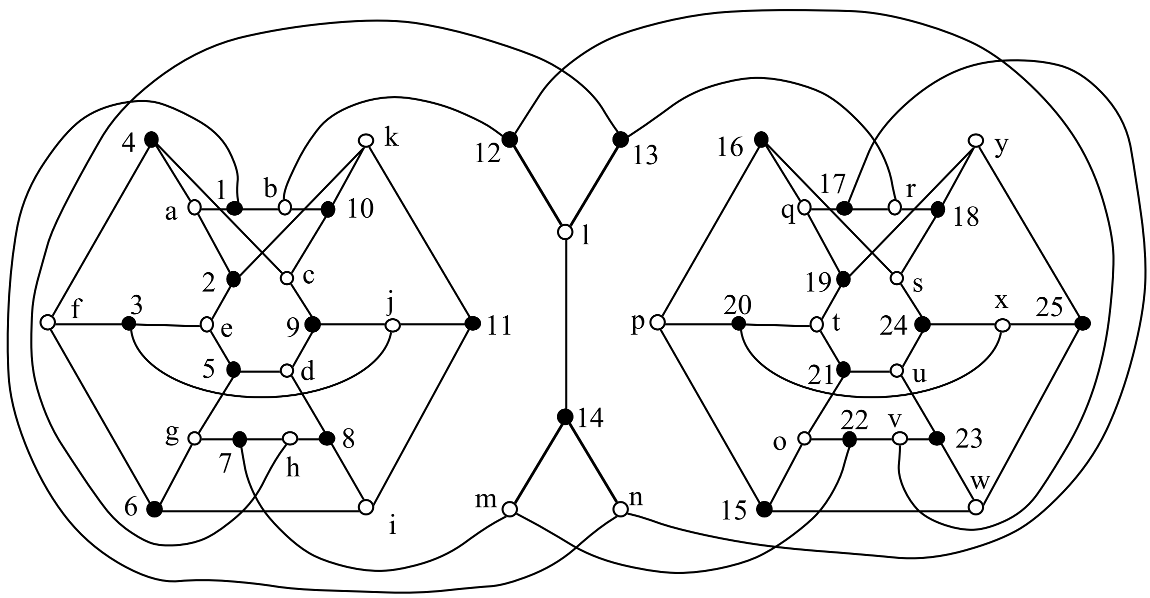

Figure 5.2.6. A labeling of the Georges graph that allows us to interpret it as the Levi graph of a combinatorial configuration. The task is simplified by the fact that there are sufficiently many characters to label all lines.

a	b	c	d	e	f	g	h	i	j	k	l	m
1	1	4	5	2	3	5	7	6	3	2	12	7
2	10	9	8	3	4	6	8	8	9	10	13	14
4	12	10	9	5	6	7	13	11	11	11	14	22

n	o	p	q	r	s	t	u	v	w	x	y
1	15	15	16	13	16	19	21	12	15	20	18
14	21	16	17	17	18	20	23	22	23	24	19
17	22	20	19	18	24	21	24	23	25	25	25

Table 5.2.1. A configuration table of the Georges configuration, as read off Figure 5.2.6.

a	b	c	d	e	f	g	h	i	j	k	l	m
1	10	4	5	2	3	6	7	8	9	11	12	14
2	12	9	8	3	4	5	13	6	11	0	14	7
4	1	10	9	5	6	7	8	11	3	2	13	22

n	o	p	q	r	s	t	u	v	w	x	y
17	15	16	19	13	18	20	21	22	23	24	25
1	22	15	17	18	16	21	24	23	25	20	19
14	21	20	16	17	24	19	23	12	15	25	18

Table 5.2.2. An orderly configuration table for the Georges configuration.

first column. In cases like the present one, where the first multilateral does not exhaust the columns, the first remaining column is used to start a new polygon. In the case of the George configuration, the first is a 22-gon, the second a triangle. Figure 5.2.7 provides an illustration of the two circuits in the Levi graph, that correspond to two multilaterals.

a	f	i	j	e	g	m	o	t	y	r	q	s
1	3	8	9	2	6	14	**15**	20	25	13	19	18
2	4	6	11	3	5	7	22	21	19	18	17	16
4	6	11	3	5	7	22	21	19	18	17	16	24

u	v	b	n	l	h	d	c	k	p	x	w
21	22	10	17	12	7	5	4	11	16	24	23
24	23	12	1	14	13	8	9	10	**15**	20	25
23	12	1	14	13	8	9	10	2	20	25	15

Table 5.2.3. A rearrangement of the columns of the Georges configuration used to show a decomposition into multilaterals. The boxed labels are explained in the text.

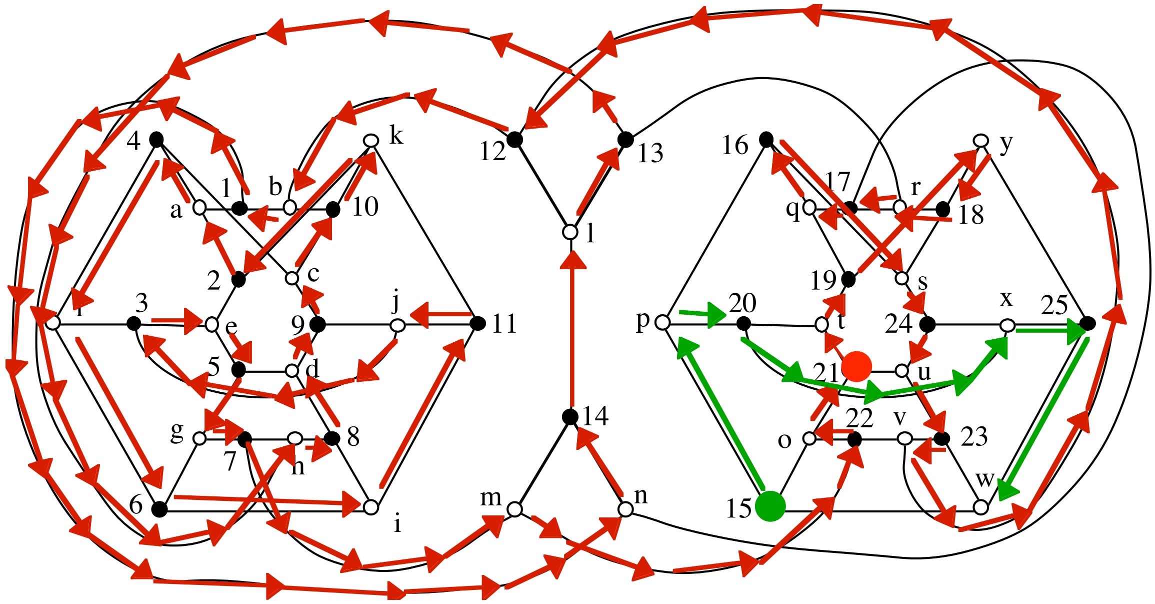

Figure 5.2.7. The two multilaterals from Table 5.2.3 that lead to a multilateral decomposition of the Georges graph.

As a last step in the Steinitz algorithm before the geometric construction, we permute the columns (lines) once more. Some vertex of a column of the last multilateral (the second in this case) must be a vertex that appeared in a previous multilateral, since otherwise the configuration would not be connected. (In the present case, we chose the vertex labeled 15.) We place that column as the first of the last multilateral, and place as the last column of the previous multilateral its column that contains the same vertex. The other columns in both multilaterals are permuted accordingly, so as to preserve the multilaterals present. The configuration table obtained in this step (resulting from the choice of 15 as the special vertex) in shown in Table 5.2.4.

The geometric realization now proceeds very simply. It can be followed in Figure 5.2.8 which was obtained using "Geometer's Sketchpad"™ and modified by ClarisDraw™. The idea follows the explanations given in Section 2.6: Choose the vertices as arbitrary points, except when constrained to lie on one or two previously

t	y	r	q	s	u	v	b	n	l	h	d	c
20	25	13	19	18	21	22	10	17	12	7	5	4
21	19	18	17	16	24	23	12	1	14	13	8	9
19	18	17	16	24	23	12	1	14	13	8	9	10

k	a	f	i	j	e	g	m	o	p	x	w
11	1	3	8	9	2	6	14	15	16	24	23
10	2	4	6	11	3	5	7	22	15	20	25
2	4	6	11	3	5	7	22	21	20	25	15

Table 5.2.4. A rearrangement of the columns of Table 5.2.3, needed for the application of Steinitz's geometric construction.

constructed lines. In this example we start with arbitrary points 21 and 19. Points 18 and 17 are also chosen freely, but the choice of 16 has to be on the line q through the previously determined points 19 and 17, and then 24 must be on the line s through 16 and 18; similarly, the point 23 must be on the line u = [21, 24]. The points 12 and 1 can be chosen freely, but 14 must be on the line n = [1, 17], and 13 on the intersection point of the lines r = [17, 18] and l = [12, 14]. Next, points 8, 9, 10, 2 are free, but 4 must be on the line a = [1, 2]. The point 6 is free, while 11 is the intersection point of lines i = [6, 8] and k = [10, 2], and point 3 is the intersection point of the lines f = [4, 6] and j = [9, 11]. Similarly, 5 is the intersection point of d = [8, 9] and e = [2, 3], and 7 is the intersection point of h = [8, 13] and g = [5, 6], while 22 is the intersection point of lines v = [23, 12] and m = [7, 14]. This completes the construction of the first multilateral. To start with the next (the trilateral), we select 15 on the line o = [22, 21], then 20 as the intersection point of t = [21, 19] and p = [16,15]. The only remaining problem is the selection of point 25, which should be at the intersection of **three lines**, namely y = [18, 19], x = [24, 20], and w = [23, 15]. It is to be expected that three lines do not have a common point. This is quite general, and it is the final solution given by Steinitz, with the selections made: the last line may need to be taken as a circle (or a parabola).

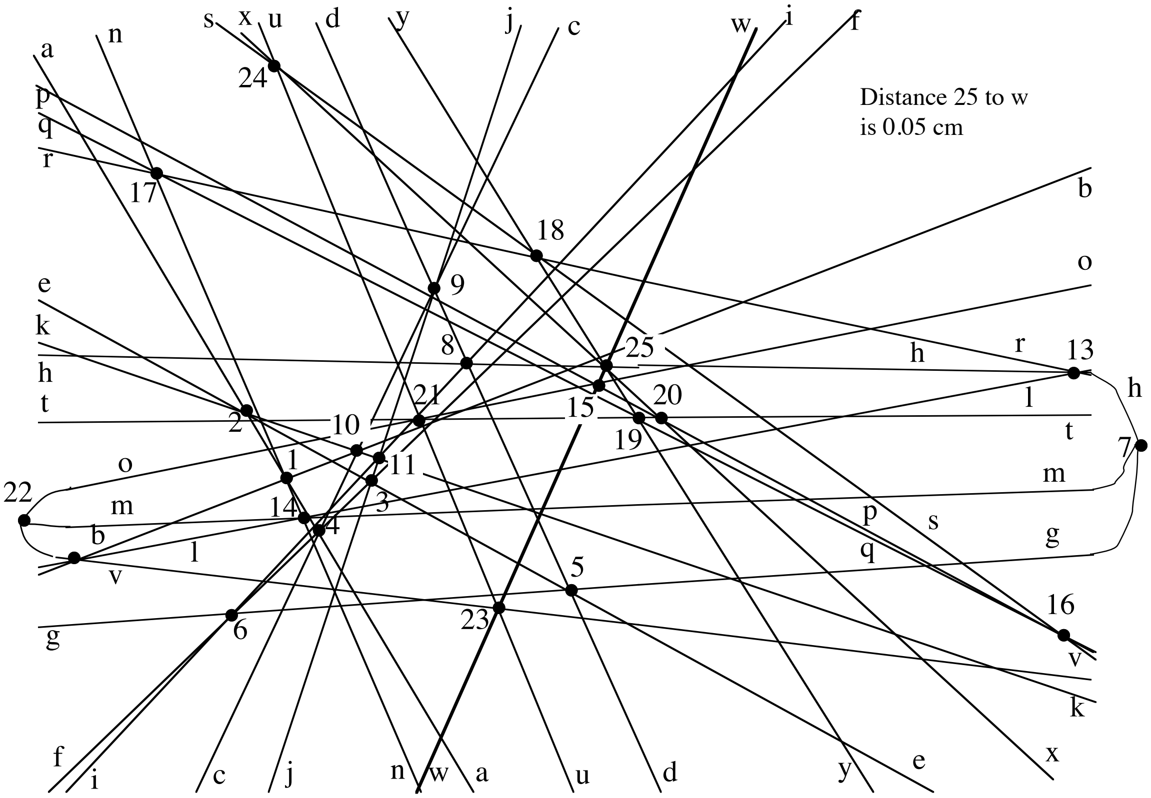

Figure 5.2.8. The construction of a 3-connected geometric configuration that does not admit a Hamiltonian multilateral. In the actual construction shown (using "Geometer's Sketchpad"™) the final incidence was missed by about 0.5 mm due to the discreteness of the underlying software. The curves at left and right are meant to indicate that the triplets of lines do meet at the points they are supposed to – but too far for inclusion in an intelligible version of the diagram.

In fact in this case — just as for many other configurations — by judicious choices of the free parameters one may find selections in which the point 25 is on a certain side of the line w, as well as selections where it is on the other side. By continuity, this implies that there is a position of incidence. The final conclusion, therefore, is that the Georges configuration can be realized geometrically, by points and straight lines. Hence it is a 3-connected non-Hamiltonian geometric configuration (25_3). ¨

The result of Theorem 5.2.1 can be extended to geometric configurations of almost all sizes:

Theorem 5.2.2. For each $n \geq 33$ there exist 3-connected geometric configurations (n_3) that do not admit Hamiltonian multilaterals.

Proof. Selecting any of the vertices of the (25_3) configuration of Theorem 5.2.1, (see Figure 5.2.9(a), where the gray oval stands for the rest of the Georges configuration (25_3), denoted here by G. We delete that vertex and replace it by the three intersection points of the lines incident with the deleted vertex and a new line L (see Figure 5.2.9(b)). We denote this truncated version of G by G'. Taking now the similarly truncated version H' of any configuration (p_3), we make its three lines incident with the three points on L (Figure 5.2.9(c)). Since both (7_3) and (8_3) have truncated versions realizable by straight lines, the above results in a configuration with $n \geq 24 + 3 + 6 = 25 + 1 + 7$ points and lines. This configuration is clearly 3-connected but it cannot be Hamiltonian.

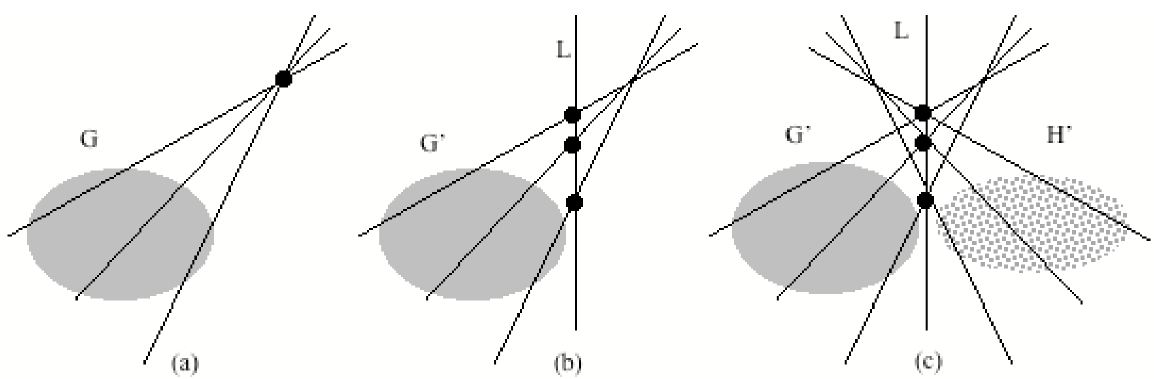

Figure 5.2.9. The construction establishing Theorem 5.2.2.

Indeed, any Hamiltonian multilateral would have to use L and two of its points, one of which must be on a line toward G', the other towards H'. The third point on L must be on one line from G' and another from H'. But then there would be a multilateral in G' using L, and therefore – by identifying the three points of G' that are on L – we would get a Hamiltonian multilateral of G. ¨

A few remarks on the background of Theorem 5.2.1. I learned of Georges' paper [G1] and his non-Hamiltonian graph from Gropp [G9]. But Gropp but makes no connection between the Georges graph and configurations, and in particular, does not observe the fact that the Georges graph has girth 6 and is therefore the Levi graph of a configuration (combinatorial at least). It should also be noted that in [G1], the rendition of the Ellingham-Horton graph, shown above in Figure 5.2.3, is missing one of the special edges.

Gropp [G9], [G19] also mentions that a result similar to Georges' has been found earlier by Kelmans [K8]. This may well be the case; however, I find the presentation in [K8] (both in the Russian original, and in the translation) too confusing to be able to decide whether the graph he constructs has girth 6. Like Georges, Kelmans does not mention girth, or configurations. The claim in [G9] that Kelmans' 50-vertex graph is the same as Georges seems quite unjustified. The expanded version of Kelman's paper (see [K9]) remains inscrutable to the present author. Moreover, there is no mention in [K9] of girth 6, of Levi graphs, or of any type of configurations

* * * * *

A natural question that can be asked in view of Theorems 5.2.1 and 5.2.2 is: Does every geometric 4-configuration admit a Hamiltonian multilateral?

As we shall now show, there is a negative partial answer:

Theorem 5.2.3. There exist 2-connected geometric 4-configurations that do not admit any Hamiltonian multilaterals.

Proof. We provide a conceptually simple construction that, unfortunately, leads to such configurations but of relatively large sizes. We use configurations (or, more precisely, fragments of configurations) such as the halves of the configurations in Figure 5.1.5. With the same conventions, we can assume that we start with an arbitrary (n_4), for example, the C(4) used in Section 5.1. (We note that the configuration (18_4) in Figure 3.3.4 could be used, but at the cost of some detailed arguments about cross-ratios.) As indicated in Figure 5.2.10, we start with eight copies, delete from each a line and take suitable projective transforms so that the points that are on three lines each are aligned as shown in Figure 5.2.10, using an additional line L and an additional point P. Since each of L and P can be used only once in any multilateral, there is no possibility for involvement of more that two of the groups of four starting configurations. Hence this non-Hamiltonian configuration is a (4087_4); if we start with (18_4) the result is a still formidable (289_4). A somewhat smaller example can be found by replacing, in each set of four, one configuration by a single point; this would lead to a non-Hamiltonian (3077_4) if using C(4), or (221_4) if using (18_4).

It is easy to see that a similar construction can be applied to k-configurations for all k ≥ 5, starting with any single geometric configuration of that kind; for example, C(k) can be used. Obviously, the configurations obtained will be monstrously large.

An unsolved problem is the question whether there are non-Hamiltonian geometric 4-configurations that are 3- or 4-connected.

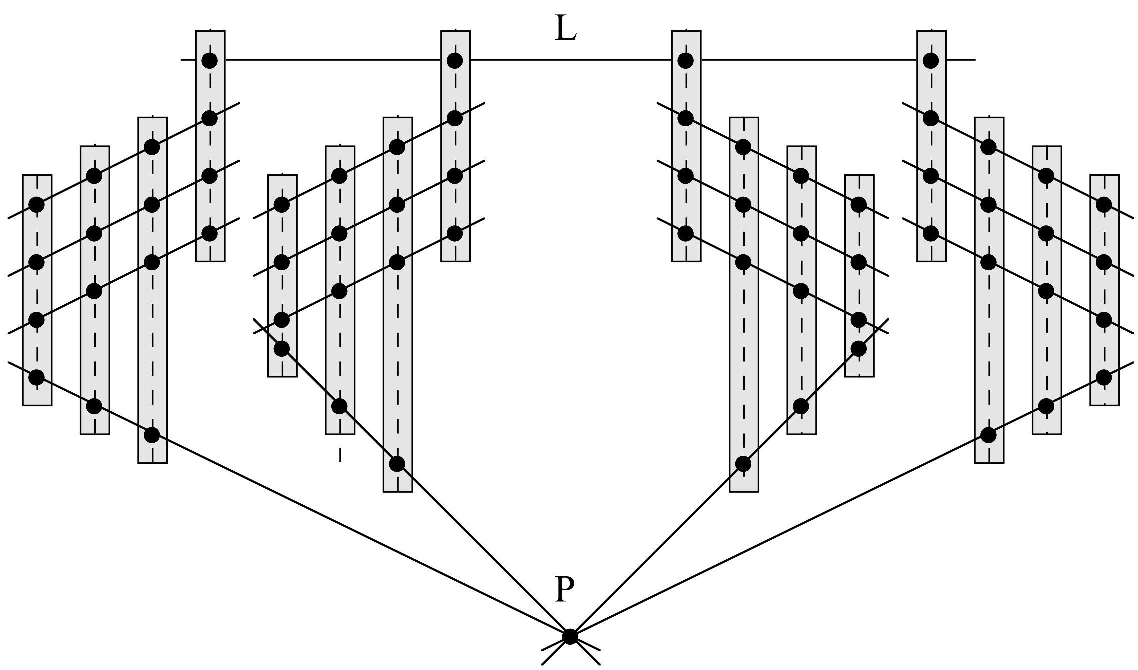

Figure 5.2.10. The scheme of the construction of a 2-connected geometric configuration (289_4) that does not admit any Hamiltonian multilaterals. Each gray rectangle represents an (18_4) configuration from which one line has been omitted.

A different direction in the study of Hamiltonian multilaterals in configurations is opened by the following generalization:

Definition. A [q,k]-configuration has a Hamiltonian multilateral if and only if it has a multilateral M such that:

(i) Every element of one of the two kinds (points or lines) is contained in the multilateral M;

(ii) Every element (of both kinds) is incident at most once with the multilateral M.

Obviously, this definition reduces to the standard one in case of balanced configurations (q = k).

Except for a few examples of type [3, 4] with Hamiltonian multilaterals, there seems to be no information available on this topic.

It also seems that the concept has not been studied in the context of bipartite graphs — to which it obviously applies.

* * * * * *

A different type of questions and results arises if we inquire about Hamiltonicity of some restricted families of configurations. For example, if we consider astral 3-configurations, one may inquire about Hamiltonian multilaterals that have the same symmetries as the configuration; we shall call them **symmetric Hamiltonian multilaterals**. In Figure 5.2.11 we show an example of a symmetric Hamiltonian multilateral in the astral (10_3) configuration. Note that both parts show the same symmetric multilateral — multilaterals are concerned with lines, not segments.

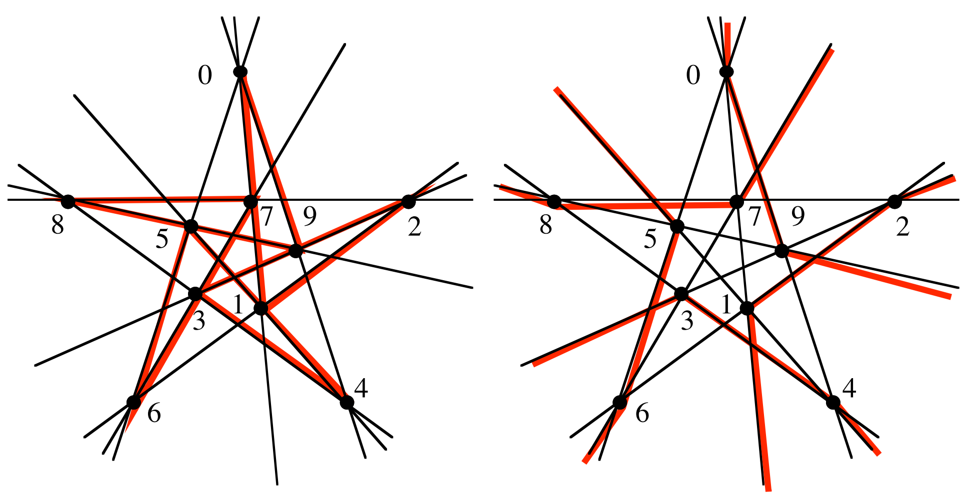

Figure 5.2.11. A symmetric Hamiltonian multilateral in the astral configurations (10_3). Both parts represent the same multilateral.

In Figure 5.2.12 we show four different symmetric Hamiltonian multilaterals on the astral (10_3) configuration. Using the notation for astral 3-configurations we introduced in Section 2.7, and which is illustrated in Figure 5.2.13, we can assert:

Theorem 5.2.4. Cyclic astral configuration m#(b, c; d) may have four types of symmetric Hamiltonian circuits:

$B_0 \to C_0 \to B_d \to ...$

$B_0 \to C_0 \to B_{d-c} \to ...$

$B_0 \to C_{-b} \to B_{d-b} \to ...$

$B_0 \to C_{-b} \to B_{d-b-c} \to ...$

A symmetric Hamiltonian circuit of one of these types exists if and only if m is relatively prime to d, d–c, d–b, or d–b–c, respectively.

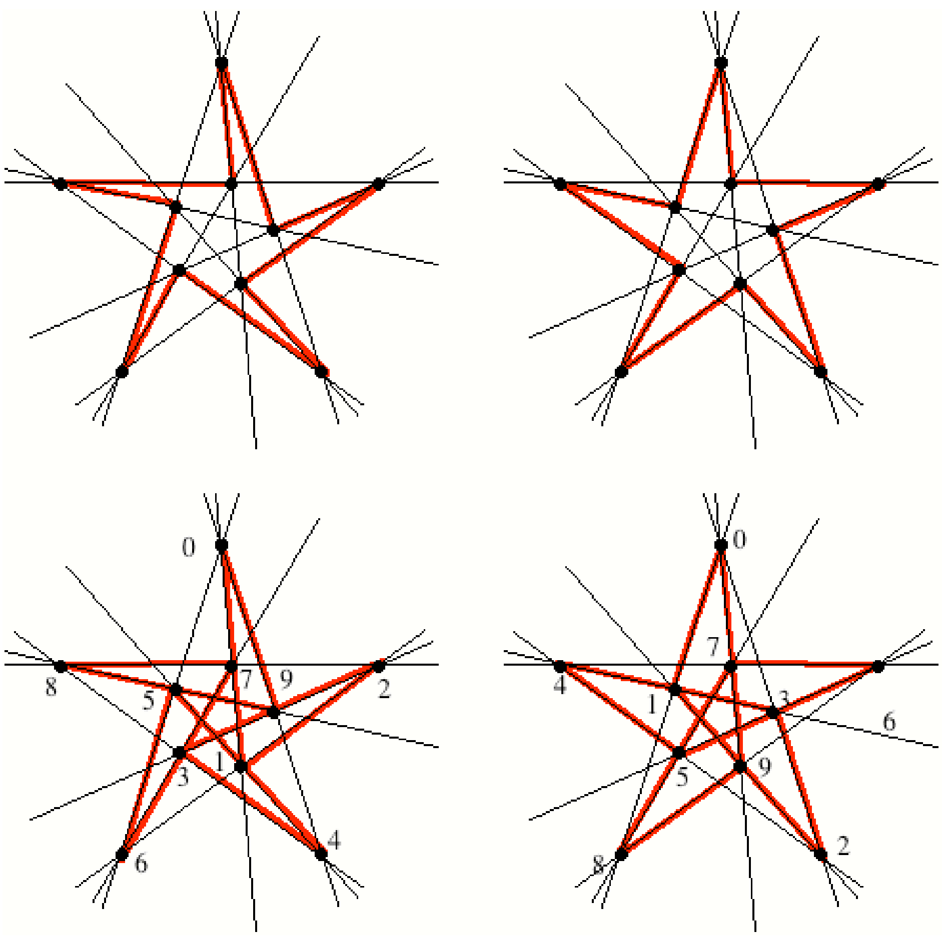

Figure 5.2.12. Four different symmetric Hamiltonian multilaterals in the astral configuration (10_3).

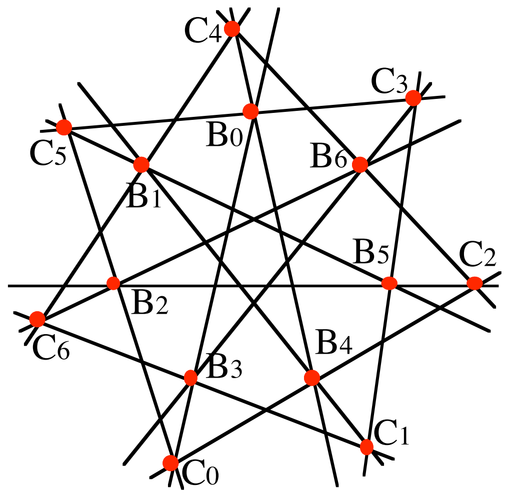

Figure 5.2.13. A reminder of the notation for astral 3-configurations, illustrated for m#(b,c;d) = 7#(3,2;4)

A related result of Hladnik *et al.* [H5], based in part on work of Alspach and Zhang [A1], should be mentioned here: Every connected cyclic 3-configuration (combinatorial or geometric) is Hamiltonian. Unfortunately, it is not clear to me exactly what is here meant by "cyclic 3-configuration".

Exercises and problems 5.2.

1. Decide the validity of the following open conjecture: Every astral 3-configurations admits a Hamiltonian multilateral.

2. Find four symmetric Hamiltonian multilaterals in the (14_3) astral configuration 7#(3,2;1).

3. Prove Theorem 5.2.4. Apply it to the configuration 8#(3,2;1).

4. The two configurations in Figure 5.2.14 do not have any symmetric Hamiltonian multilateral. If the result of [H5] mentioned above relates to them, they have Hamilto-nian multilaterals. In any case, either find a Hamiltonian multilateral, or else show that there is none such.

5. Determine whether the three 3-astral configurations (9_3) in Figure 1.1.6 admit symmetric (or any) Hamiltonian multilaterals.

6. In configurations with dihedral symmetry group one can not expect any Hamiltonian multilateral to have the same symmetry group. At most, one may look for cyclically symmetric Hamiltonian multilaterals. In Figure 5.2.15 we show three examples of this situation. Determine whether the six astral 4-configurations (36_4) shown in Figure 3.6.3 admit cyclically symmetric Hamiltonian multilaterals.

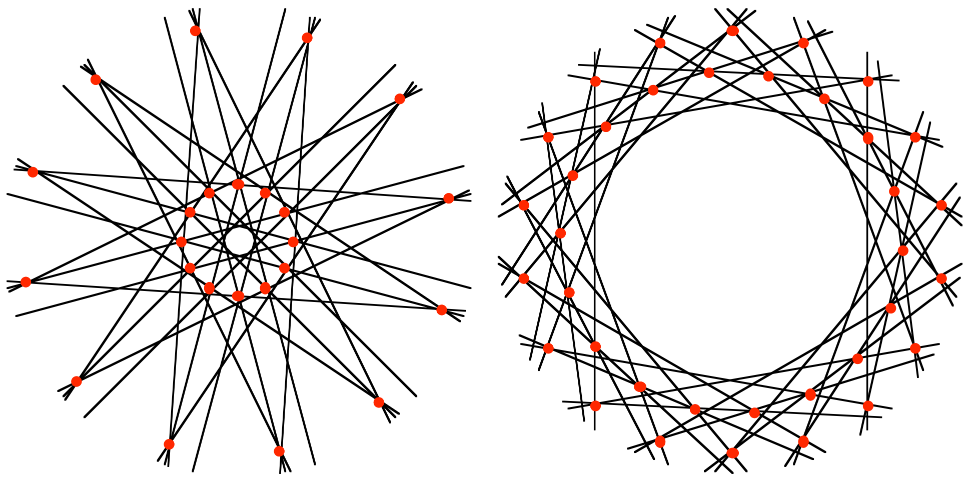

Figure 5.2.14. Selfpolar astral configurations 12#(5, 5; 2) and 18#(5, 1; 3) have no symmetric Hamiltonians. Do they have any Hamiltonian circuits at all?

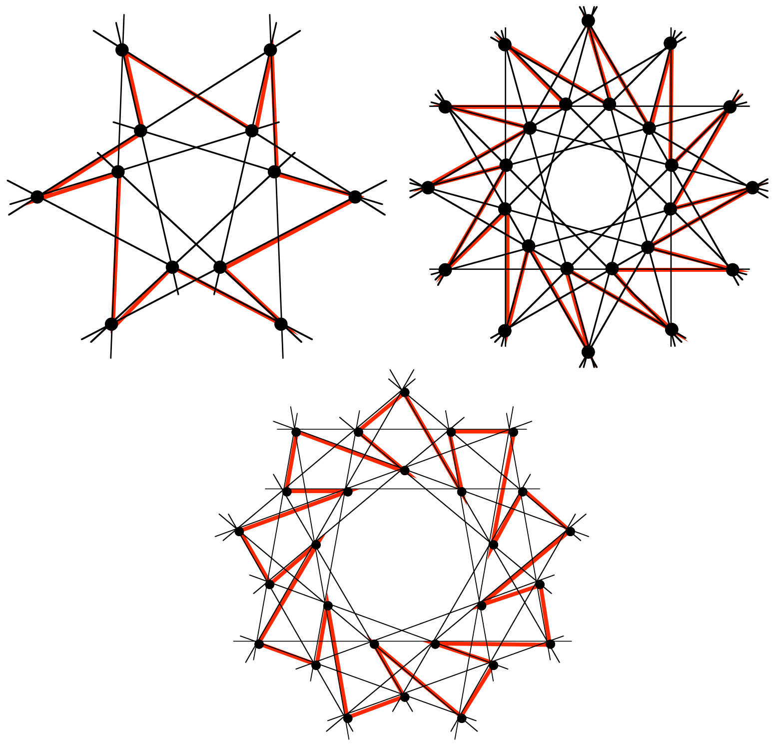

Figure 5.2.15. Examples of cyclically symmetric Hamiltonian multilaterals in three configurations with dihedral symmetry.

7. Following [D7] we say that set of points of a configuration C is a **blocking set** it if contains a point of every line of C but not all points of any of the lines. Show that the (22_3) configurations shown in Figure 5.2.2 contains no blocking set. (This example disproves two conjectures in [D7].) For more information about blocking sets and blocking set-free configurations see [G11], [G24].