4.6 TOPOLOGICAL CONFIGURATIONS

Studies of topological configurations have begun only in the very recent past. While in many ways analogous to geometric configurations, there are significant differences that deserve to be investigated in more detail. Here I will try to present the material that is available at this time.

The distinction between geometric and topological configurations became evident long ago, through Schroeter's proof [S6], [S8] that one of the ten combinatorial configurations (10_3) cannot be geometrically realized; see Section 2.1 for more details. The fact that it *almost* can be realized geometrically (as in Figure 1.2.2, with lines just a bit bent) means that it is topologically realizable. However, neither this, nor the fact that it is not known whether there exist geometrically non-realizable 3-connected (n_3) configurations with $n > 10$ that are topologically realizable, resulted in any consistent effort to find clarification. It took almost forty years after Schroeter's discovery for Levi [L3] to even define the appropriate concepts.

Another rather frustrating aspect of the situation concerning topological 3-configurations comes about through Steinitz's theorem (see Section 2.6). In the case of *topological* 3-configurations unintended incidences pose no problem, and one may formulate the resulting statement as follows:

Theorem 4.6.1. Every connected combinatorial 3-configuration with $n \geq 9$ can be realized by pseudolines if the incidence of an arbitrary point-line pair is disregarded.

Naturally, just as in the case of the Steinitz theorem itself, the unfulfilled incidence can always be restored by allowing a curve of degree at most 2. But there is no guarantee that this curve can be chosen in such a way that we obtain a topological configuration. As we have seen in Section 2.1, for $n = 7$ or 8 this is, in fact, impossible and there is no topological realization of these configurations.

A separate question is whether in certain families of 3-configurations (such as astral, or 3-astral, or others) there exist topological configurations that cannot be realized by geometric ones of the same character. An affirmative answer to one of these questions arises from the examples in Section 2.7 (in particular, see Figure 2.7.6). However, the full extent of such situations for connected astral 3-configurations has not been determined. More precisely, in Figure 4.6.1 we show four different astral 3-configurations of pseudolines which arise from unintended incidences in geometric astral configurations — all four resulting in the same astral 4-configuration (24_4).

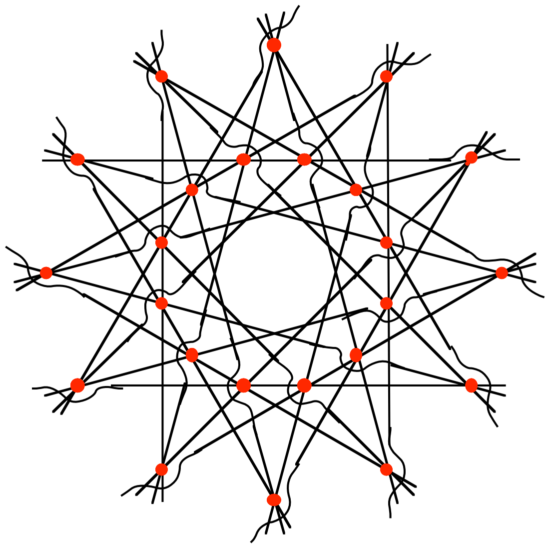
A different situation happens with the astral 3-configuration 12#(5,1;3). Its drawing does not produce either the intended (24_3), nor a (24_4). Instead, the resulting family of points and lines has some points on three lines and some on four, while some line are incident with just three points and some with 4. This is illustrated in Figure 4.6.2(a). Again it is possible to avoid unintended incidences by replacing one orbit of lines by pseudolines, as indicated in Figure 4.6.2(b).

In all these cases it is not known whether actual geometric realizations of the 3-configurations can be obtained if one does not impose symmetry restrictions.

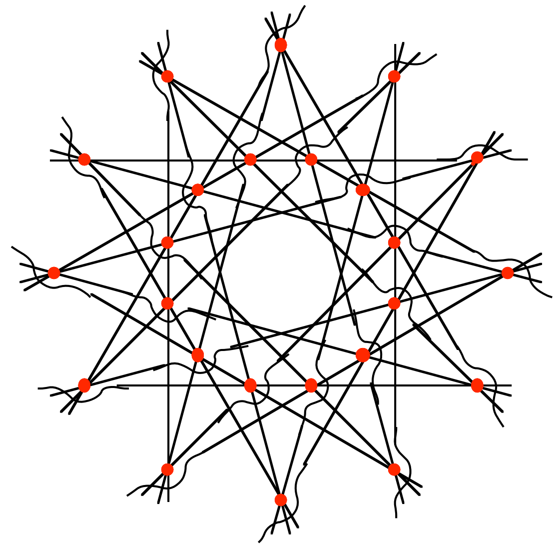
Concerning topological 4-configurations, we have already discussed in Section 3.2 the non-existence of topological (n_4) configurations for $n \leq 16$ and the fact that for every $n \geq 17$ there exist topological (n_4) configurations. Very recently, L. Berman [B4], determined the conditions for the existence of astral (that is, 2-astral) configurations of pseudolines with dihedral group of symmetries. The main result of [B9] is the following:

Theorem 4.6.2. Astral topological configurations (n_4) exist if and only if n is even and $n \geq 22$.

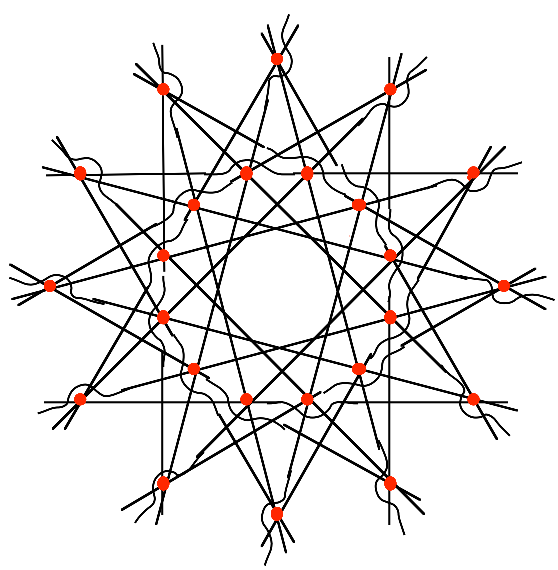
For the existence part of the proof it is sufficient to provide examples. An astral (22_4) configuration of pseudolines was first shown in [G50], and has been reproduced in several other publications; see Figure 4.6.3. Applying the notation we used in Sections 3.5 and 3.6 to topological configurations, this is 11#(5,4;1,4). It can be used as a template for all even $n = 2m \geq 22$: For each $m \geq 11$, the symbol m#(5,4;1,4) represents such a configuration. An example (with m = 17) is provided in Figure 4.6.4. To establish the inequality for n, it is necessary to first notice that due to the requirements for topological



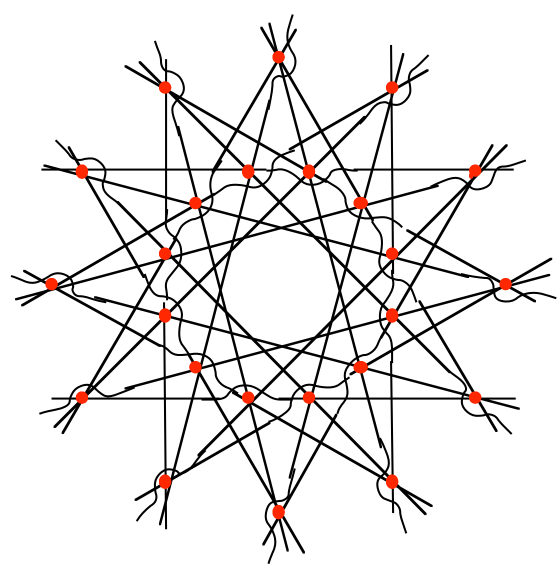
12#(5, 1; 3)



12#(5, 1; -1)



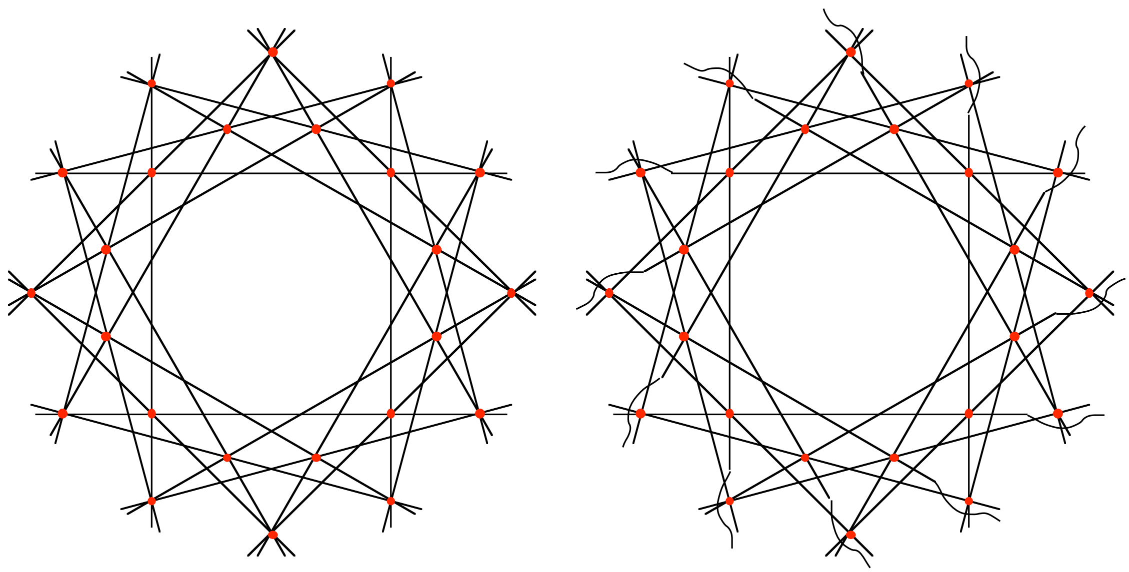
12#(4, 4; 2)



12#(4, 4; 1)

Figure 4.6.1. Four instances where an astral geometric 3-configuration (12_3) leads to the astral 4-configuration (24_4). The pseudolines can avoid the unintended incidences.

astral configurations, one can assume the configuration to be connected, have its points coincide with the vertices of two concentric regular m-gons, and have the concept of "span" of diagonals available — just as for geometric configurations. Then it is easy to verify that the shorter span must be at least 4, hence the larger span at least 5, and therefore m greater than twice 5. (This is an abbreviated version of the detailed arguments in [B3].) ♦



(a) (b)

Figure 4.6.2. A drawing (a) of the astral 3-configuration 12#(5,1;3) produces no geometric configuration, but can be modified to a topological configuration (b).

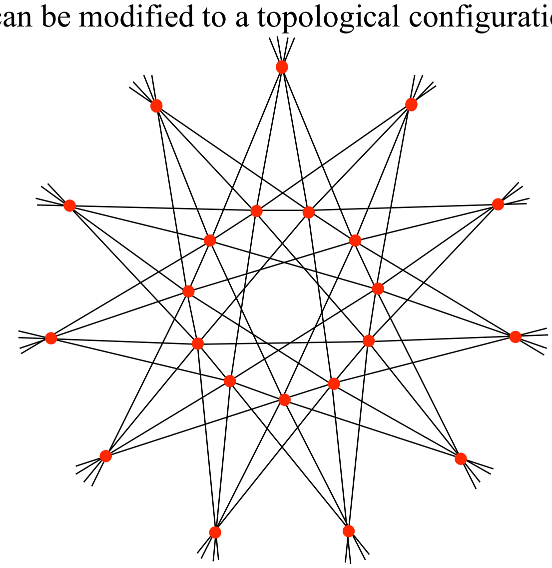


Figure 4.6.3. A topological astral configuration (22_4), that can be described as 11(5,4;1,4).

A more detailed description of astral topological 4-configurations is given in [B9] as well. It concentrates on those with dihedral symmetry. With a slight modification of the notation in [B9], we may summarize the results as follows. Using the symbol m#(b,c;d,e) in the same meaning as explained in Sections 3.5 and 3.6, we note that:

- The configuration points of the inner orbit can be situated on a circle of a radius that can vary between certain limits;
- The points of the inner orbit are either aligned with those of the other orbit (Type 1), or else situated at positions that enclose with them angles that are odd multiples of π/m (Type 2).
- m#(b,c;d,e) and m#(d,e;b,c) are equivalent; moreover $c \neq d$ and $b \neq e$; we conventionally assume that $b < e$;
- It follows that $c < b$ and $d < e$, and $b - c > e - d$;
- For Type 1 configurations we have $b - c \equiv e - d \equiv 0 \bmod 2$, and

  For Type 2 configurations we have $b - c \equiv e - d \equiv 1 \bmod 2$.

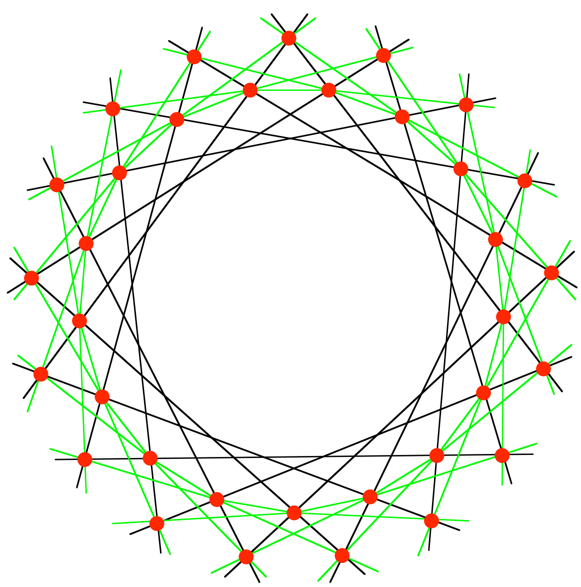
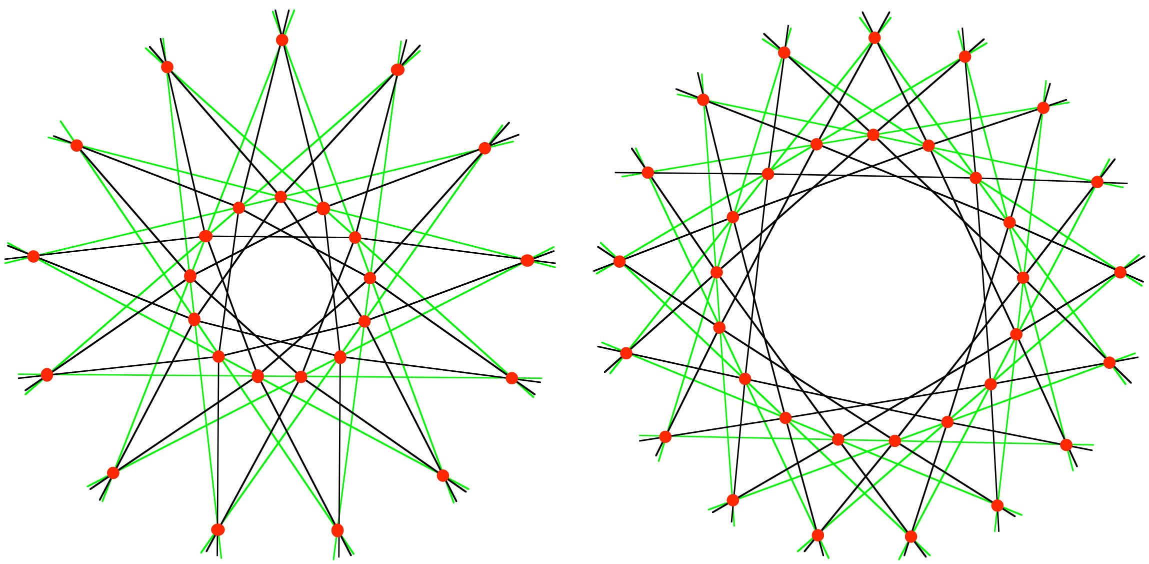


Figure 4.6.4. An astral topological configuration (34_4) of Type 2. It can be specified as 17#(5,4;1,4).

While these conditions impose many restraints on astral topological configurations, it is also clear that most of them cannot be "straightened" or "stretched" into geometric astral configurations. The reason in that the geometric m#(b,c;d,e) configurations exist only if m is a multiple of 6, while no such restriction holds in the topological case.

The smallest topological astral configurations is 11#(4,1;4,5) shown in Figure 4.6.3 above. It is the only astral configuration (22_4), and is of Type 2. The smallest astral topological configuration of Type 1 is 13#(5,1;4,6), shown together with 17#(5,1;4,6) in Figure 4.6.5.

Even when m is divisible by 6 there are topological astral configurations m#(b,c;d,e) that are not stretchable. The smallest such configuration is 18#(6,1;5,8), shown in Figure 4.6.6.



(a) (b)

Figure 4.6.5. Two astral topological configuration of Type 1. (a) The configuration 13#(5,1;4,6), the smallest such configuration. (b) Another (34_4) topological astral configuration, that can be specified as 17#(5,1;4,6).

Berman's paper [B9] contains a number of other results that we cannot get into here. It should only be mentioned that there are examples of essentially chiral configurations, that is, configurations that are astral under a cyclic symmetry group but are not even isomorphic to an astral configuration with mirror symmetries. Such configurations do not exist for geometric astral configurations. An example of an essentially chiral astral configuration is shown in Figure 4.6.7. A complete description of such configurations is still lacking, as is also any treatment of k-astral topological configurations for $k \geq 3$.

An interesting conjecture in [B9] can be formulated as follows:

Conjecture 4.6.1. If the outer orbit of points in a astral topological configuration m#(b,c;d,e) with dihedral symmetry is on a circle of radius 1, then the inner orbit is on a circle of radius r, where

$$0 < r < \cos((b-c-1)\pi/m)/\cos(\pi/m).$$

For a study of simplicial arrangements of pseudolines see [B8].

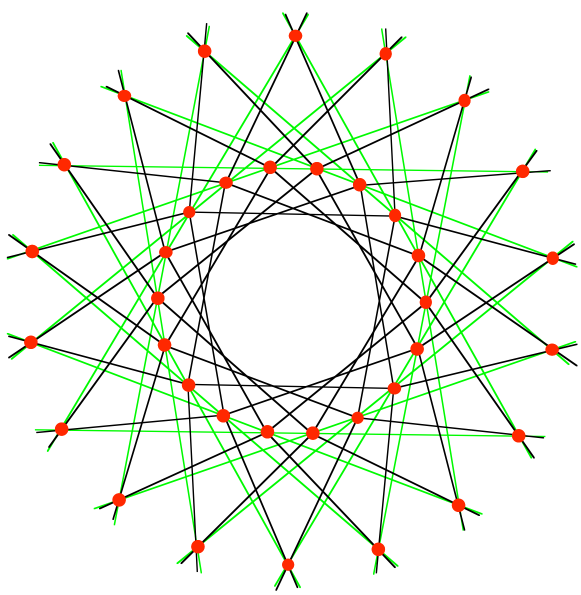


Figure 4.6.6. The astral topological configuration 18#(6,1;5,8), the smallest configuration with m divisible by 6 that is not a geometric astral configuration.

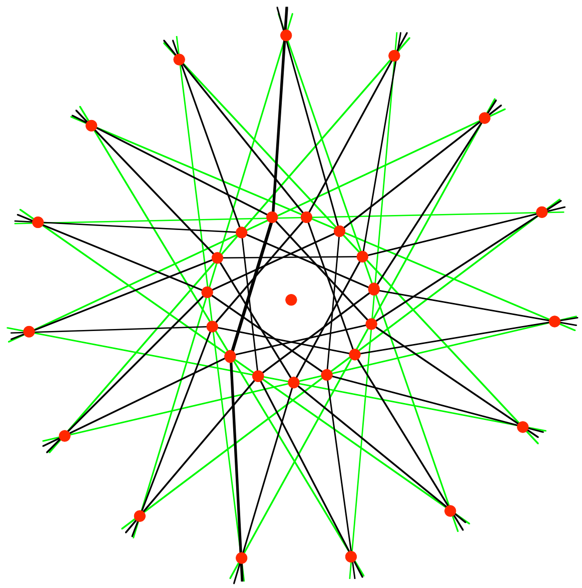


Figure 4.6.7. A chiral astral configuration (30_4) that is not isomorphic to any astral configuration that admits mirror symmetries. One of the pseudolines is drawn by heavy segments. It can be labeled 15#(6,1;5,7), and the fact that $6 - 1 \equiv 1 \bmod 2$ while $7 - 5 \equiv 0 \bmod 2$ shows that it cannot be dihedral of either Type 1 or Type 2.

Exercises and problems 4.6

1. Justify the claims that the configurations in Figures 4.6.5a and 4.6.6 are the smallest of their kind.

2. What is the smallest topological 5-configuration you can find?

3. How many distinct astral topological configurations (26_4) and (30_4) can you find?

4. What are the smallest topological 3-astral 4-configurations you can find?

5. Generalize the statement (in the proof of 4.6.2) that the symbol m#(5,4;1,4) describes a valid topological astral 4-configuration for each $m \geq 11$. What about analogous statements for 3-astral configurations?