4.7 UNCONVENTIONAL CONFIGURATIONS

In this section we shall consider several families of objects that we shall call "configurations" even though they do not fit the definition of that word accepted in all the other sections of this book.

The first of these families are "configurations of points and circles". Some examples are shown in Figures 4.7.1 and 4.7.2. In analogy to configurations of points and lines we may denote them by a symbol such as (p_q, n_k), where p, n are the numbers of points and of circles, and q, k are the number of circles incident with each point and the number of points incident with each circle; in case the numbers are equal, we use the notation (n_k). Hence the three configurations shown are (4_3), (8_3,6_4), and (10_4).

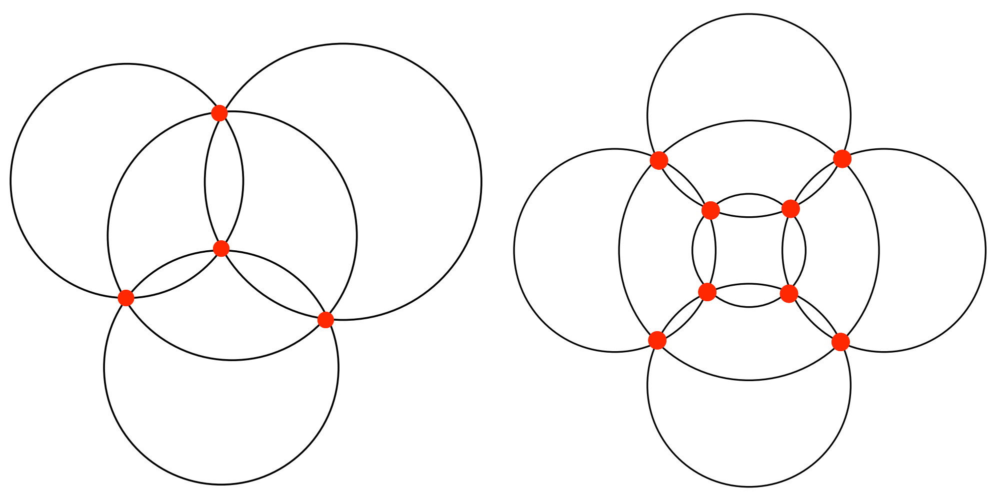
Several aspects of configurations of points and circles deserve notice.

First, such configurations are generalizations of configurations of points and lines in a very direct way: Every configurations of points and lines in the projective plane can be shown as a configuration of antipodal pairs of points and great circles in the model of the projective plane on the sphere; a stereographic projection then maps this into a configuration of points and circles in the plane. However, these are only very special cases of such configurations — none of those in Figures 4.7.1 and 4.7.2 is of this kind.

Second, in all but name, configurations of points and circles made their appearance before configurations of points and lines. For example, the configuration in Figure 4.7.1b is an illustration of a theorem of A. Miquel [M19a] dating to 1838, asserting that if four pairwise intersections of four circles are concyclic, the other four intersections of the same pairs are concyclic as well. This is one of several results of Miquel, some of which have been greatly generalized by many writers, starting with Clifford [C2**] in 1871 and de Longchamp [L4*] in 1877. One of the achievements are the so-called "chains of theorems" bearing the names of Clifford and de Longchamps. The former establishes the existence of configurations of points and circles ($(2^{n-1})_n$) for all $n \geq 1$. The cases n = 1 or 2 are trivial, and n = 3 is shown in Figure 4.7.1a. For more recent works on this topic and its relatives see, for example, Ziegenbein [Z8*], Rigby [R3*], Longuet-Higgins [L5*], Longuet-Higgins and Perry [L5**], and references given there to other works.

Third and last — why is there no greater activity regarding these configurations? I venture to guess that the preoccupation with just a few specific results (such as the "chains of theorems") tended to discourage more general inquiries. There are various subclasses of circle configurations that may well be worth investigating: Are pairs of circles required to intersect twice, are touching circles allowed, can disjoint circles appear, are straight lines admitted, does one wish to consider symmetries in the inversive plane, – – the choices and possibilities are very wide and almost entirely unexplored. (The inversive plane seems an appropriate setting for many of the considerations of symmetries of configurations of points and circles; see, for example Coxeter [C7], Eves [E2], Yaglom [Y1].)

The configuration (10_4) in Figure 4.7.2 is an example of configurations $((2n)_n)$ that exist for all $n \geq 5$ and exhibit remarkable symmetry in the inversive plane. The (10_4) configuration has a single orbit of points and a single orbit of circles under inversive transformations. I do not know what other configurations are as symmetric, but probably there are many additional ones.



(a) (b)

Figure 4.7.1. Configurations of points and circles. (a) A (4_3) configuration. (b) A $(8_3,6_4)$ configuration.

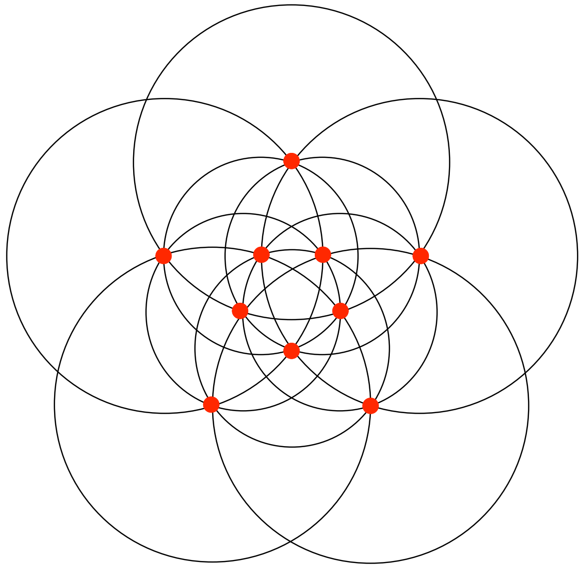


Figure 4.7.2. A (10_4) configuration of points and circles.

The second family of "unconventional configurations" is illustrated by the examples in Figures 4.7.3 and 4.7.4. The objects in this family are the traditional points and

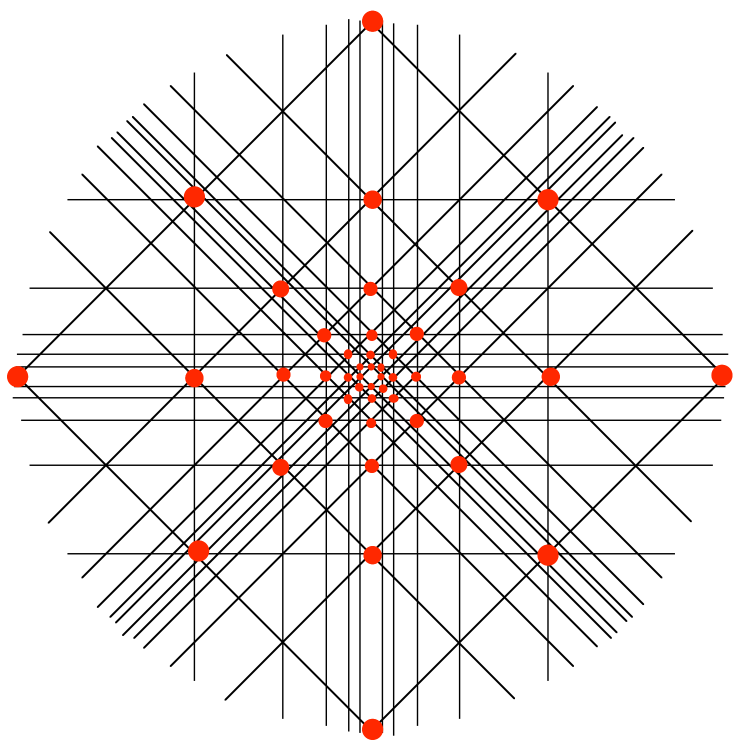


Figure 4.7.3. An infinite 3-configuration with 4-fold dihedral symmetry and single transitivity classes of points and of lines under the group of similarity transformations.

lines of the Euclidean plane, and the configurations satisfy all the conditions assumed throughout the book — except the requirement that there are only finite numbers of points and of lines. More precisely, we are now looking at infinite families of points and lines such, for some finite k, each point is incident with k lines, each line with k points, and the family is discrete in the sense that every point [line] has a neighborhood that contains no other point [line] of the family. We shall call a family of this kind an *infinite k-configuration.*

While many different kinds of infinite k-configurations (or of analogously defined infinite [q,k]-configurations) can be contemplated, the two examples we show have few orbits of points and of lines under similarity transformations.

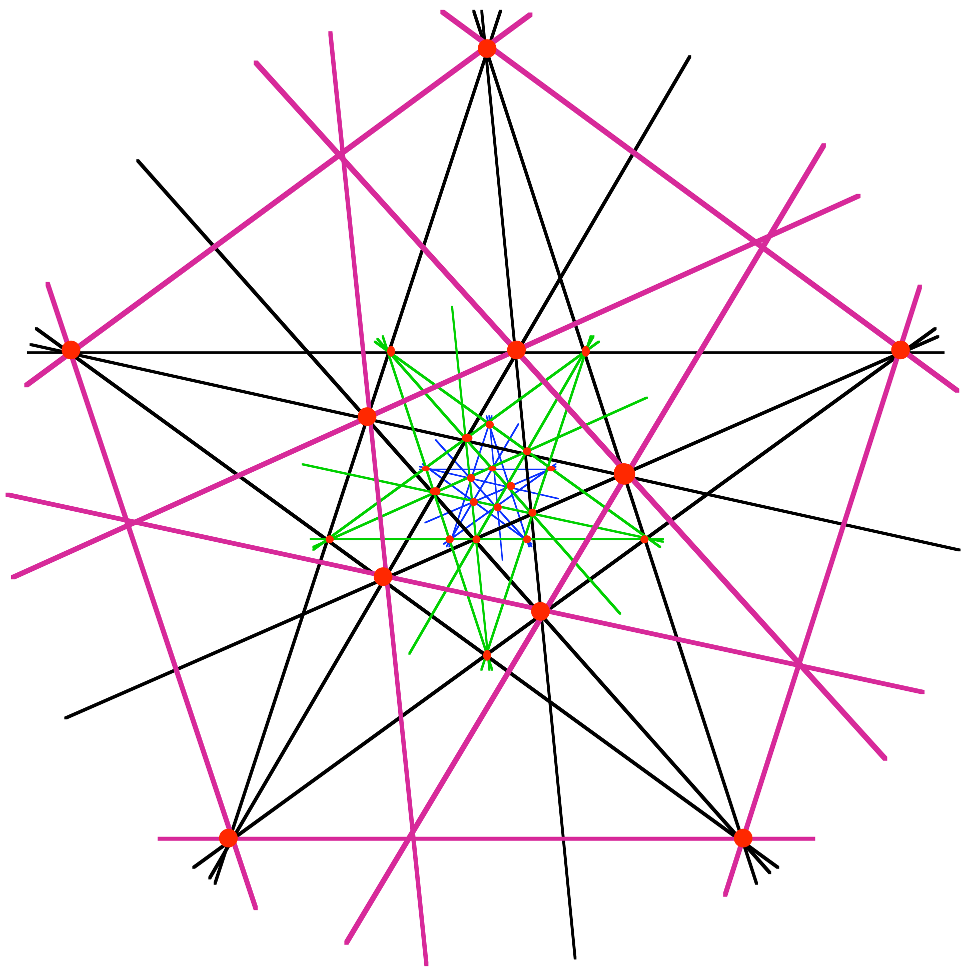


Figure 4.7.4. An infinite 5-configuration obtained by repeated inscribing/circumscribing of copies of the astral configuration 5#(2,2;1). The copies are distinguished by colors.

These configurations can be interpreted as an iterative analogue of the (4m) construction we considered in Section 3.3. The infinite 3-configuration in Figure 4.7.3 arises by repeatedly inscribing (4_2) configurations in each other. A construction of this type can be performed starting with any regular m-lateral, leading to an infinite 3-configuration with m-fold dihedral (or cyclic — with a suitable placement of the m-laterals) symmetry. Such configurations can therefore be considered as infinite analogues of the families of inscribed/circumscribed multilaterals we shall consider in Section 5.3.

The infinite 5-configuration in Figure 4.7.4 arises in the same way from repeated inscription/circumscription of copies of the astral configuration (10_3) shown in Figures 1.3.3 and 1.5.4. It is the only example of this kind that I found in the literature; it is explicitly mentioned in van de Craats' paper [V1]. It is clear that this type of construction can be carried out with other astral 3-configurations.

The third (and last) family of unconventional configurations is illustrated by the remaining figures of this section. In these configurations the role of points and lines are

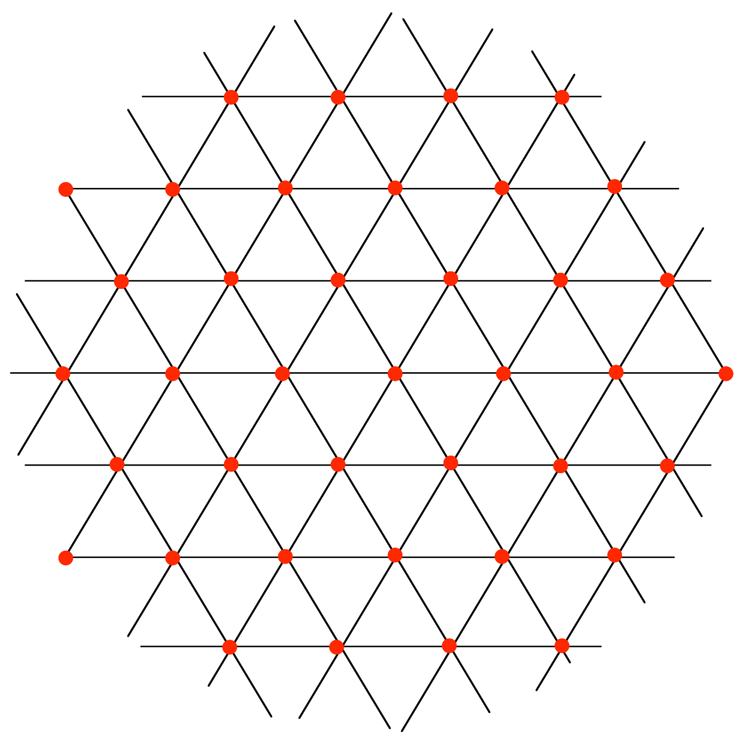


Figure 3.7.5. An infinite [3]-configuration with a single orbit of points and of lines under isometric symmetries of the plane.

different, although there are infinitely many of both: Each point is on precisely k lines for some finite k, but each line contains infinitely many points. We call such configurations *infinite [k]-configurations.* To avoid complications we also require that there be no accumulation points or lines. It is again convenient to consider configurations with a high degree of symmetry under the group of isometric maps of the plane. It is easy to verify that infinite [k]-configurations exist for all $k \geq 1$.

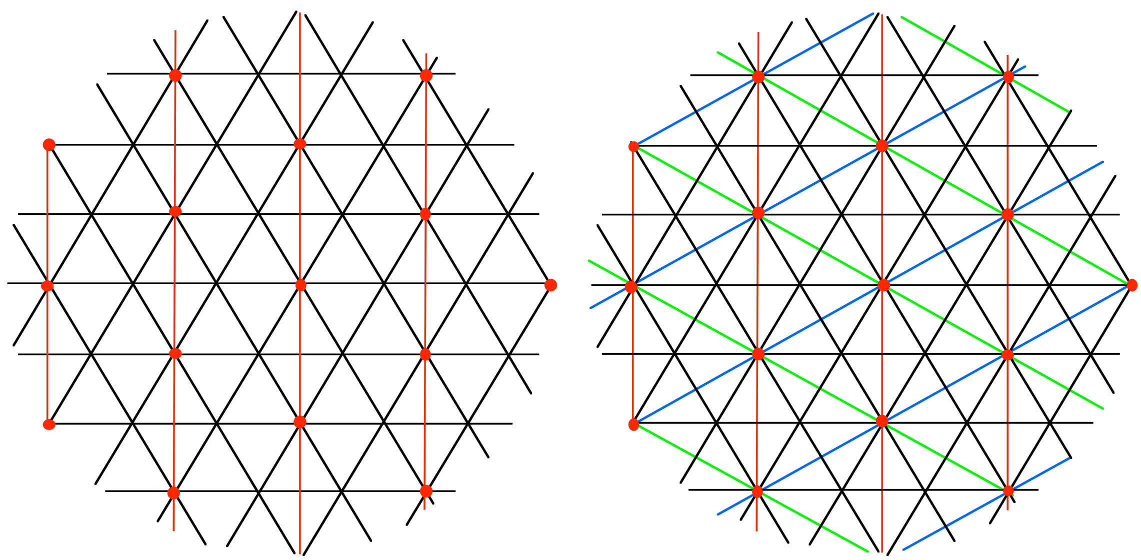


Figure 4.7.6. Examples of infinite [4]- and [6]-configurations.

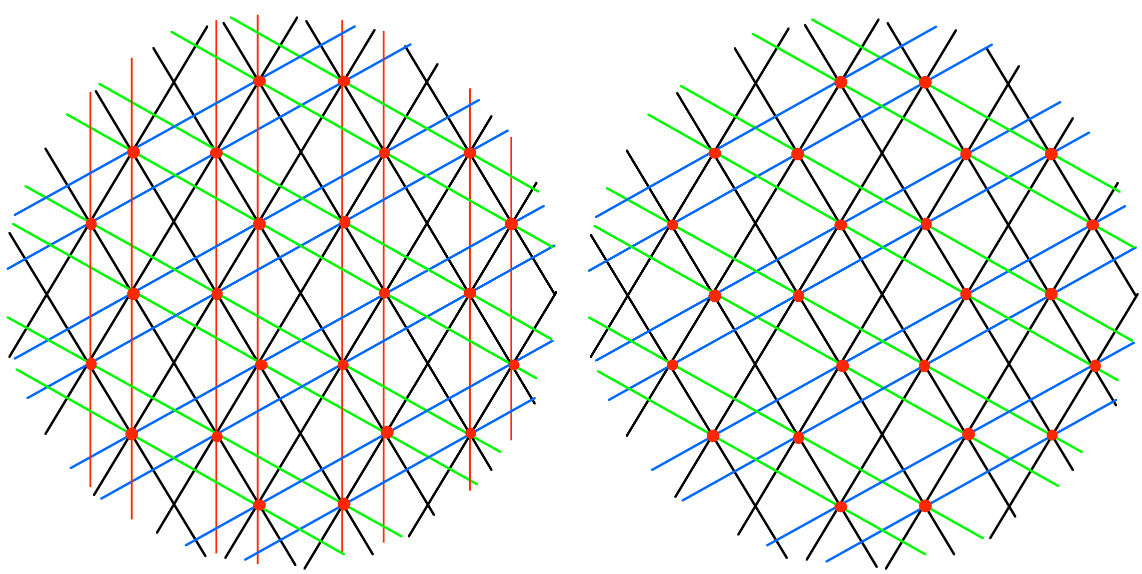


Figure 4.7.7. Examples of infinite [5]- and [4]-configurations.

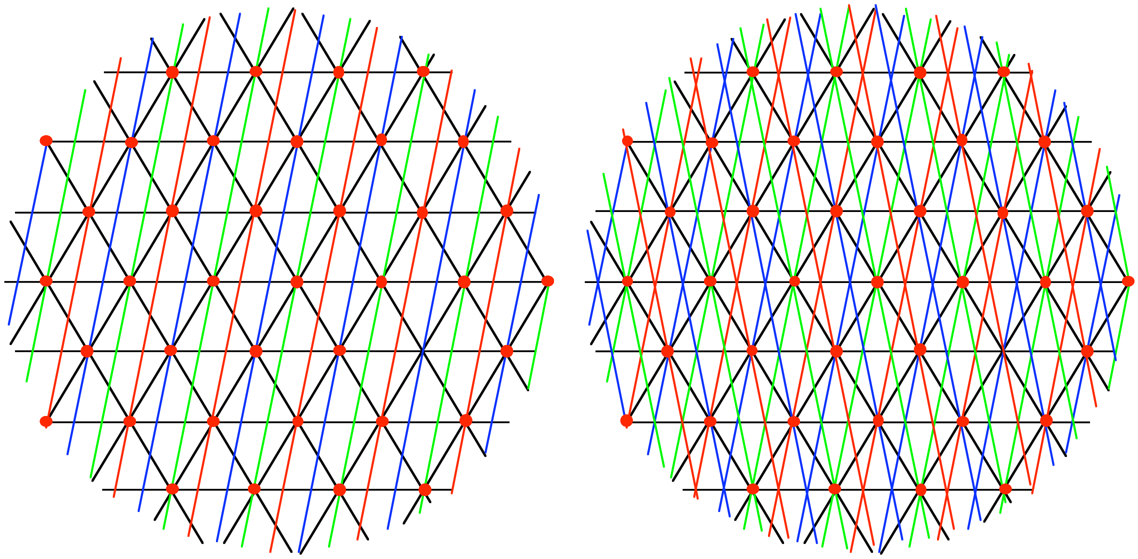


Figure 4.7.8. Additional example of infinite [4]- and [5]-configurations.

Exercises and problems.

1. Construct the (12_4) and (14_4) analogues of the configuration of points and circles in Figure 4.7.2.

2. Decide whether there are configurations (n_k) of points and circles for arbitrarily large k.

3. Modify the construction in Figure 4.7.3 to obtain a *chiral* infinite 3-configuration (that is, with cyclic symmetry group) and with a single orbit of points and one of lines.

4. Justify the claim that the van de Craats construction fan be carried out for other astral 3-configurations.

5. Is there any infinite k-configuration such that its points have no accumulation point?

6. Find infinite [k]-configurations that differ in some essential aspect from the ones shown here.

7. Construct infinite configurations of points and circles that share some features with the infinite configurations of points and lines described above.