4.5 FLORAL CONFIGURATIONS

Floral configurations provide a means of visualizing some rather large configurations in a pleasing and visually accessible way. The topic was initiated by J. Bokowski late in 2006, in an email message asking whether the configuration attached to the message has been already found by anybody. The configuration in question is shown in Figure 4.5.1. It was a completely new type of configuration, and the curiosity it engendered quickly led to a wealth of configurations analogous in some sense. Collectively they became known as "floral configurations". The results of the early investigations of these configurations have been presented in [B12]; most will be reviewed here, together with new developments. Many of the latter arose in discussions with the coauthors of [B12], and I owe them sincere gratitude.

Loosely speaking, a floral configuration is a connected configuration that has a number of parts, called florets, arranged within the configuration in a symmetric way.

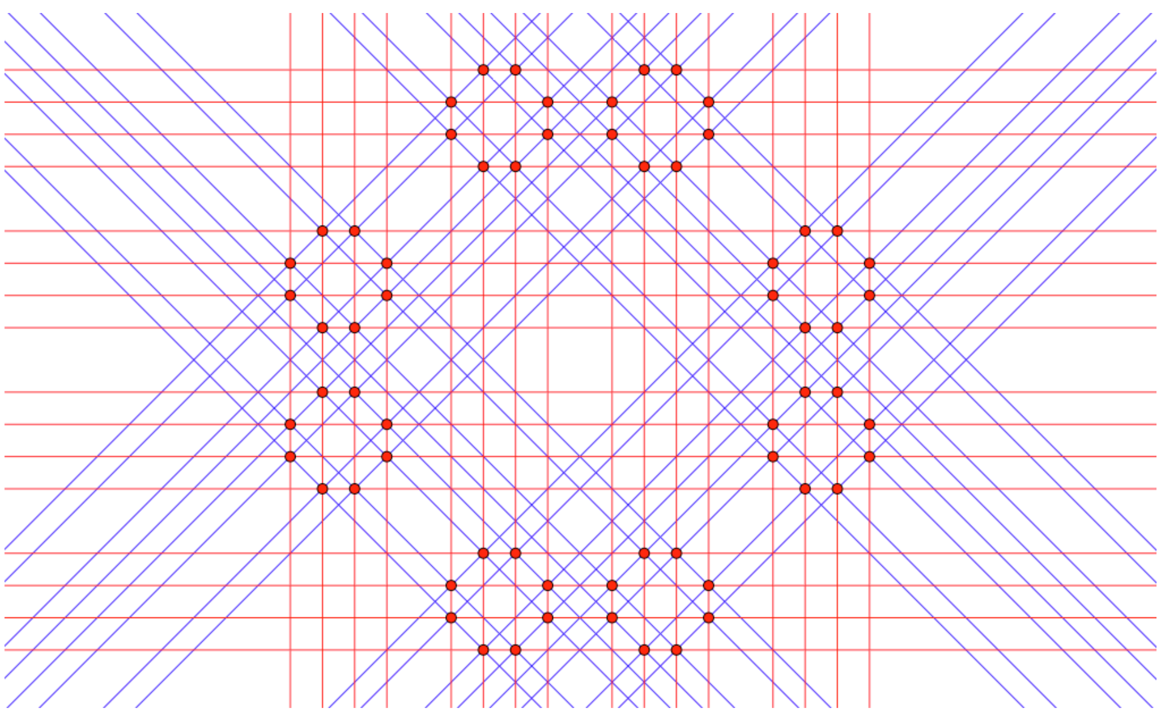


Figure 4.5.1. The first "floral" configuration, from an email by J. Bokowski on October 28, 2006.

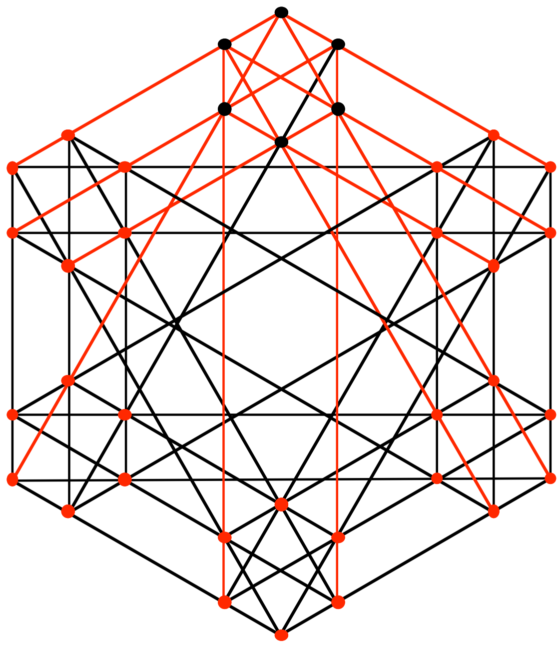
Reasonable people may (and do) differ regarding what level of generality is reasonable in the context of floral configurations. For our purposes the following approach seems most appropriate; it is more general than the approach in [B12], with which we shall compare it near the end of this section.

Definition 4.5.1. A **floret** is a collection of points and lines, with prescribed incidences. A **floral configuration** is a configuration that consists of a collection of florets, such that the symmetry group of the configuration acts transitively on the florets.

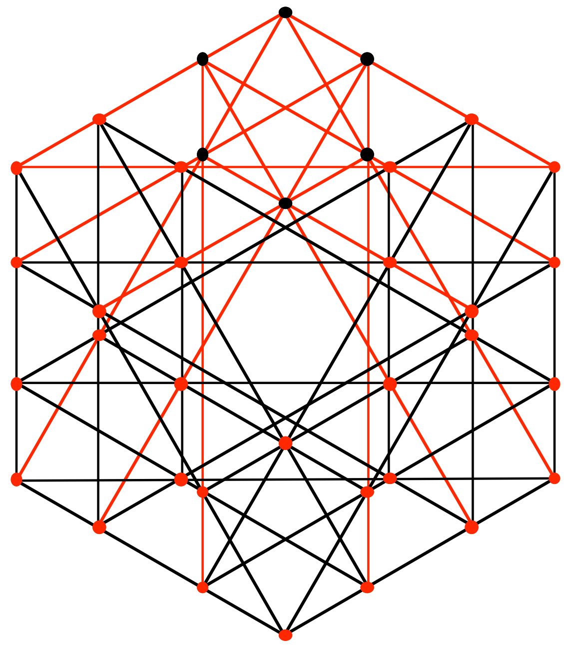
In view of the generality of the concept, it is not surprising that one may distinguish several varieties of floral configurations. To begin with, there is the question of what symmetry group is being considered. It turns out that dihedral groups are much more productive in this context, and we shall devote most of the section to them. The cyclic groups will be considered briefly afterwards.

Even before discussing methods for the construction of floral configurations, a limitation of their appeal needs to be discussed. It is quite clear that the florets in Bokowski's configuration are easily picked out, as are the ones in parts (a) and (b) of Figure 4.5.2. However, this becomes increasingly more difficult in the other parts of that illustration, and the question arises whether it is appropriate to call all of them "floral". One may wish to restrict consideration to only those configurations in which each floret is contained in a single sector determined by the mirrors of the symmetry group, or in two such sectors – but there really is no obvious and natural delimitation. Hence we shall not make any such restriction, although we shall endeavor to present examples in which the florets can be readily discerned.

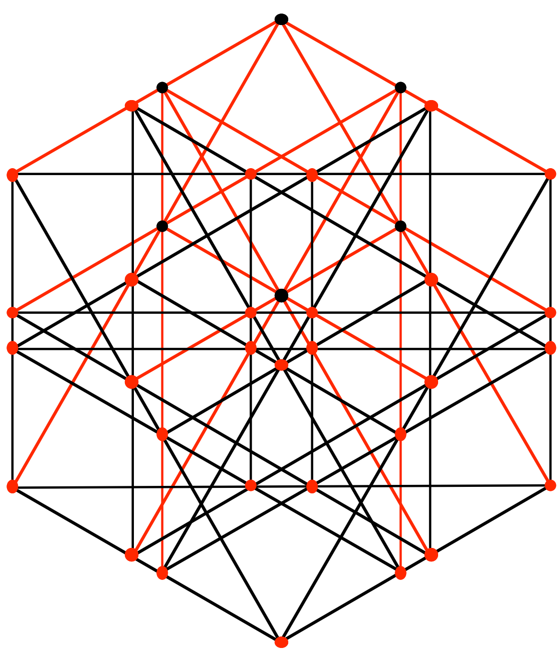
Four construction methods seem to furnish all known examples of floral configurations. Since they depend on the **m**irrors of the dihedral groups, we denote them as constructions $(\mathcal{M}_1)$, $(\mathcal{M}_2)$, $(\mathcal{M}_3)$ and $(\mathcal{M}_4)$.



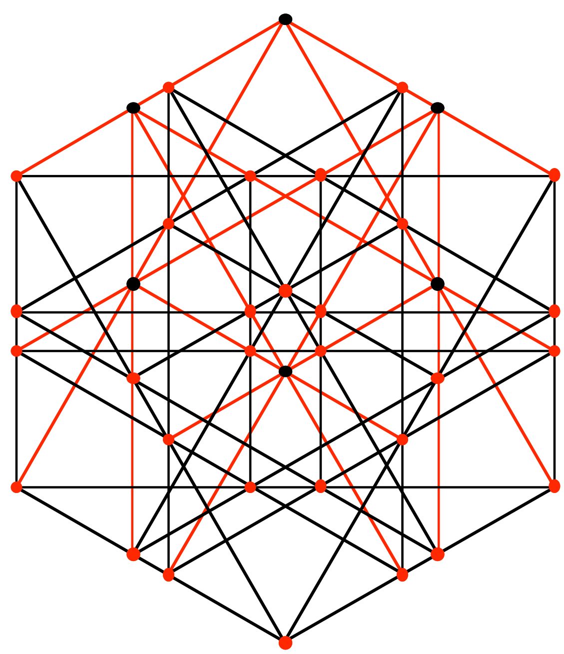
(a)



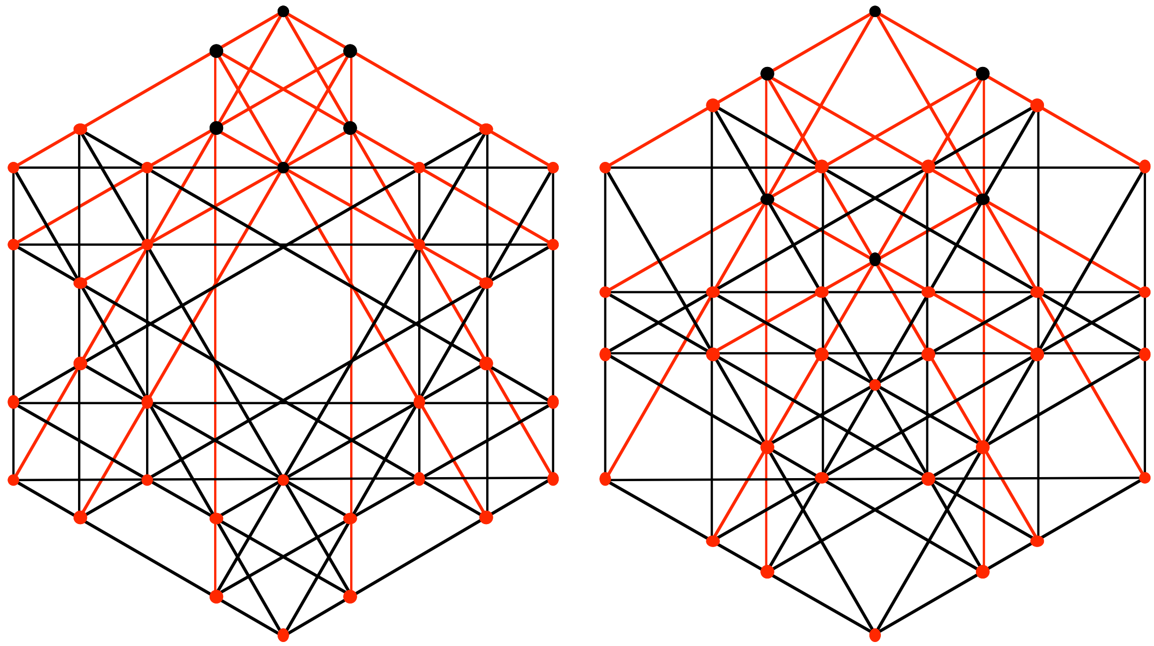
(b)



(c)



(d)



(e) (f)

Figure 4.5.2. The first four floral (36_4) configurations are isomorphic, and differ only in the size of the florets. The florets of the configuration are clearly distinguishable in (a) and (b), but less easily in (c) and especially in (d). In fact, in all four cases the situation is complicated by the fact that two different sets of florets can be picked out. When constructing the configurations, the top florets in (a), (b), and (c) had their top three points on the upper half of the top sides of the overall hexagon, while in (d) they reached beyond the top half. However, with a bit of contemplation it is easy to reverse the perception in all four of the configurations. The diagrams in (e) and (f) show two representations of the same configuration – with some unintended incidences.

Construction ($\mathcal{M}_1$)**.** Let S be a family of s concurrent lines, equiinclined to each other, so that they represent the s mirrors of a dihedral group d_s. Let the **protofloret** F be a [q,t]-configuration such that each of the lines in F is perpendicular to one of the mirrors in S, and no mirror in S is a mirror for F. Then images of the protofloret F under all reflections in the s mirrors of S create *in general* a floral (q,2t)-configuration with 2s florets.

The "in general" part refers to two possibilities of failure of the construction:

- The resulting configuration may be disconnected, hence cannot qualify as a floral configuration;
- There may be some accidental incidences, which make this a representation of the underlying combinatorial configuration – but not a realization of it

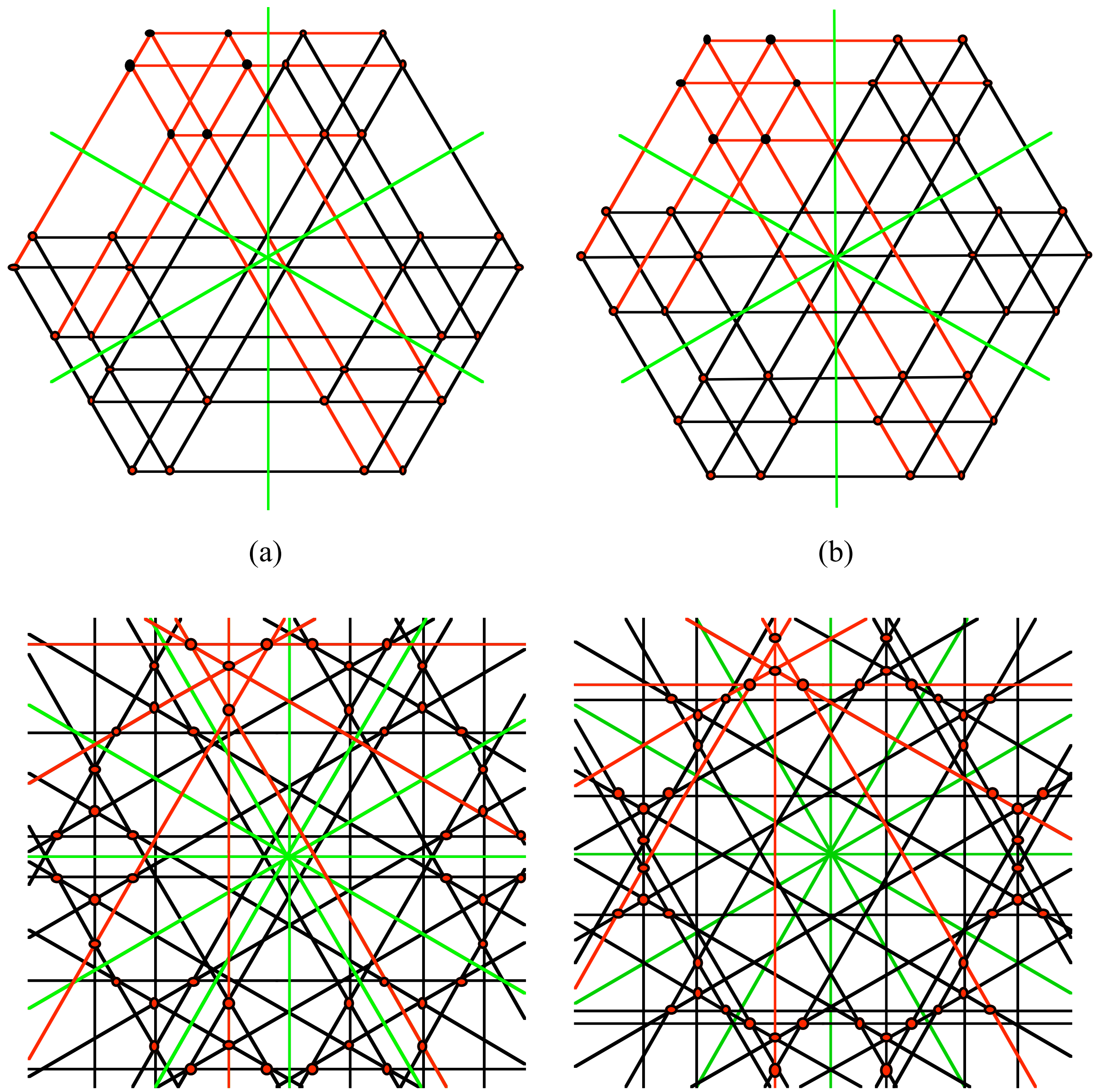
It should be stressed that neither the points of an individual floret, nor its lines, are required to have any non-trivial symmetries – although in many cases they do have them.

An illustration of the construction ($\mathcal{M}_1$) is provided in Figure 4.5.3. There s = 3 (in parts (a) and (b)) or s = 6 (in parts (c) and (d)), with q = 3 and t = 2. The lines of S are shown green. The protofloret F is shown with black points and red lines, while the other points are red and lines are black. Bokowski's original floral configuration, shown in Figure 4.5.1, can also be obtained by the ($\mathcal{M}_1$) construction.

The color-coding in Figure 4.5.3 will be used throughout the present section for all constructions with method ($\mathcal{M}_1$).

It should also be stressed that all floral configurations obtained by the construction ($\mathcal{M}_1$) have at least two degrees of freedom. This is most easily seen by observing that if the distance of the center of symmetry from the centroid of the protofloret is kept fixed, the size of the protofloret can be changed continuously, as can its position with respect to the mirrors. The configuration in Figure 4.5.2 shown earlier was constructed by the ($\mathcal{M}_1$) method as well, and the influence of the size of the protofloret can be discerned easily.

Construction ($\mathcal{M}_2$)**.** As before, let S be a family of s concurrent lines, equiinclined to each other, so that they represent the s mirrors of a dihedral group d_s. Let the **protofloret** F be a [q,t]-configuration such that each of the lines in F is perpendicular to one of the mirrors in S, and precisely one mirror M in S is a mirror for F. Moreover, let no line of F be perpendicular to M. Then images of the protofloret F under all reflections in the s mirrors of S create in general a floral [q,2t]-configuration with s florets.

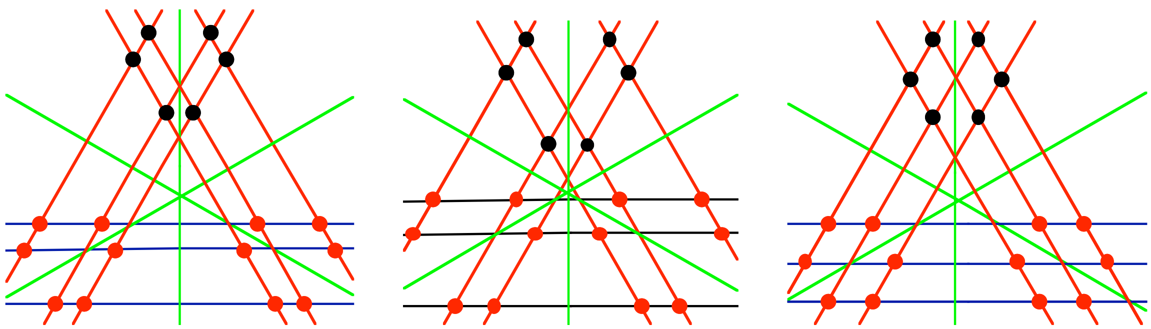


(a) (b) (c) (d)

Figure 4.5.3. Four floral (3,4)-configurations formed using construction ($\mathcal{M}_1$). In (a) and (b) the protofloret F is a (6_3, 9_2) configuration, with points at the vertices of an isogonal hexagon. In each case the result is a (36_3, 27_4) configuration with symmetry group d_3; the two configurations are isomorphic. In (c) and (d) the configurations are (48_3, 36_4) with symmetry group d_6, and the protofloret F consists of the vertices of an equilateral triangle and its center, the sides of the triangle, and its mirrors; hence it is a (4_3, 6_2) configuration. In all parts the lines incident with F are shown in red, the mirrors in S are shown green.

An example of construction ($\mathcal{M}_2$) is provided in Figure 4.5.4. There s = 3, q = 2, and t = 2. The result is a floral [2,4]-configuration $(18_2, 9_4)$. Figure 4.5.2 also shows configurations obtained by method ($\mathcal{M}_2$). As with ($\mathcal{M}_1$), unintended incidences may occur. This happened, for example, in Figure 4.5.2, parts (e) and (f).

The floral configurations constructed using ($\mathcal{M}_2$) have at least one degree of freedom for continuous changes — the size of the protofloret relative to its distance from the center of symmetry. This is illustrated in parts (a) and (b) of Figure 4.5.4. In some cases the floret itself may have continuous changes in shape; this is shown in part (c).



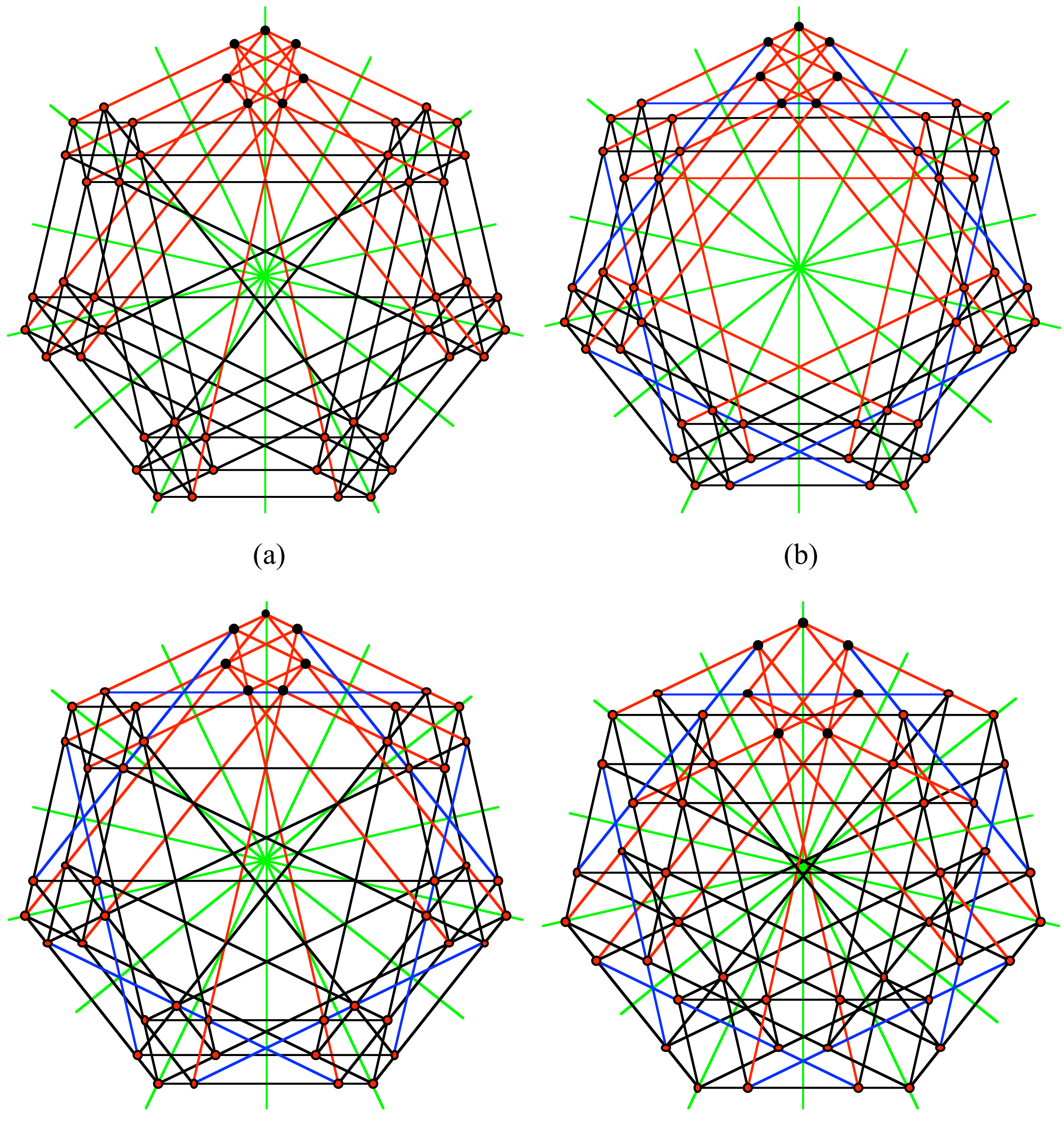
(a) (b) (c)

Figure 4.5.4. Three isomorphic floral configurations $(18_2, 9_4)$ obtained by method ($\mathcal{M}_2$). The florets in (a) and (b) differ only in size, those in (c) differ in shape.

Construction ($\mathcal{M}_3$) starts with a floral [q,k]-configuration C constructed by method ($\mathcal{M}_2$). Assuming, for ease of formulation, that the mirror M of the protofloret F is vertical, we look for the uppermost points on florets F' and F" symmetric with respect to M. If each of these two florets has a single highest point X' and X", respectively, we focus on the corresponding points Y' and Y" in the protofloret F, and on a pair of lines, symmetric with respect to M, which go to points Z' and Z" on F that are lower that Y' and Y". We create a new protofloret F* by deleting from F the two lines just mentioned, and introducing a horizontal line H containing the two points Z' and Z". The protofloret F* is not a configuration, since it has two points (Y' and Y") that are incident with only q–1 lines. Mirroring the changes we made to get F* from F to all the florets of C, we now utilize the presence of a degree of freedom in floral configurations constructed like C by method ($\mathcal{M}_2$), and change the size of the protofloret F* until the line H passes through the highest points on F' and on F" — if this is possible. Then, if there are no unintended incidences, this renders the whole collection of points and lines into a floral [q,k]-configuration.

An illustration of the ($\mathcal{M}_3$) construction is shown in Figure 4.5.5. In part (a) we have a (49_4) floral configuration constructed by the ($\mathcal{M}_2$) method, and in the other parts are three floral configurations obtained from it by ($\mathcal{M}_3$). The newly introduced lines in these configurations are shown in blue. In distinction from the other floral configurations we have seen, configurations obtained by ($\mathcal{M}_3$) have lines that are incident with three florets each. It also should be noted that although in many cases the florets F' and F" are the ones nearest to F, this is not always the case. In Figures 4.5.6 and 4.5.7 we show floral configurations (56_4) in which the points of the protofloret are seven of the eight vertices of a regular octagon. In the second of these the special lines go to non-adjacent florets.

Construction ($\mathcal{M}_4$) is analogous to ($\mathcal{M}_3$), but it starts with a floral [q,k]-configuration C constructed by either method ($\mathcal{M}_1$) or method ($\mathcal{M}_2$). Assuming, for ease of formulation, that a mirror M mapping the protofloret F to its adjacent floret F° is vertical, we look for the uppermost points on florets F' and F" symmetric with respect to M, and adjacent to F and F°, respectively. If each of these two florets has a single highest point X' and X", respectively, we focus on the corresponding points Y' and Y" in the protofloret F and the floret F°, and on a pair of lines, symmetric with respect to M, which go to points Z' and Z" on F and F° that are lower than Y' and Y". We create a new protofloret F* by deleting from F the line just mentioned and omitting its companion from F°, and introducing a horizontal line H containing the two points Z' and Z". The protofloret F* is not a configuration, since it has a point (namely Y') that is incident with only $q-1$ lines. Mirroring the changes we made to get F* from F to all the florets of C, we now utilize the presence of a degree of freedom in floral configurations constructed like C by method ($\mathcal{M}_1$), and change the size of the protofloret F* until the line H passes through the highest points on F' and on F" — if this is possible. Then, if there are no unintended incidences, this renders the whole collection of points and lines into a floral [q,k]-configuration.



(a) (b) (c) (d)

Figure 4.5.5. A floral configuration (49_4) obtained using $(\mathcal{M}_2)$ is shown in (a). From it three distinct configurations (49_4) are obtained by construction $(\mathcal{M}_3)$. The special lines resulting from the construction are shown in blue.

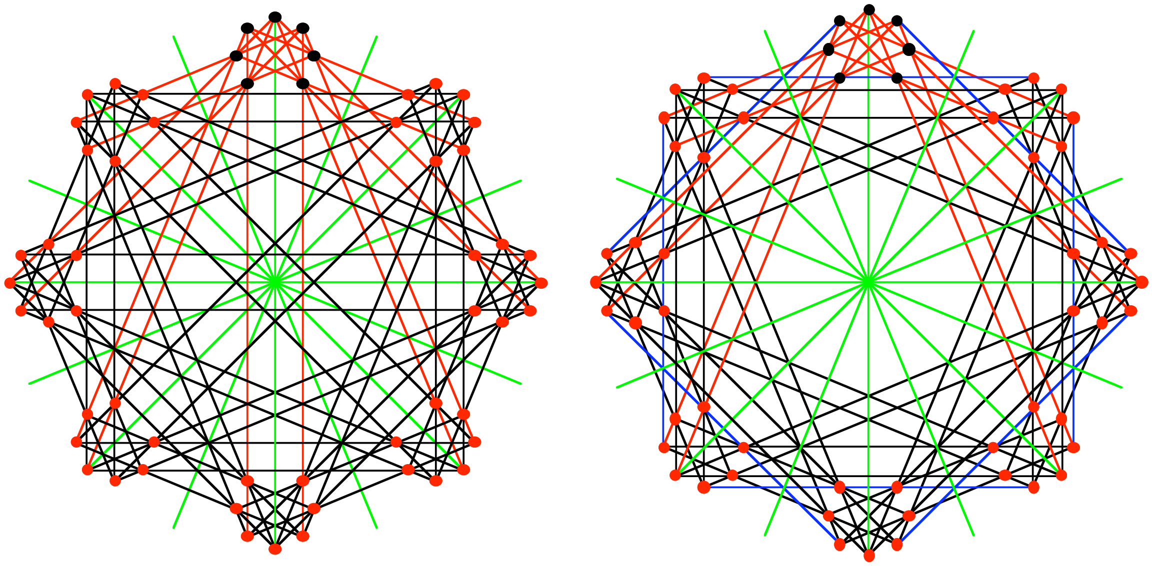


Figure 4.5.6. A floral (56_4) configuration obtained by the $(\mathcal{M}_2)$ construction, and another (56_4) configuration resulting from it by the $(\mathcal{M}_3)$ method. The points of the protofloret are seven of the eight vertices of a regular octagon. The protofloret has d_1 symmetry, the configurations have d_8 symmetry.

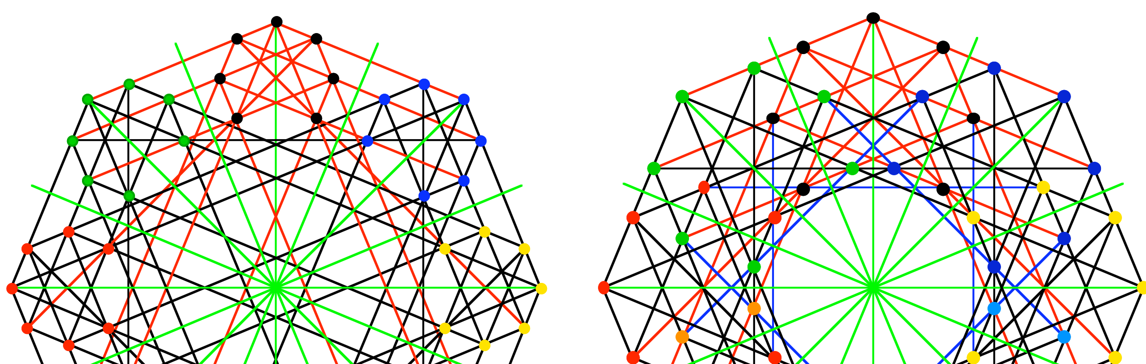


Figure 4.5.7. The same procedure as in Figure 4.5.6, except that the special line determined by two points of the protofloret was aimed not at adjacent florets but at a pair of more distant florets. To achieve the intended incidence, the size of the protofloret had to be increased, leading to a situation similar to that in Figure 4.5.2(d). As a visual aid, the points of each floret were given distinct colors, matched in the two parts.

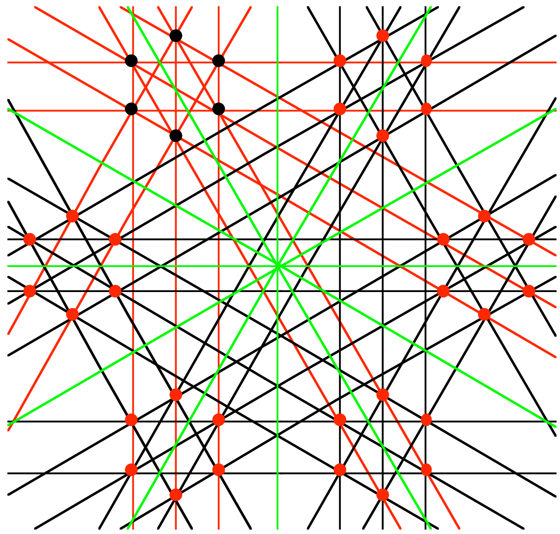
Construction ($\mathcal{M}_4$) is illustrated in Figures 4.5.8 and 4.5.9, by floral configurations (36_4) and (128_4). It should be noted that in contrast to the other constructions, ($\mathcal{M}_4$) leads to configurations in which some lines are incident with four florets. It is also worth mentioning that, as in construction ($\mathcal{M}_3$), instead of "adjacent" florets in some cases florets lying farther away may be used.

In order to illustrate the esthetic appeal of floral configurations, and their great variety, we shall now present a number of examples. Most deal with the more versatile ($\mathcal{M}_1$) construction.

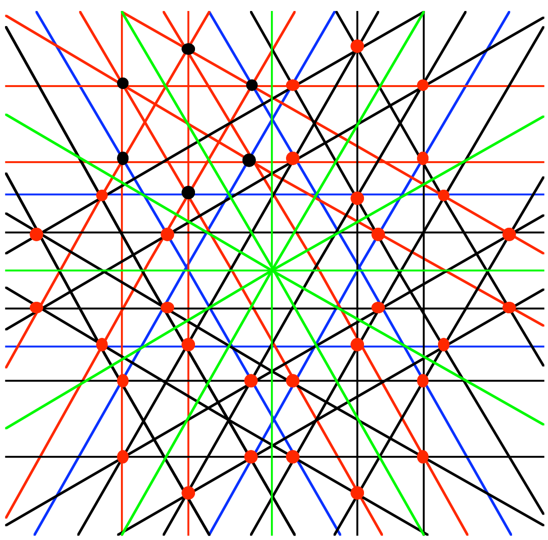
Figure 4.5.10 shows a floral [5,4]-configuration (120_5, 150_4), while Figure 4.5.11 has a floral [3,4]-configuration (72_3, 54_4), both obtained using construction ($\mathcal{M}_1$). A floral configuration (72_4) obtained by the same method is shown in Figure 4.5.12. Figure 4.5.13 shows a (98_4) floral configuration in which the points of the protofloret are at vertices of a regular heptagon, and the lines are diagonals of that heptagon. In contrast, the protofloret in Figure 4.5.14 is a (6_4, 12_2) configuration without any symmetry, used to construct by method ($\mathcal{M}_1$) a floral (108_4) configuration with d_9 symmetry.

Construction method ($\mathcal{M}_2$) is illustrated by Figure 4.5.15 that shows a floral (150_3, 75_6) configuration with d_5 symmetry, in which the protofloret is a (15_3) configuration with d_5 symmetry; analogs of this configurations are the only example found so far of floral configurations in which each line is incident with six points. Construction method ($\mathcal{M}_2$) also yields the (72_4) configuration in Figure 4.5.16, in which the points of the protofloret are at the vertices of an isogonal dodecagon; the shape of the protofloret is variable.

Another method of constructing floral configurations starts with a floral [q,k]-configuration F_2, with protofloret F_1. Using F_1 as protofloret we can construct a **3-strata** floral configuration F_3. With this terminology, F_2 would be a 2-strata configuration, and F_1 a 1-stratum configuration. This construction method is illustrated in Figure 4.5.17. It shows a 3-strata floral (4,8)-configuration (250_4,125_8), in which the protofloret F_1 has points at the vertices of a regular pentagon. The second stratum is a (25_4) floral configuration obtained by the ($\mathcal{M}_3$) method; with it as protofloret the complete configuration is obtained by the ($\mathcal{M}_1$) construction.



(a)



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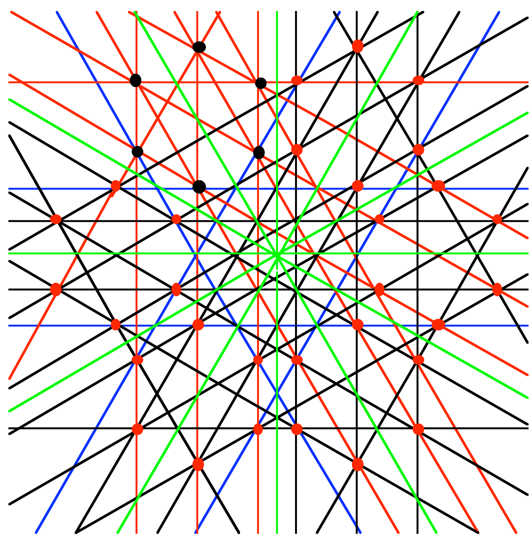

(c)

Figure 4.5.8. Floral configurations (36_4). The protofloret in (a) has points at the vertices of a regular hexagon, and symmetry group d_2; the configuration is obtained by the ($\mathcal{M}_2$) method and has symmetry group d_6. The other two configurations are obtained from it by the ($\mathcal{M}_4$) construction, which involves changing the size of the protofloret, and deleting different lines from the original.

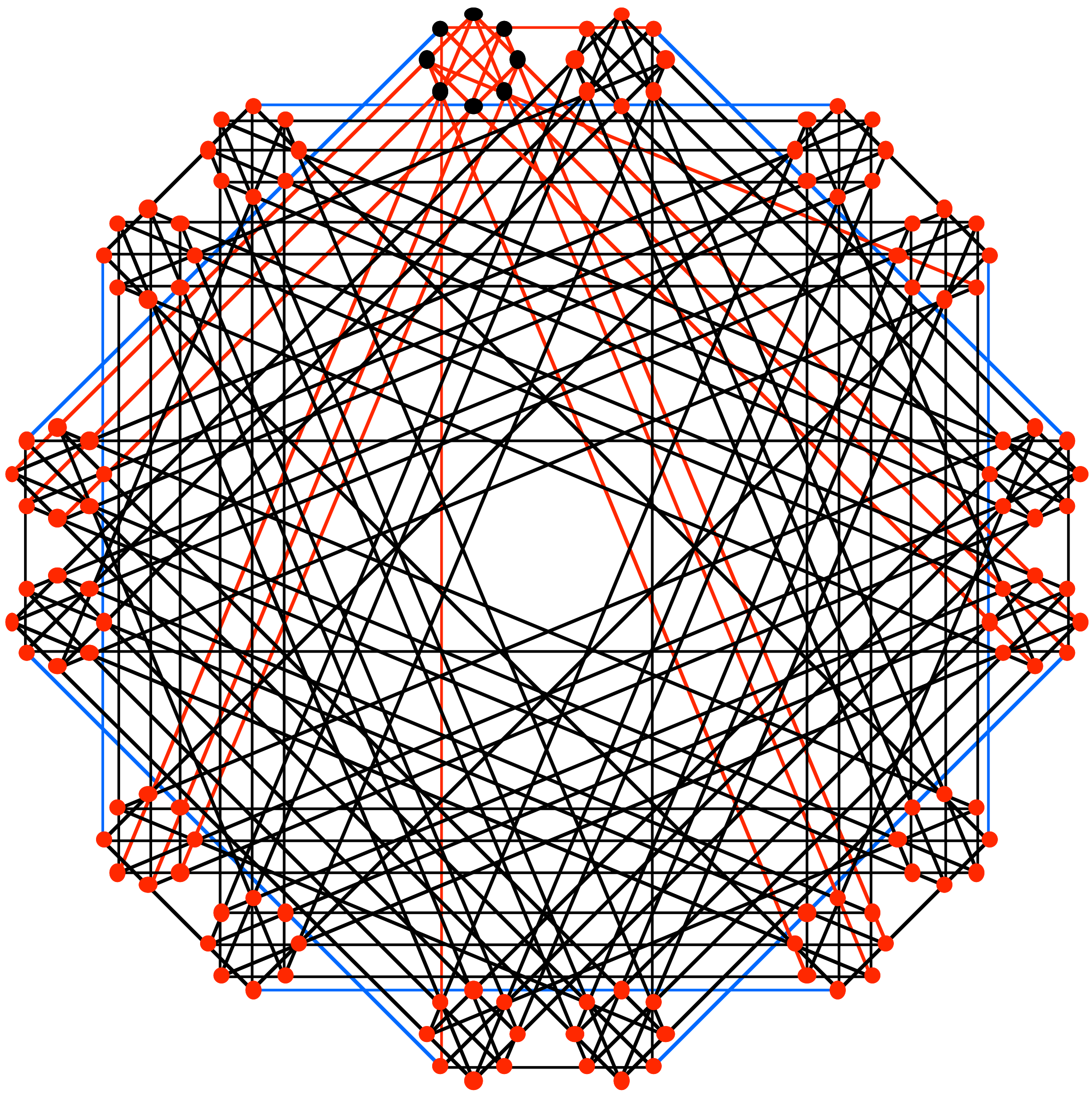

Figure 4.5.9. A floral configuration (128_4) with 16 florets and symmetry group d_8, obtained by construction $(\mathcal{M}_4)$ from a (128_4) configuration reached through the $(\mathcal{M}_1)$ method. The special lines are again shown in blue, but the mirrors are not shown.

Clearly, this procedure could be repeated to 4-strataconfigurations, and so on; however, the diagrams become for to crowded for visual comprehension.

Here is a brief comparison of our floral configurations with the presentation in [B12]. The main and fundamental difference is that in [B12] the protoflorets are considered as only sets of points that are restricted to coincide with either the vertices of a

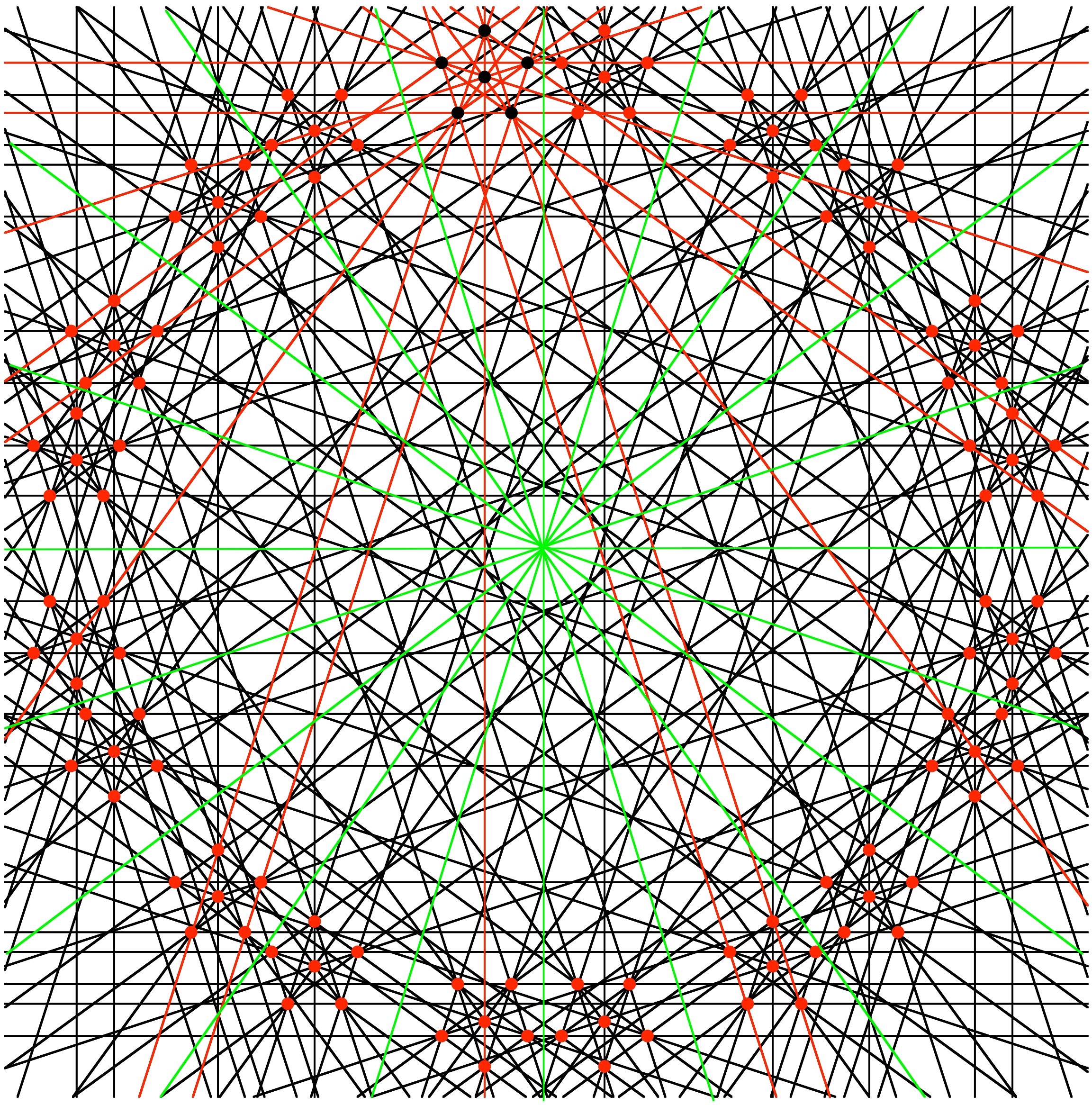

Figure 4.5.10. A floral [5,4]-configuration (120_5, 150_4), constructed by method ($\mathcal{M}_1$). The protofloret is a (6_5, 15_2) configuration consisting of the vertices of a regular pentagon and its center, and all the lines determined by these six points. The protofloret has symmetry group d_5, the configuration has d_{10}.

regular polygon, or the vertices of an isogonal but not regular polygon; a further restriction related the symmetries of the protofloret to those of the configuration. The main exposition in [B12] is restricted to 4-configurations, with more general types mentioned

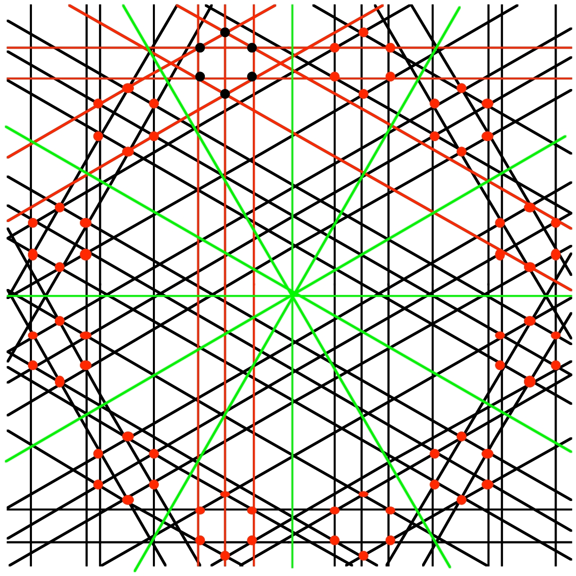

Figure 4.5.11. A floral [3,4]-configuration (72_3, 54_4) obtained using construction ($\mathcal{M}_1$), and with symmetry group d_6. The protofloret is a (6_3,9_2) configuration, with symmetry group d_2; there are 12 florets. The color conventions are the same as in Figure 4.5.3.

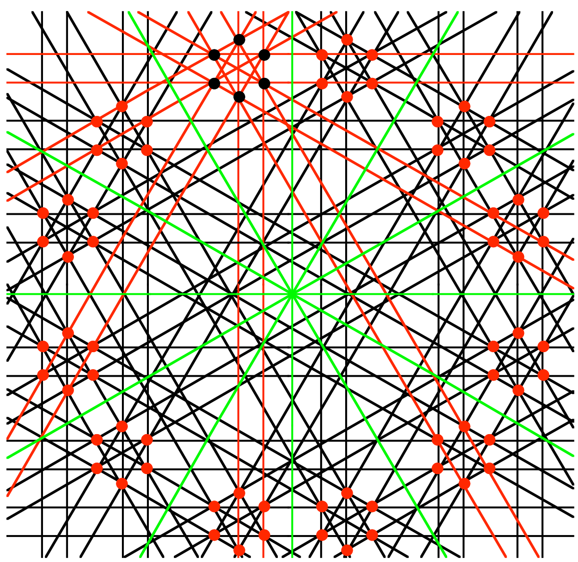

Figure 4.5.12. A floral configuration (72_4) obtained by method ($\mathcal{M}_1$). The protofloret has points at vertices of a regular hexagon, and symmetry group d_3. The configuration has twelve florets and symmetry group d_6.

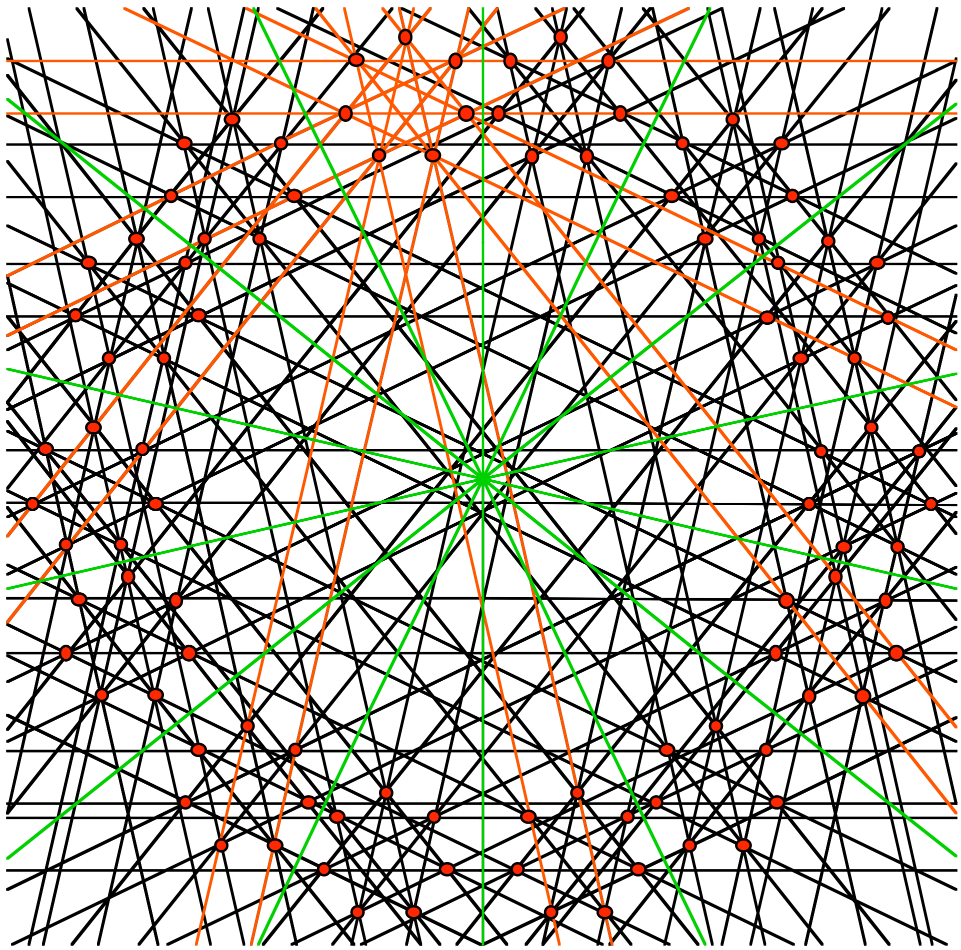

Figure 4.5.13. A floral configuration (98_4) obtained by method ($\mathcal{M}_1$). Both the protofloret and the configuration have symmetry group d_7.

only briefly as "generalized floral configurations". The lines of the protofloret are left to be examined in each case. With these differences in mind, the classification of floral configurations into five varieties can be explained in our terminology as follows. Varieties (A) and (C) of [B12] are obtained by the ($\mathcal{M}_1$) construction, (B) and (D) by the ($\mathcal{M}_2$) method. The protoflorets in (A) and (B) have points coinciding with vertices of isogonal, non-regular polygons, those in (C) and (D) at vertices of regular polygons. Variety (E) consists of configurations obtained by the ($\mathcal{M}_3$) method; they have no degrees of freedom beyond similarities.

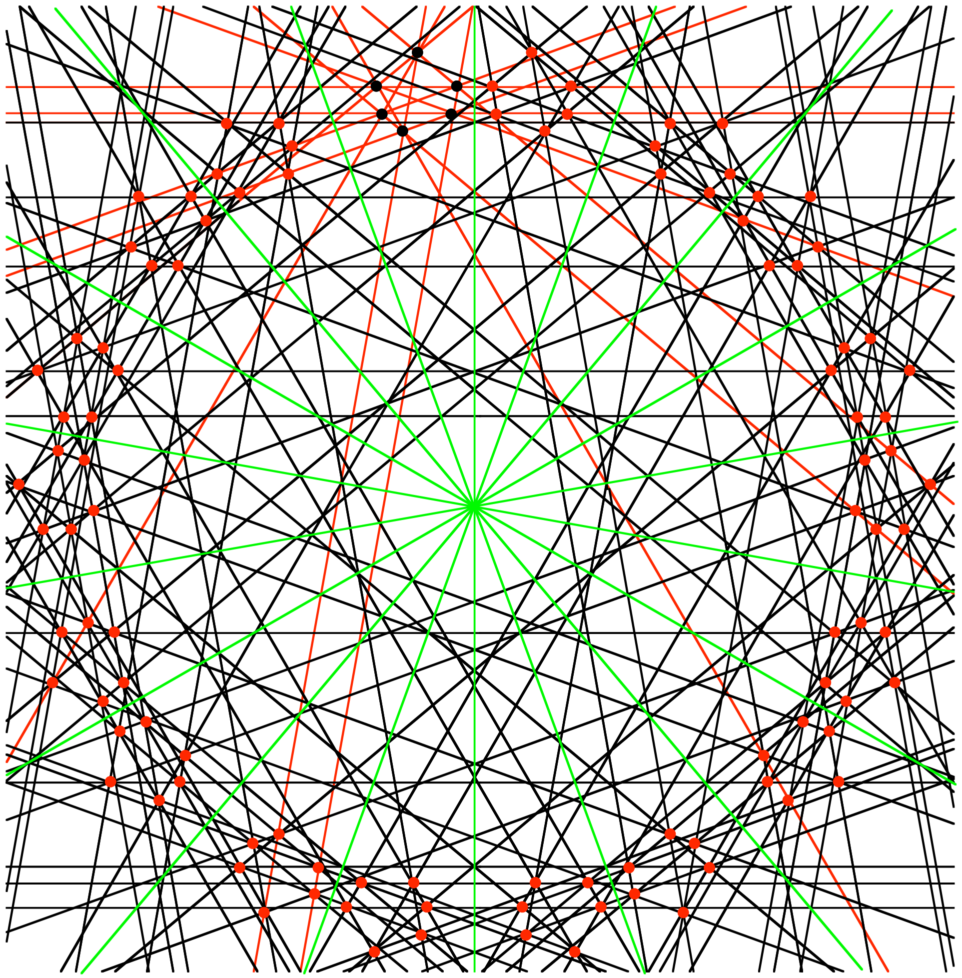

Figure 4.5.14. A floral (108_4) configuration with protofloret a $(6_4, 12_2)$ configuration devoid of any symmetry. The configuration was constructed using method $(\mathcal{M}_1)$ and has 18 florets and symmetry group d_9.

Many of the configurations illustrated in this section do not fit into the classification of [B12], since they do not satisfy the definition of floral configurations adopted there. Those that do are: Figure 4.5.1 is of variety (A), Figure 4.5.16 of variety (B), Figures 4.5.12 and 4.5.13 are of variety (C), Figures 4.5.2, 4.5.5(a) and 4.5.8(a) are of variety (D), and 4.5.5(b,c,d) of variety (E). Figure 4.5.17 is analogous to the generalized floral configuration $(512_4, 256_8)$ in [B12].

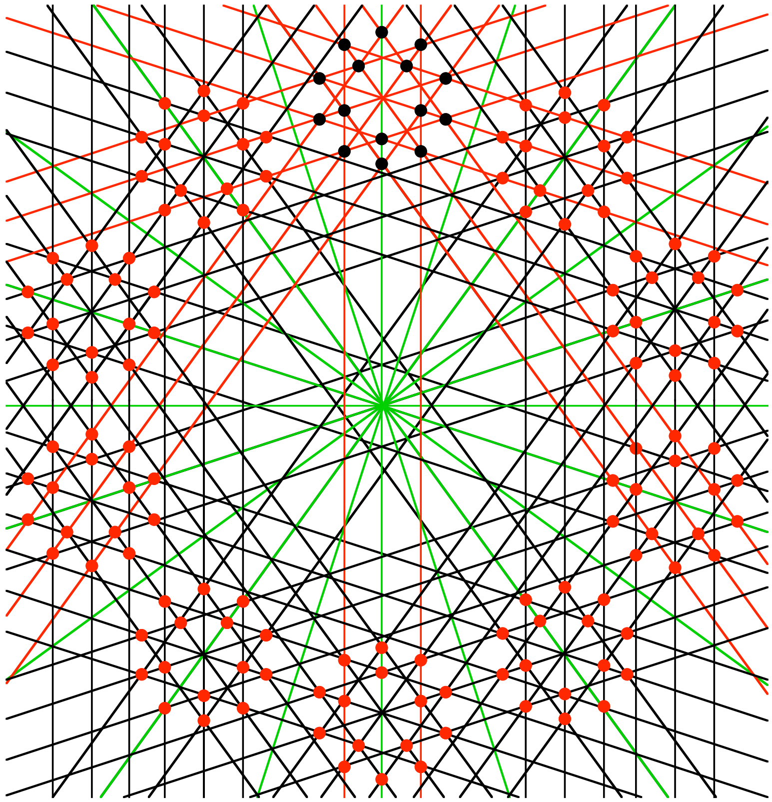

Figure 4.5.15. A $(150_3, 75_6)$ floral configuration with d_{10} symmetry, resulting from the $(\mathcal{M}_2)$ construction using as protofloret a (15_3) configuration with d_5 symmetry.

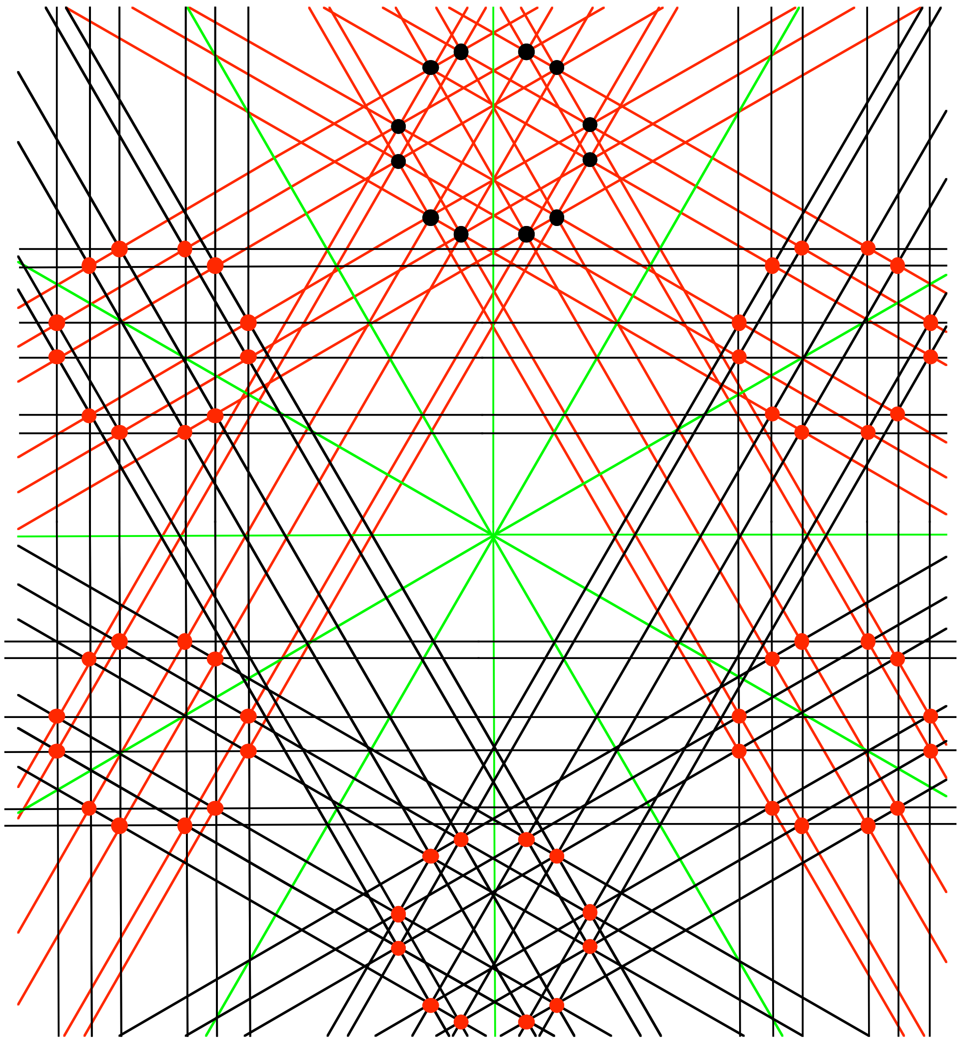

Figure 4.5.16. A floral configuration (72_4) obtained by method $(\mathcal{M}_2)$. The points of the protofloret are at the vertices of an isogonal 12-gon, and the protofloret has symmetry group d_2.

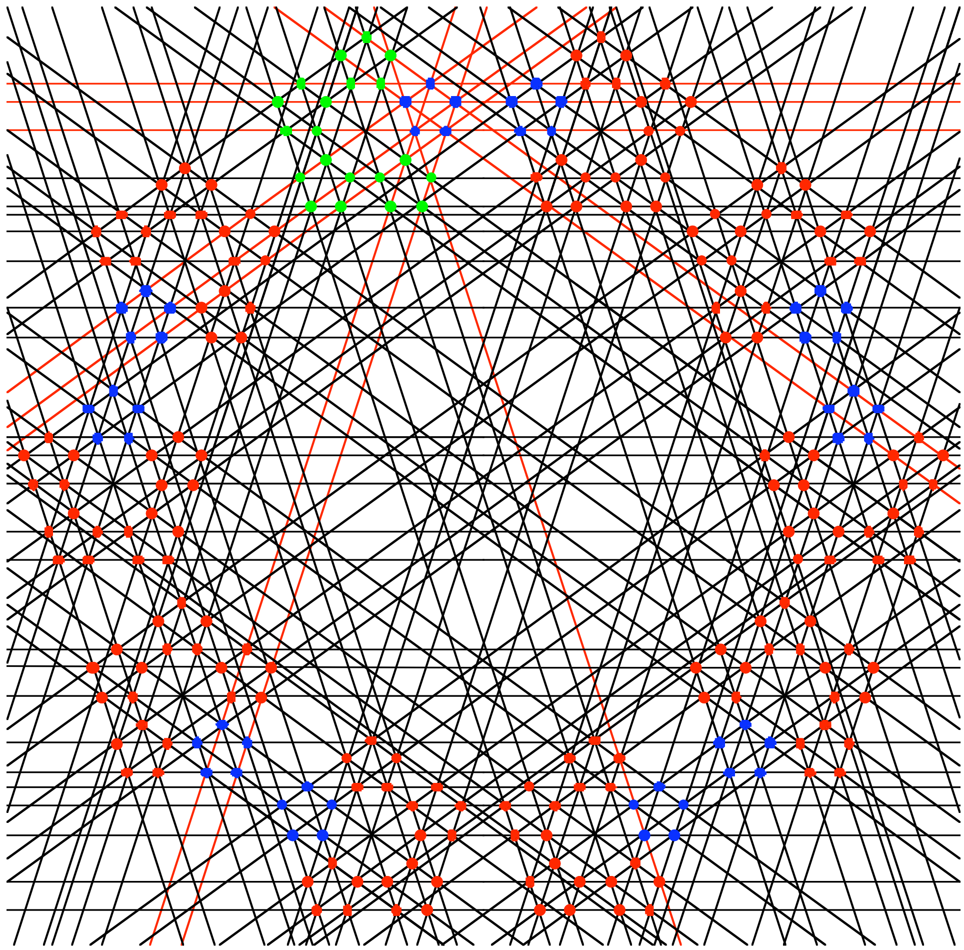

Figure 4.5.17. A floral [4,8]-configuration $(250_4,125_8)$ with three strata. The protofloret F_0 of the first stratum has points at the vertices of a regular pentagon. Using method $(\mathcal{M}_2)$, five copies of the protofloret form a second stratum F_1, which is a floral configuration (25_4). By method $(\mathcal{M}_1)$, ten copies of F_1 form the third stratum F_2. The protoflorets of the first two strata have symmetry group d_5, as does the complete configuration. The protofloret of the first stratum in each second stratum floret is shown in blue, and one second stratum protofloret is shown in green (and blue). The lines incident with one floret of the first stratum are shown in red.

We conclude with a brief description of chiral floral configurations. We have already encountered a wide class of these, when investigating the chiral astral 3-configurations in Section 2. In Figure 4.5.18 we show two examples of such configurations. Many additional examples can be found in Sections 2.7 (a protofloret consists of two points and two lines) and 2.9 among the multiastral 3-configurations.

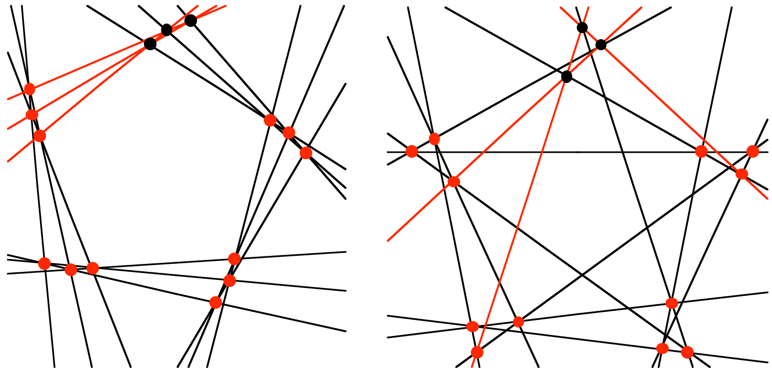

Figure 4.5.18. Two 3-astral configurations (15_3) that are also chiral floral configurations.

A different example of a chiral floral 4-configuration was first presented in [B12], it is reproduced in Figure 4.5.20b. A general method of generating such floral configurations is based on the $(\mathcal{M}_1)$ construction, and illustrated in a simple case in Figure 4.5.19. The crucial step is the replacing of one-half of the florets in the dihedral configuration by their mirror images.

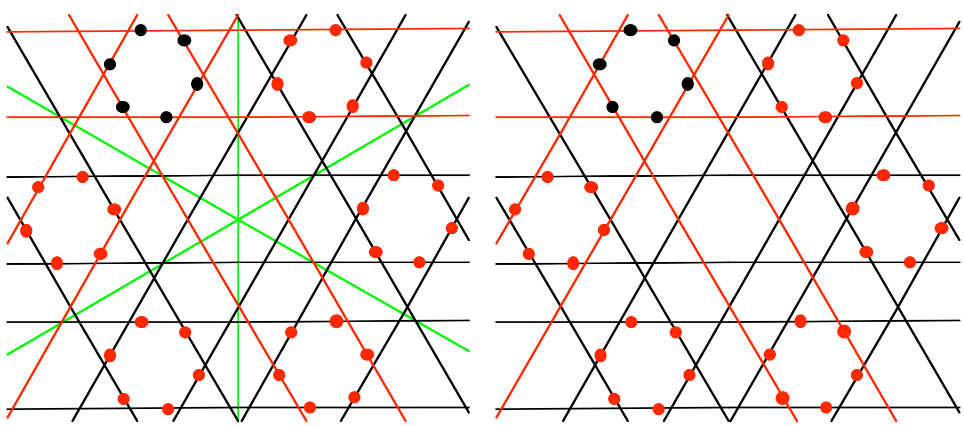

Figure 4.5.19. (a) An application of the $(\mathcal{M}_1)$ construction to a (6_1) configuration (that is, a protofloret consisting of six points and six lines). (b) Replacing one half of the florets by their mirror images yields a chiral floral configuration.

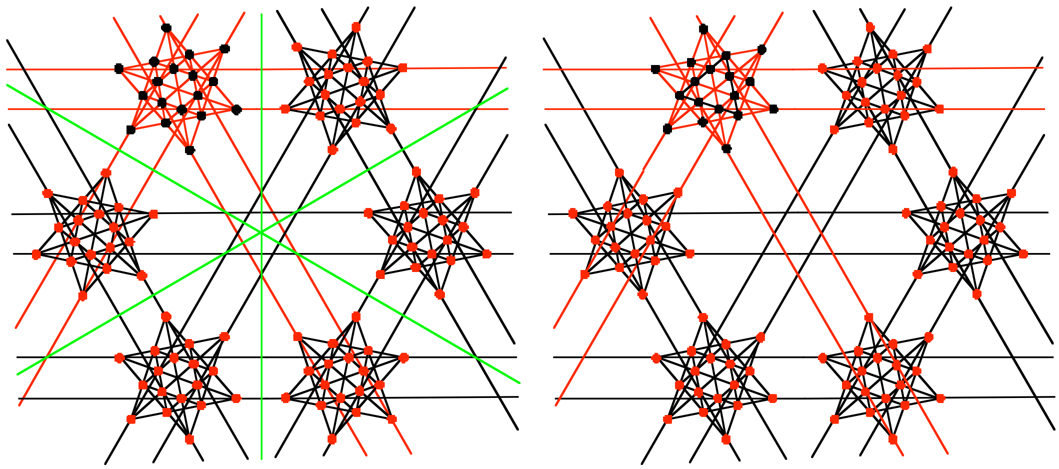

Figure 4.5.20. (a) An adaptation of the ($\mathcal{M}_1$) construction to the case in which not all lines in a protofloret are perpendicular to one of the mirrors. (b) Replacing half of the florets by their mirror images yield a chiral floral configuration.

Exercises and problems 4.5

1. Construct your own floral configurations, using each of the four methods ($\mathcal{M}_i$) .

2. Using the Martinetti "module" shown in Figure 2.4.2, show that a chiral floral (geometric) configuration (n_3) can be constructed for every $n = 10m$, $m \geq 3$.