5.1 CONNECTIVITY OF CONFIGURATIONS

We start by recalling from Section 1.4 the concept of Levi graph L(C) of a configuration C. This is a bipartite graph (that is, there are two sets of nodes — black and white) with no edge connecting vertices of the same color. Usually the black nodes of L(C) correspond to the vertices of C, while the white ones correspond to the lines of C. A black node is connected to a white node by an edge of L(C) if and only if the corresponding vertex of C is incident with the corresponding line. The advantage of L(C) is that the graph L(C) represents the configuration *faithfully* — that is, knowing the Levi graph of a configuration enables one to determine the (combinatorial) configuration uniquely. An example of the Levi graph of a (12_3) configuration is presented in Figure 5.1.1.

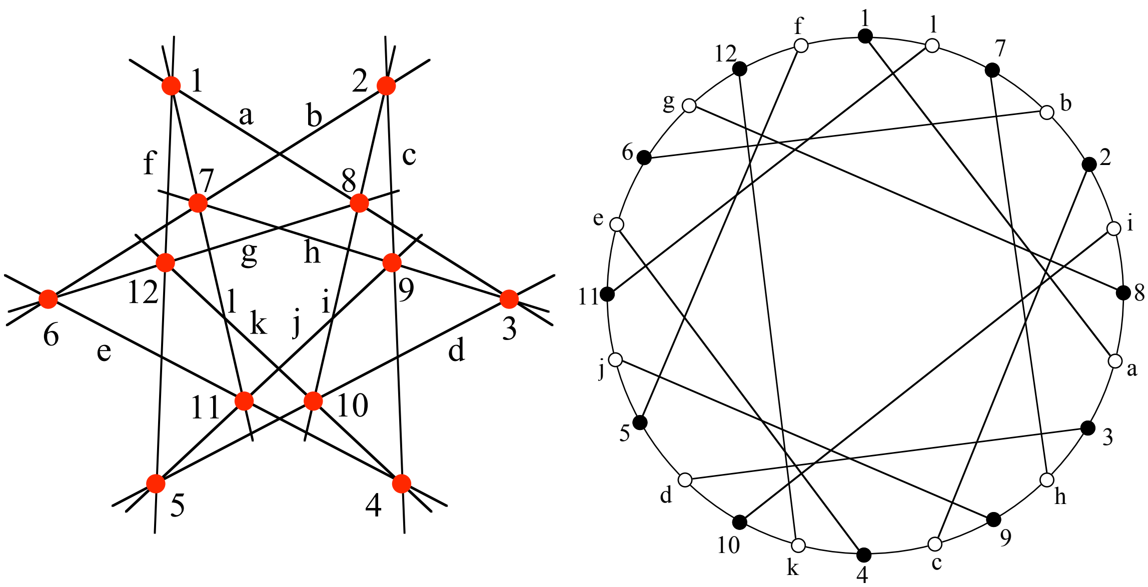

Figure 5.1.1. A configuration (12_3) and its Levi graph. Since the graph admits an incidence-preserving color reversal, which yields a graph isomorphic to the original (by reflection in the line bisecting the segments 7 h and 12 k, for example), the configuration is self-dual.

For a given configuration C, the Levi graph L(C) is uniquely determined. However, L(C) may admit various presentations, with different properties. For ex ample, in Figure 5.1.2 is shown another rendition of the Levi graph of the configuration (12_3) in Figure 5.1.1; it can be understood as an imbedding of the Levi graph in the torus. This presentation shows that all vertices of this configuration form one orbit under automorphisms of the configuration, and that all lines form one orbit as well. This is not easily visible from either the drawing of the configuration or its Levi graph in Figure 5.1.1.

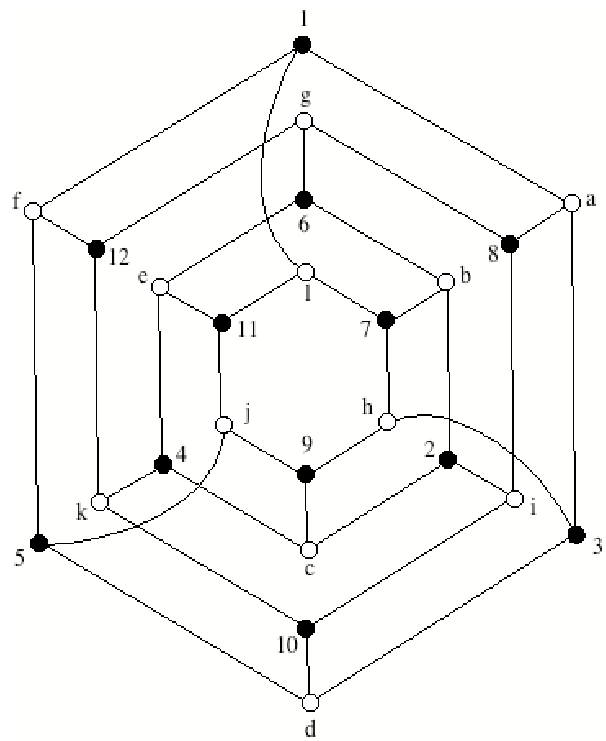

Figure 5.1.2. Another rendition of the Levi graph of the configuration (12_3) in Figure 5.1.1. It is easy to visualize this graph embedded in a torus in such a way that the combinatorial equivalence of all vertices of the configuration is obvious.

The main utility of Levi graphs comes from the fact that the graph-theoretic properties of L(C) may be used to define or determine properties of the configurations involved. We shall return to this topic in the next section, in the context of multilaterals.

Levi graphs are particularly useful in connection with questions about the connectivity of configurations. Concepts such as "connected", "k-connected", etc. for a configuration are defined by asking whether its Levi graph has the property in question. It is clear that these concepts can be defined directly in the configurations, but the formulations, distinctions, relevance, and familiarity are in many cases more easily perceived on the Levi graphs.

Theorem 5.1.1. Every connected combinatorial or geometric k-configuration C is 2-connected.

Proof. Note that a configuration is connected but not 2-connected (that is, the deletion of a single point from the Levi graph of C disconnects the graph) if and only if the dual configuration has the same property. Hence for a proof of the theorem by contradiction we may assume that there is a line L whose removal disconnects the configuration. At least one of the connected components resulting from the removal of L has at most h points incident with L, where $1 \leq h \leq k/2$. Then the number of incidences of the m lines of this component with the p points of the component is, on the one hand, equal to km, but on the other hand equal to $k(p - 1) + h$, since L was incident with h points. These numbers should be equal, but as one of them is divisible by k and the other is not, a contradiction was reached. ¨

Theorem 5.1.1 is due to Steinitz [S1], as is the idea of its proof. We have seen earlier (Corolary 2.5.3) a different proof of this result. But that proof relied on the construction of an orderly configuration table, which was a rather deep result. The approach here provides a good example of the utility of introducing graph-theoretic concepts (in particular, the Levi graph), in considerations of configurations. Steinitz did not have such tools, and as a consequence he needed more than a page of densely printed (and clumsily formulated) arguments to state and prove Theorem 5.1.1.

As a strengthening of Theorem 5.1.1 one might conjecture that each connected k-configuration, $k \geq 3$, is 3-connected. However, this is not the case. A counterexample is shown in Figure 5.1.3. It is known that all combinatorial configurations (n_3)

with $n \leq 13$ are connected and, moreover, are 3-connected. This is best possible since for n = 14 there are counterexamples to both parts. The combinatorial configuration consisting of two disjoint copies of the Fano configuration is disconnected, while the configuration in Figure 5.1.3 is connected but not 3-connected. Any disconnected geometric (or topological) configuration (n_3) must have $n \geq 18$.

For 4-configurations the corresponding numbers are:

- There are disconnected combinatorial configurations (n_4) if and only if $n \geq 26$.
- There are disconnected topological configurations (n_4) if and only if $n \geq 34$.
- There are disconnected geometric configurations (n_4) if and only if $n \geq 36$.

The example in Figure 5.1.3 shows that there are 2-connected 2-configurations that are not 3-connected. This leads to the following problem:

If $k \geq 4$ and $2 \leq j < k$, do there exist j-connected k-configurations that are not (j+1)-connected?

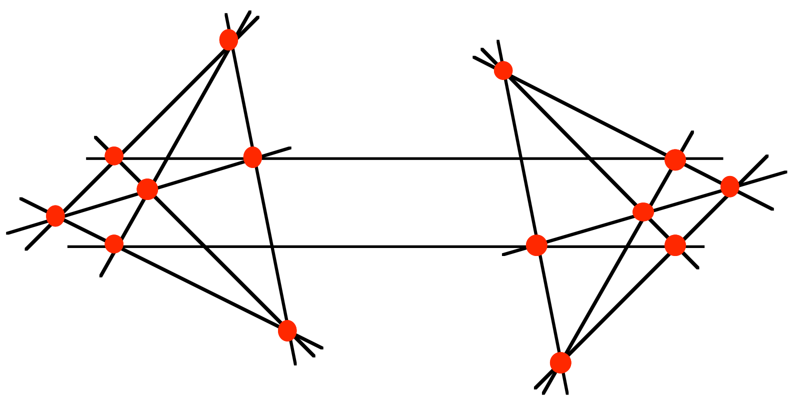

Figure 5.1.3. A connected configuration (14_3) which is not 3-connected.

An affirmative answer is given by the construction described in the proof of the following theorem:

Theorem 5.1.2. For each $k \geq 4$ and each j with $2 \leq j < k$ there exist geometric k-configurations that are j-connected but not (j+1)-connected.

Proof. We consider first the case j = 2. The text deals with arbitrary k, and the illustration in Figure 5.1.4 presents in parallel the case k = 4. We start with copies of configurations LC(k) described in Section 1.1 (see also [P9]), with a slight modification. As described in Section 1.1, LC(k) consists of an array of k^k points of the integer lattice in the Euclidean k-space E^k, with all coordinates in range [0, k-1], together with the lines parallel to the coordinate axes through these points. The modification we need here is that in one of the directions the coordinate k–1 is replaces by a convenient other integer. Our attention focuses on one of the lines in that direction, and the k points on it. (If desired, we way think of these configurations as projected into the plane.) In Figure 5.1.4 this situation is schematically indicated; one of the gray rectangles represents the modified configuration LC(k), the dashed line within the rectangle represents the chosen line, and the four dots represent the four points of LC(k) on that line. We need 2k configurations of this type, indicated by the gray rectangles, and positioned in such a way that two of the solid lines connect the k configurations at left with the k configurations at right, while each of the other 2k–2 solid lines connects the k copies in each half among themselves. Note that in each half, a single configuration is placed differently than the other k–1. Finally, we delete the dotted lines, thus creating a k-configuration. It is obvious that the resulting configuration is 2-connected, but the deletion of the two lines running between the two halves disconnects it; hence the configuration is not 3-connected.

For j > 2 we proceed analogously, but with an additional step. The construction is illustrated in Figure 5.1.5.

The first modification of the above construction is that the left half now has j–1 dots "above" the rest, and the right half has k–j+1 such points. Naturally, the corresponding numbers are the "bottom" are k–j+1 and j–1. Also, we do not insert the bottom connecting line between the two halves. Instead, we take a stack of k copies of what we constructed so far (it is best to imagine these copies to be in parallel planes, stacked above each other), and connected the corresponding points that were originally connected by the omitted bottom line. This is the desired configuration. It is clearly j-connected,

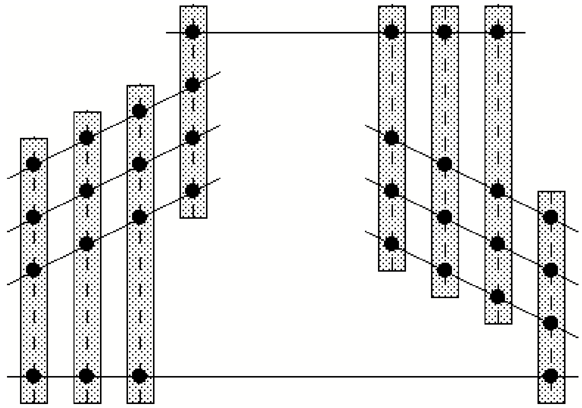

Figure 5.1.4. The construction of a geometric 4-configuration that is 2-connected but not 3-connected.

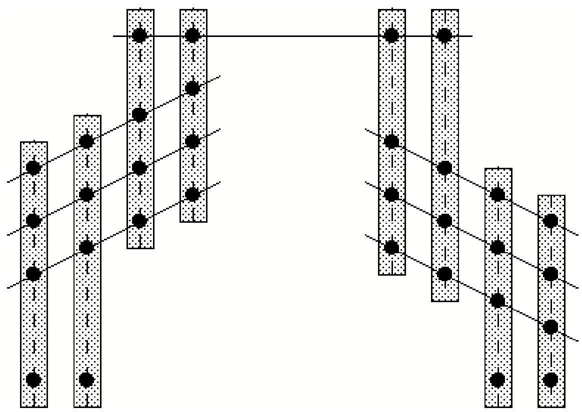

Figure 5.1.5. The construction of a geometric 4-configuration that is 3-connected but not 4-connected.

but the right part at each level can be disconnected from the rest by the omission of the j–1 lines connecting it to the other levels, and the line connecting it to the other half at its own level. ¨

It is clear that these constructions lead to very large configurations even in the smallest cases: (2048_4) in Figure 5.1.4, and (8192_4) in Figure 5.1.5. There probably exist much smaller configurations with the same properties — but justifying the existence of the appropriate projective images to yield the alignments necessary may be more involved. Rather obviously, much smaller combinatorial configurations with the properties discussed in Theorem 5.1.2 may exist, but this seems of rather marginal interest. On the other hand, even though one may expect to find combinatorial configurations of this type that are smaller than the corresponding geometric ones, no actual examples seem to be available.

* * * * *

In course on configurations I gave in the 1990 I made several conjectures that aimed at extending the result of Theorem 5.1.5 above to unbalanced [q,k]-configurations with $3 \le q \ne k \ge 3$. Xin Chen, a student in that course, produced various counterexamples; among them one proving that there exist [4,3]-configurations which are 1-connected but not 2-connected.

A small modification of Chen's procedure leads to:

Theorem 5.1.3. Combinatorial [q,k]-configurations that are 1-connected but not 2-connected exist if and only if $q \ne k$.

Proof. As we have seen in Theorem 5.1.5, every connected k-configuration 2-connected. For the other direction, due to duality, if $q \ne k$ it is enough to consider the case $q > k$. We start by forming a [q,k]-configuration, **cyclic** as far as possible. By this is meant that one uses as many cyclic sequences as necessary -- in the illustration below we need two such cycles. These configurations are generalization of the cyclic configurations $\mathbf{C}_3(n)$ we introduced on In Section 2.1. (The use of "cyclic" configurations comes only to simplify the checking that the tables which will be constructed are actually configuration tables.) Then we use a Martinetti-type construction (see Section 2.4): we select some $k-1$ lines with a property specified below, add one more point, and form a fragment in which all points except the new one are incident with q lines, and the new point is incident with k lines. The selected lines (which are then omitted) should be such that their points can be grouped in a way that no pair occurs in any other line; the new point is "cross-connected" to the points in these lines. As an illustration, the case $q = 4$, $k = 3$ is explicitly presented in detail . Here we take $n = 12$, the chosen lines are 1 2 4 and 7 8 10, and the additional point is 0. This way the configuration

1	2	3	4	5	6	**7**	8	9	10	11	12	1	2	3	4
2	3	4	5	6	7	**8**	9	10	11	12	1	5	6	7	8
4	5	6	7	8	9	**10**	11	12	1	2	3	9	10	11	12

yields the subfiguration

0	**0**	**0**	2	3	4	5	6	8	9	10	11	12	1	2	3	4
1	**2**	**4**	3	4	5	6	7	9	10	11	12	1	5	6	7	8
7	**8**	**10**	5	6	7	8	9	11	12	1	2	3	9	10	11	12

If three copies of this fragment are taken, distinguished by the number of dashes, and a line 0 0' 0" is added, we get a combinatorial $(39_4, 52_3)$ configuration which is connected but not 2-connected.

In the general case, the additional point 0 of the fragment will be on k lines only, thus having a deficit of q–k lines. To supply these we find a connected [q–k, k]-configuration C. Then we take as many copies of the fragment as there are points in C, and identify the point 0 of one copy of the fragment with each point of the configuration C. (In the above example, C is the [1, 3]-configuration which consists of just three points and one line.) This clearly yields a [q,k]-configuration of type which is connected but not 2-connected since each copy of the point 0 disconnects the configuration. ¨

It is not known whether the $(39_4, 52_3)$ configuration is the smallest of this type.

Theorem 5.1.3 deals with combinatorial configurations, and the question arises whether there exist connected geometric [q,k]-configurations, with $q \neq k$, that are not 2-connected.

A partial affirmative answer is given by the following result.

Theorem 5.1.4. For every q and k, with $\min\{q,k\} \geq 3$ and $q \neq k$, there exist geometric [q,k]-configurations that are connected but not 2-connected.

Proof. We first consider the case [4,3]-configurations. We start with a tricyclic 3-configuration C_1 shown in Figure 5.1.6. (This particular configuration is (54_3), but it is likely that smaller configurations of the same general type could be used in the construction.) The significance of the two heavily drawn lines will be explained soon. As in some of the other constructions, the next step is best explained by thinking of the configuration C_1 as contained in a plane of the 3-dimensional space. By adding two congruent copies, situated perpendicularly above and below C_1, and adding vertical lines through all points of C_1, we obtain a configuration $(162_4, 216_3)$, which we designate

C_2. Now we delete the two heavily drawn lines from the configuration C_1 — but not from the two copies of it, which we used in C_2. Instead of these two lines we introduce three new lines, as shown in Figure 5.1.7, and a new point incident with all three of these. (The existence of such a triplet of lines depends on the variability afforded to tricyclic configurations by the presence of an arbitrary parameter.) This step leads from C_2 to a prefiguration C_3. All lines in C_3 are incident with three points, and all points of C_3 except the newly introduced point are incident with four lines.

In the final step we take two additional copies of C_3 and connects the three exceptional points by a line. This results in a [4,3]-configuration which is connected but obviously is not 2-connected.

An easy modification of this construction for works for $q > 4$. All that is needed is to use copies of C_2 to construct (in 4-space, for greater comfort) a [5,3]- or [6,3]- etc. configuration, before proceeding to C_3 and to the final configuration.

Clearly, the polars of these configurations (or, more precisely, of their projections into the plane) yield the appropriate connected but not 2-connected [3,k]-configurations, with $k \geq 4$.

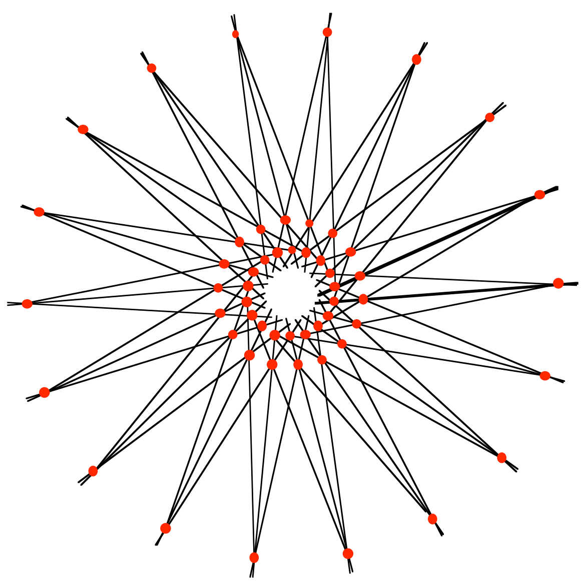

Figure 5.1.6. The tricyclic configuration C_1 used in the proof of Theorem 5.1.4.

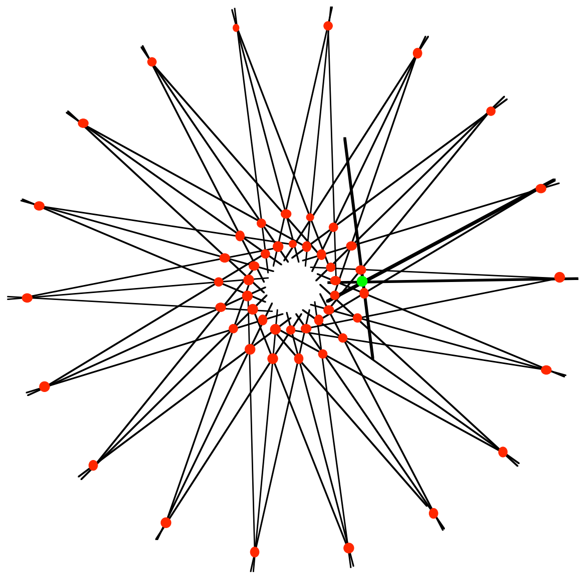

Figure 5.1.7. The prefiguration C_3 used in the proof of Theorem 5.1.4. The new point is indicated by the green dot.

Now, if min{q, k} ≥ 4, we need an additional step in the construction. We shall assume that q ≥ k, since otherwise we could construct the dual configurations. We start again with the (54_3) configuration C_1 shown in Figure 5.1.6. By repeatedly using the procedure we designated (5m) in Section 3.3, we generate from C_1 a [q,k]-configuration C*. In Section 3.3 we went only one step, from 3- to 4-configurations. But an easy modification allows the construction of a k-configuration C** that contains the original C_1 as a subconfiguration. By stacking k copies of C** and connecting them with lines through corresponding points, a [k+1,k]-configuration is obtained; repetition of this step leads to the required C*. Now we replace – as before – the two lines drawn heavily in Figure 5.1.6 by the three lines and a point as shown in Figure 5.1.7. The newly introduced point O is the only point that is on just q-1 lines. Now taking k-1 additional copies of C* and connecting the k points (O and its images in the other copies of C*) by a line gives the desired [q,k]-configuration which is connected but not 2-connected.

*   *   *   *   *   *

Two elements (points or lines) of a configuration are said to be **independent** if they are:

- Two points that are on no line of the configuration;
- Two lines if they are not incident with a point of the configuration;
- A point and a line if they are not incident.

A family of elements in a configuration is called **independent** if any two of its elements are independent.

A configuration C is said to be **unsplittable**[1] if the deletion of any independent family of its elements leaves a connected configuration. In other words, if for every independent family F, any two elements that do not belong to the family F are in a multilateral that does not use any element in F. Equivalently, C is **splittable** if it can be disconnected by an independent family of elements.

For example, the (12_3) configuration in Figure 5.1.8 has several 5-element independent families but no 6-element independent families. It is easy (even if tedious) to check that the configuration is unsplittable. Similarly, for the (15_3) configuration in Figure 5.1.9, the maximal number of elements in an independent family is 6, and the configuration is unsplittable. These and other examples lead to

Conjecture 5.1.1. The maximal number of elements in an independent family in a connected configuration (n_k) is $[n/k] + 1$.

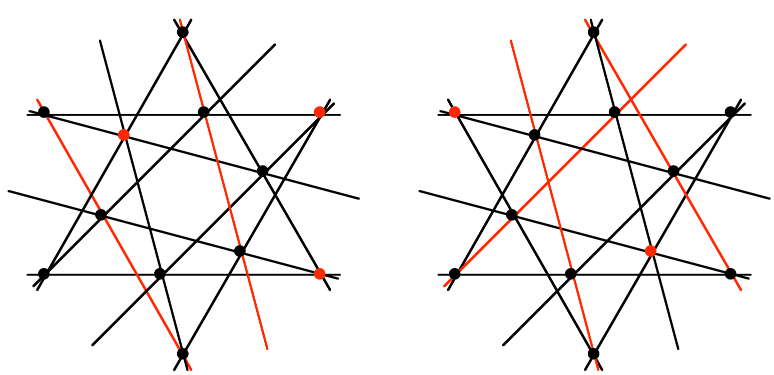

Figure 5.1.8. Two independent families of five elements each (shown in red), that do not disconnect the (12_3) configuration.

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[1] The material on independent families and unsplittable configurations is part of an ongoing collaboration with Tomaz Pisanski; it was presented in part in [P9].

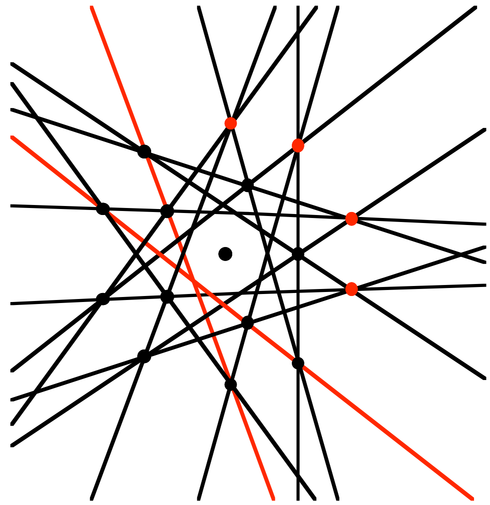

Figure 5.1.9. An independent family of six elements (red) in a (15_3) configuration.

We shall encounter unsplittable configurations in Section 5.6. Here we shall conclude with

Theorem 5.1.5. Every unsplittable 3-configuration is 3-connected.

Proof. Assume that a connected configuration C is not 3-connected; we shall show it is splittable. As a consequence of Steinitz's theorem 2.5.1 it is possible to present C in an orderly configuration table, hence it must be at least 2-connected.

If C is not 3-connected, it can be disconnected by two elements, and we have the two possibilities:

(i) Both elements are of the same kind; without loss of generality we can assume that the disconnecting set consists of two points.

(ii) The disconnecting set consists of one point and one line.

If either of these disconnecting sets were not a splitting set, the two points must be incident with a line of C, or the points and line must be incident, respectively.

In case (i), let the two points be A and B, and let L be the line incident with both. Let D be the third point of L. It is impossible that of the six lines incident with L, only two come from the same connected component. (If this were the case, either the two lines would be incident with the one of the points A, B, D — which would mean that C is not

2-connected; or else the two lines would be incident one each with A and B. Then taking two copies of this components, together with A, B, and L, and attaching three such systems at a point corresponding to D in all three, would again yield a configuration that is not 2-connected.) So each component has three lines incident with L. Since L is incident with only six lines, and A, B disconnect C, the arrangement must be like the one in Figure 5.1.11. But then the point B and line M are a splitting set.

In case (ii), the situation must again be as shown in Figure 5.1.10, with A and L the disconnecting elements, and again M and B are a splitting set. ¨

We may note that the converse of Theorem 5.1.5 does not hold. There are 3-connected configurations that are splittable. The smallest I know is the (15_3) configuration shown in Figure 5.1.11.

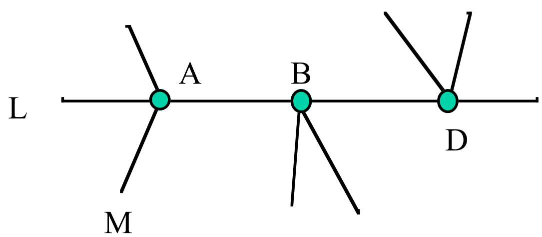

Figure 5.1.10. Schematic arrangement used in the proof of Theorem 5.1.5.

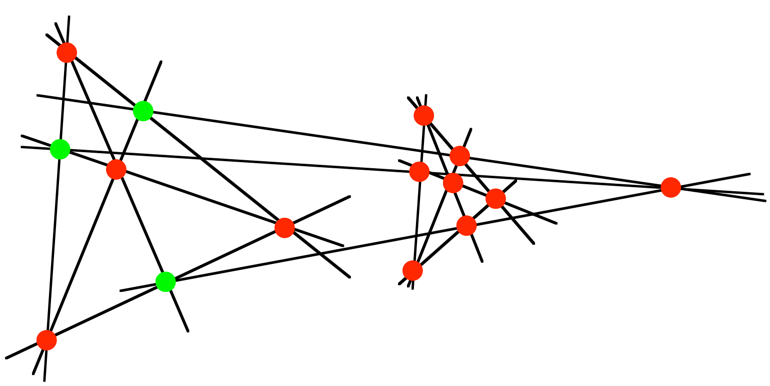

Figure 5.1.11. This 3-connected configuration (15_3) is splittable. A splitting set consists of the three green dots.

Exercises and problems 5.1

1. The geometric [q,k]-configurations constructed in the proof of Theorem 5.1.4 are quite large, even in the case of [4,3]-configurations. Find smaller examples of connected but not 2-connected geometric [4,3]-configurations.

2. Do there exist reasonably-sized 2-connected but not 3-connected geometric [4,3]-configurations? It is clear that this question can be generalized.

3. Without relying on polarity, give a detailed proof of Theorem 5.1.4 for the case of geometric [3,4]-configurations.

4. Prove that Conjecture 5.1.1 is true for k = 2, even without the connectedness assumption. Show that it is invalid for k = 3 if connectedness is not assumed.

5. Show that Conjecture 5.1.1 holds for all (10_3) configurations.

6. Show that the cyclic configuration $\mathbf{C}_3(n)$ (see Section 2.1 for the definition) is unsplittable.

7. Investigate which of the cyclic configurations $\mathbf{C}_3(n,a,b)$ (see definition in Exercise 2.1.2) are unsplittable.

8. The independent families of elements of a configuration C can be characterized as corresponding to independent (that is, unconnected) sets of vertices in the **independence graph** I(C) of C. (The independence graph of C can be defined as the (graph-theoretic) **square** of the Levi graph L(C); this is the graph in which two vertices are connected by an edge is they are connected in the original graph, or if they share a common adjacent vertex. Equivalently, I(C) is the union of L(C) with the edges of the Menger graph M(C) described in Section 1.4, and the edges of M(C*), where C* is the configuration dual to C.)