4.4 UNBALANCED [q,k]-CONFIGURATIONS WITH [q,k] ≠ [3,4]

Very little has been published about geometric [q,k]-configurations with q ≠ k and {q,k} ≠ {3,4}. As mentioned in Section 4.2, a few results were found by Cayley [C2*], using planar sections of configurations of flats of various dimensions generated by families of points in general position. Specific instances will be mentioned below. Some of these methods have been used (mostly in special cases) by later writers.

There is more information about the corresponding combinatorial configurations, much of it due to H. Gropp. Here is a survey of what is known.

For combinatorial [3,5]-configurations (p_3, n_5) the necessary conditions for existence are $3p = 5n$, $p \geq 13$, and $n \geq 11$. Therefore we must have $p = 5r$ and $n = 3r$ for some integer r, so that we are looking at $((5r)_3, (3r)_5)$ configurations with $r \geq 4$. A combinatorial configuration $(20_3, 12_5)$ is shown in Table 4.4.1. From results on the "orchard problem" (see [B33]) it is known that 12 lines determine at most 19 triple points; it follows that no geometric $(20_3, 12_5)$ configuration is possible. Unfortunately, I do not know of any simple proof of the orchard problem result.

1	1	1	2	2	3	3	4	4	5	5	8
2	6	10	6	9	7	13	8	9	6	7	13
3	7	11	14	10	11	17	12	11	10	12	15
4	8	12	15	16	14	18	14	18	17	15	16
5	9	13	18	19	16	19	17	20	20	19	20

Table 4.4.1. A $(20_3, 12_5)$ combinatorial configuration.

There are interesting connections between combinatorial configurations $(12_5, 20_3)$ and Steiner triple systems S(2,3,13). We recall that a Steiner triple system S(2,3,v) is a collection of triplets from a v-element set, such that each pair of elements occurs in one and only one triplet. It is well known that a Steiner triple system S(2,3,v) exists if and only if v º 1 or 3 (mod 6). (For general information about Steiner triple systems see, for example, [B29, Section 10.3] or [R5, pp. 388 – 390].) The unique system S(2,3,7) is one of the incarnations of the combinatorial configuration (7_3), (the Fano plane) which we discussed in Section 2.1. There is unique system S(2,3,9). There are two (and only two) non-isomorphic systems S(2,3,13), which are of interest here. Information about them is presented in Tables 4.4.2 and 4.4.3, taken from [M4]. For a history of the S(2,3,13) see Gropp [G12].

1	1	1	1	1	1	2	2	2	2	2	3	3
2	4	6	8	10	12	4	5	8	9	11	4	5
3	5	7	9	11	13	6	7	10	12	13	8	12

3	3	3	4	4	4	5	5	5	6	6	7	7
6	7	9	7	10	11	6	8	9	8	9	8	10
10	11	13	9	13	12	13	11	10	12	11	13	12

Orbit: {1,2,3,4,5,6,7,8,9,10,1,12,13}. Automorphisms group has order 39.
Generators: (1 2 13 5 3 11 6 12 7 9 8 10 4) (1 2 3) (4 10 7) (9 13 12)

Table 4,4.2. The Steiner triple system $S(2,3,13)_1$.

1	1	1	1	1	1	2	2	2	2	2	3	3
2	4	6	8	10	12	4	5	8	9	11	4	5
3	5	7	9	11	13	6	7	10	12	13	8	12

3	3	3	4	4	4	5	5	5	6	6	7	7
6	7	9	7	10	11	6	8	9	8	9	8	10
13	11	10	9	13	12	10	11	13	12	11	13	12

Orbits: {1, 2, 5, 6, 8, 13} {3, 9, 10} {4, 11, 12} {7}. Automorphisms group has order 6.
Generators: (1 2 8) (3 10 9) (4 11 12) (5 13 6) (1 5) (2 6) (3 10) (8 13) (11 12)

Table 4.4.3. The Steiner triple system $S(2,3,13)_2$.

One interesting property of Steiner systems S(2,3,13) is that the deletion of one point and the triplets containing it yields a combinatorial configuration $(12_5, 20_3)$. It is clear that the deletion of different points from the same orbit yields isomorphic configurations. As it happens, deleting points from different orbits of the Steiner systems S(2,3,13) yields non-isomorphic configurations. Hence there are five such configurations $(12_5, 20_3)$. This result is due to Novak [N2]; see also Gropp [G17].

* * * * * *

Concerning values of $r \geq 5$ we shall show that there exist geometric $((5r)_3, (3r)_5)$ configurations for all $r \geq 5$. By duality and polarity, the same is true for configurations $((3r)_5, (5r)_3)$.

Theorem 4.4.1. There exist geometric $((5r)_3, (3r)_5)$ configurations for all $r \geq 5$; moreover, they can be chosen as astral in the extended Euclidean plane.

Proof. The validity of this statement follows at once from the family of configurations illustrated in Figure 4.4.1; clearly, analogous configurations exist for all $r \geq 5$.

Additional examples of geometric [3,5]-configurations are shown in Figures 4.4.2 and 4.4.3.

Cayley [C2*] described a $(21_5,35_3)$ configuration.

* * * * * *

For combinatorial [3,6]-configurations (p_3, n_6) the necessary condition for existence are $p = 2n$ and $n \geq 13$. A combinatorial configuration $(26_3, 13_6)$ is shown in Table 4.4.4. It can also be shown (see [G20]) that combinatorial configurations $((2n)_3, n_6)$ exist for all $n \geq 13$. Gropp [G17] states that there are exactly 787 distinct $(28_3, 14_6)$ combinatorial configurations.

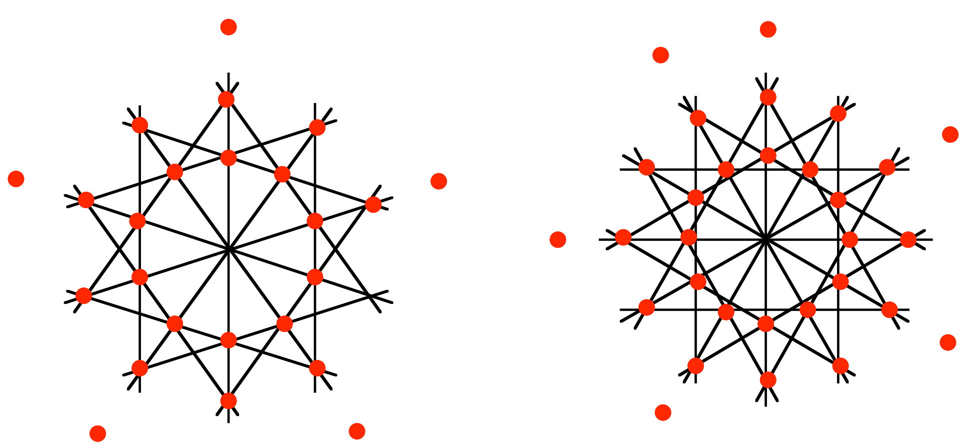


Figure 4.4.1. Typical examples of [3,5]-configurations astral in the extended Euclidean plane. The two examples correspond to r = 5 and r = 6.

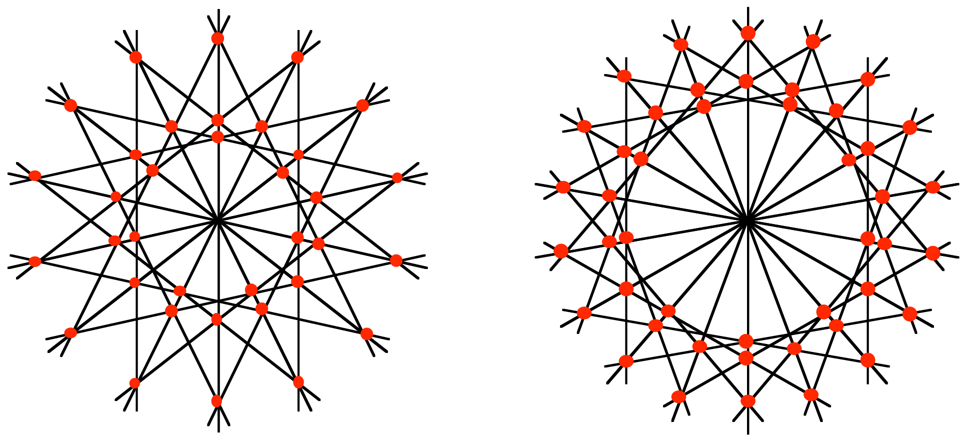


Figure 4.4.2. Examples of astral [3,5]-configurations in the Euclidean plane. These are clearly representatives of an infinite family, and several variants are possible.

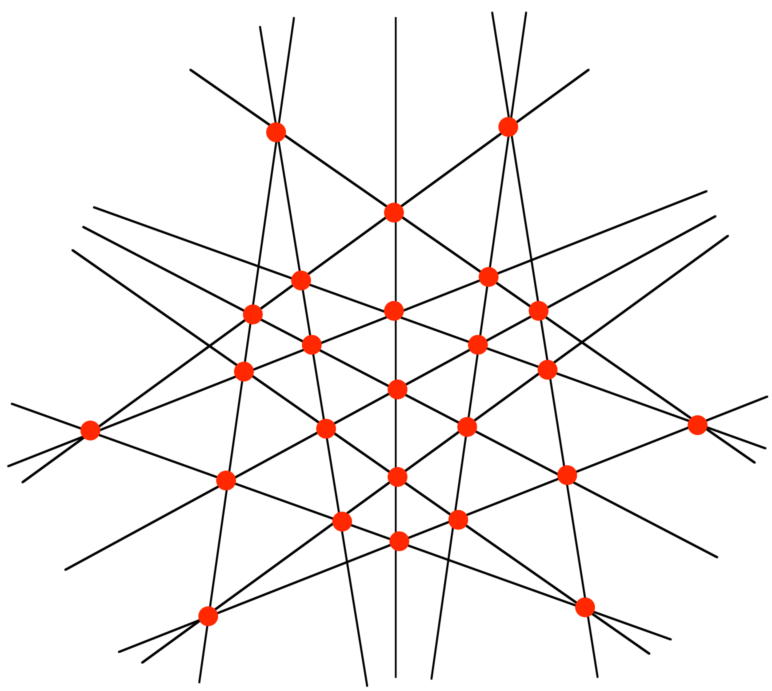


Figure 4.4.3. Another $(25_3, 15_5)$ configuration.

There seems to be no geometric $(26_3, 13_6)$ configuration, but I am not aware of any proof. Also, there is a large difference between the case of [3,6]-configurations and the [3,5]-configurations considered above. In the latter case, for all values of n that satisfy the necessary conditions and are beyond a certain limit (in fact n ≥ 15), an astral

1	1	1	2	2	3	3	4	4	5	5	6	6
2	7	12	7	8	7	8	9	10	10	11	9	11
3	8	13	12	16	15	13	14	12	16	13	15	14
4	9	14	17	20	23	17	17	22	19	18	18	21
5	10	15	18	25	24	19	20	24	21	20	19	22
6	11	16	21	26	26	22	23	25	23	24	25	26

Table 4.4.4 A $(26_3, 13_6)$ combinatorial configurations (found in 1999 by Xin Chen, at the time a student in one of my classes). The number of distinct $(26_3, 13_6)$ combinatorial configurations seems not to be known.

configuration is possible in the extended Euclidean plane. For [3,6]-configurations this is not the case. We have:

Theorem 4.4.2. For all $r \geq 5$ there exist astral $((6r)_3, (3r)_6)$ geometric configurations in the Euclidean plane.

Proof. In Figure 4.4.4 we show the two typical configurations of this kind for r = 6 and 7. The only known configuration $(30_3, 15_6)$ is not typical; it is shown in Figure 4.4.5, and we have already seen it in Figure 1.6.8. ¨

Thus, except for small values of n, there exist geometric configurations $(2n_3, n_6)$ for all n that are multiples of 3. For no other values of n are any geometric [3,6]-configurations known.

It is clear that geometric [3,k]-configurations and [k,3]-configurations can be constructed for all $k \geq 7$ in analogy to the configurations in Figures 4.4.1, 4.4.2, and 4.4.4. As these are not really interesting, and no additional information seems available, we shall not pursue this topic any farther. Instead, we turn now to [4,k]-configurations and their duals.

For [4,5]-configurations (p_4, n_5) the necessary conditions are $p \geq 17$, $n \geq 16$, $5n = 4p$. Therefore the configurations are necessarily of the form $((5r)_4, (4r)_5)$ for $r \geq 4$. According to Gropp [G20], combinatorial configurations with these parameters exist for

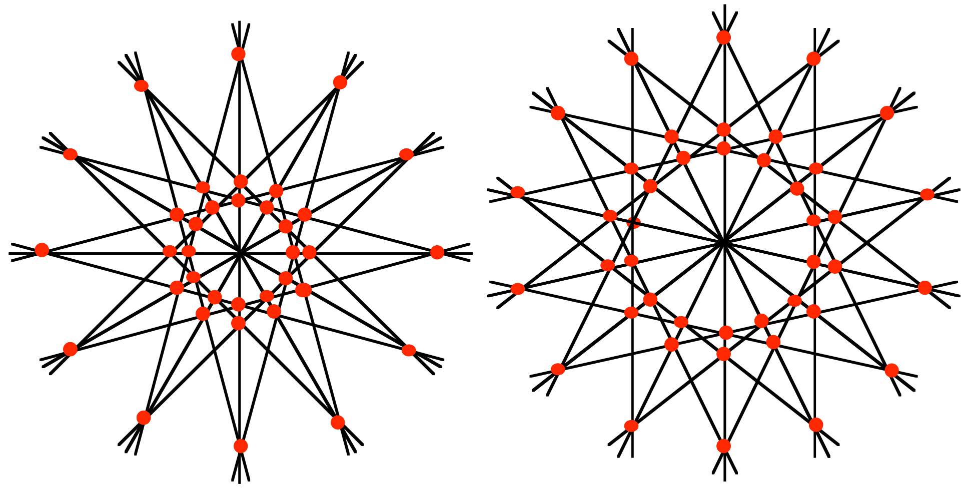


Figure 4.4.4. Configurations (36_3, 18_6) and (42_3, 21_6).

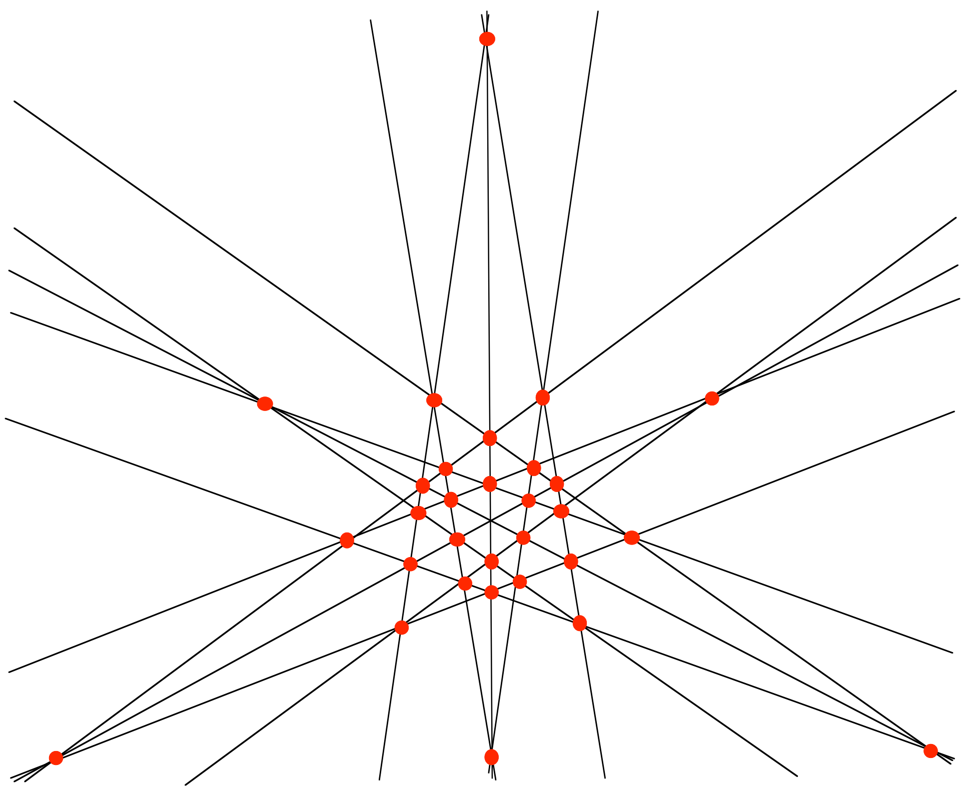


Figure 4.4.5. The only known (30_3, 15_6) configuration.

all r ≥ 4. There seems to be no information available concerning the number of distinct configurations for each value of r.

Concerning topological or geometric [4,5]-configurations, there is an elegant family of geometric configurations ($(5r)_4$, $(4r)_5$) for r ≥ 9. (I do not know whether or not there are any for $r \leq 8$.) Its two smallest members are shown in Figure 4.4.6, and their construction can be explained as follows: Starting for $r \geq 9$ from the 4-astral 4-configurations $((4r)_4)$, such as the ones denoted in Section 3.8 by 9#(3,1;2,4;3,2;3,2) or r#(3,1,2,4,1,3,4,2), additional r points are added at-infinity (in the directions of the quadruplets of parallel lines of the 4-configuration). This yields a $((5r)_4, (4r)_5)$ configuration with five orbits of points and four orbits of lines. Polars of these configurations are [5,4]-configurations. Other [5,4] configurations can be obtained by adding r mirrors to the same 4-configurations $((4r)_4)$, but only for odd $r \geq 9$. The two cases analogous to the ones in Figure 4.4.6 are illustrated in Figure 4.4.7. We used a combination of these methods in Section 4.1.

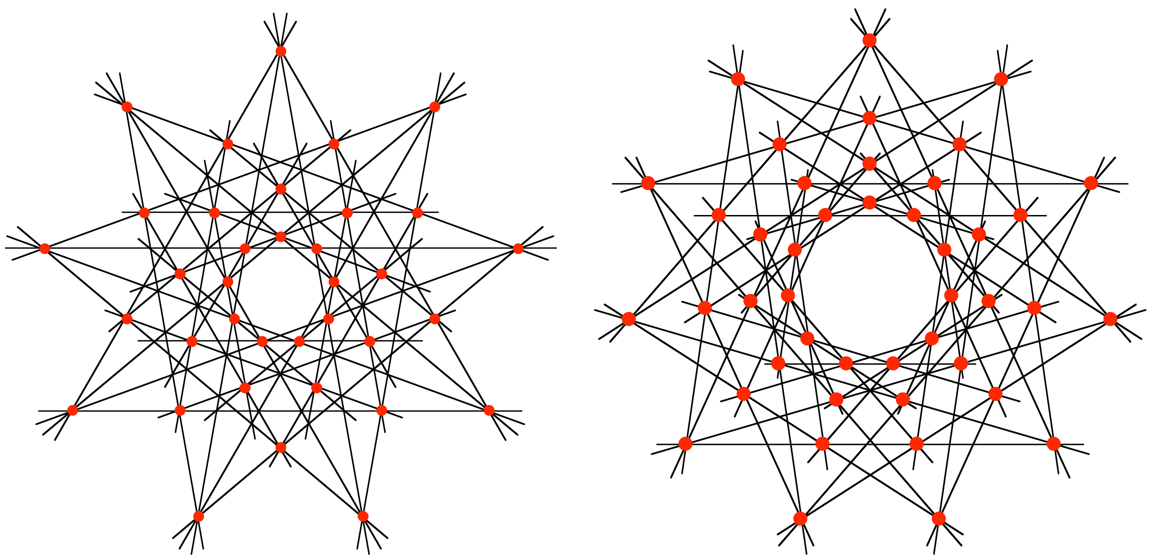


Figure 4.4.6. Typical $((4r)_4)$ configurations with symbols 9#(3,1;2,4;3,2;3,2) and 10#(3,1,2,4,1,3,4,2). Addition of r points-at-infinity to each yields configurations $((5r)_4, (4r)_5)$. Addition of r mirrors gives $((4r)_5, (5r)_4)$

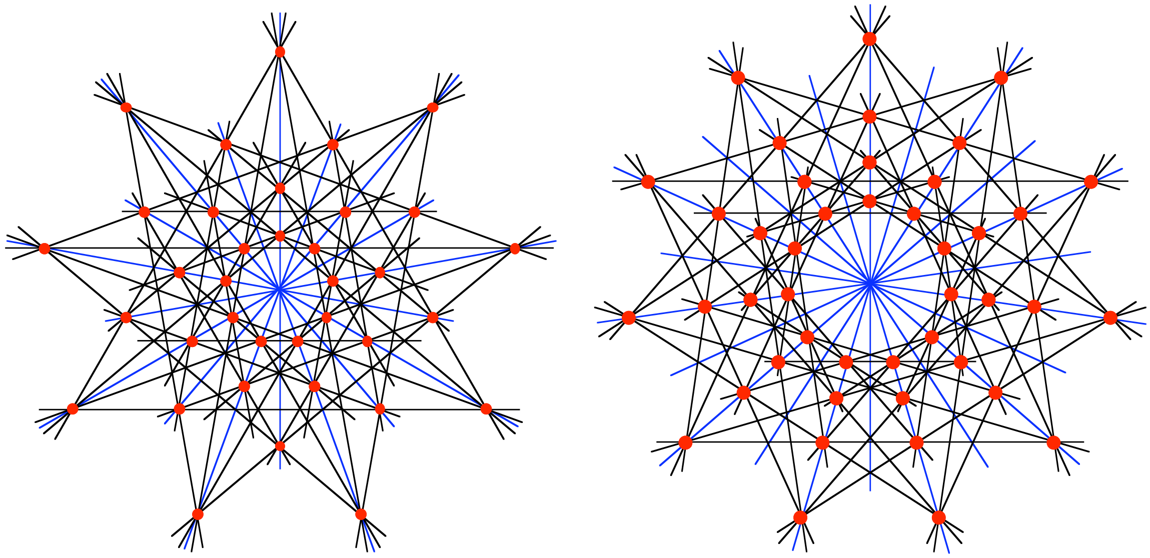


Figure 4.4.7. For odd r, adding r mirrors to 4-configurations such as 9#(3,1;2,4;3,2;3,2) and 11#(3,1,2,4,1,3,4,2) yields [5,4]-configurations $((4r)_5, (5r)_4)$.

There is very little information available about small [q,k]-configurations with still larger values of q and k. Some examples, similar to those above and possible for some particular parameter values, are shown in Figures 4.4.8 and 4.4.9.

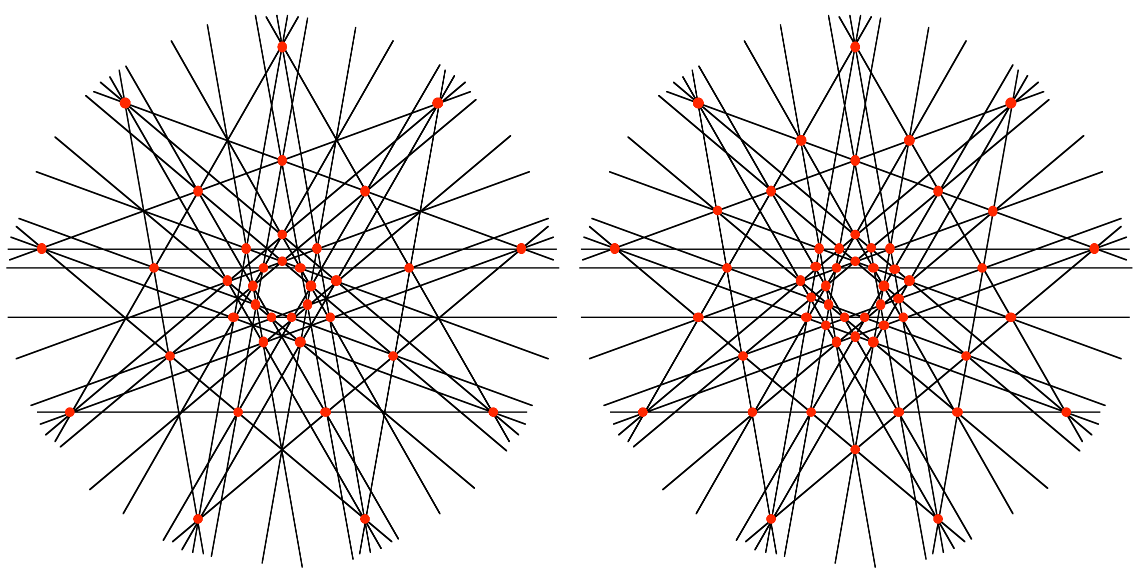


Figure 4.4.8. At left, a 4-configuration (36_4) with symbol 9#(3,1,4,2,1,3,2,4). Adding 18 points yields a (4,6)-configuration $(54_4, 36_6)$ shown at right.

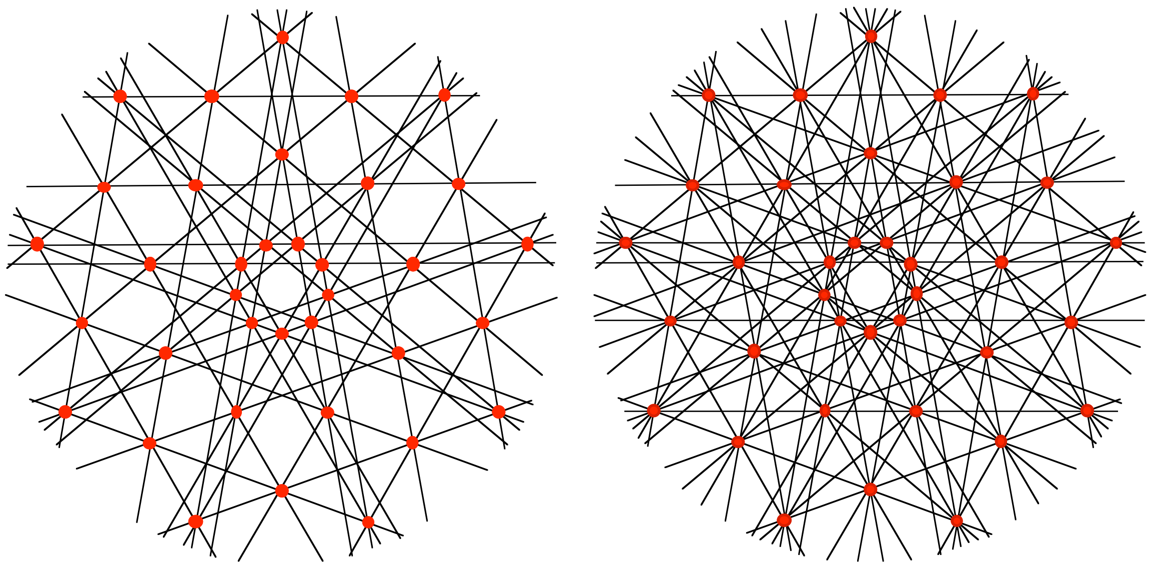
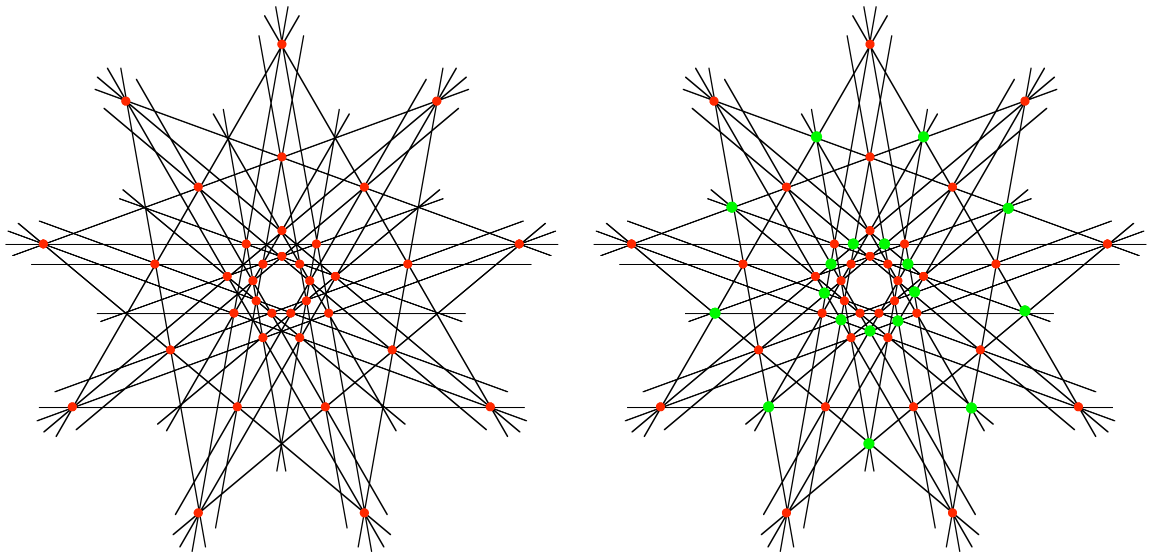


Figure 4.4.9. At left, a 4-configuration (36_4) with symbol 9#(4,3;1,4;2,1;3,2). Adding 18 lines yields a [6,4]-configuration $(36_6, 54_4)$ shown at right. Adding to it the nine lines of mirror symmetry yields a $(36_7, 63_4)$ configuration. Adding instead the nine points at infinity leads to a configuration $(45_6, 54_5)$.

Many other 4-astral 4-configurations can be used in constructions similar to the ones illustrated in Figures 4.4.7 to 4.4.10.

A complete determination of astral [6,4]-configurations (and their polars) was carried out by L. Berman [B2]. These are configurations in which each point is on six lines and each line contains four point, there being two orbits of points and three orbits of lines. As demonstrated in [B5] there are precisely five connected astral $(60_6,90_4)$ configurations, and no other connected astral [6,4]-configurations. One of these is shown in Figure 4.4.11. This configuration can be understood as superposition of three astral (30_4) configurations: the sporadic 30#(12,10;6,10), and the systematic 30#(12,10;3,9) and 30#(10,6;3,9). Similarly for the other four. Some other results on [q,k]-configurations can also be found in [B5].

The material we have presented in this section exhausts the knowledge available to us. As in most other sections, there are lots of obvious questions and open problems for which we have no guesses as to the correct answers. The hope is that some readers will take it as a challenge to enlarge the compass of known facts.



(a) (b)

Figure 4.4.10. (a) The 4-astral configuration 9#(3,1;4,2;1,3;2,4) has quadruplets of lines concurrent at points that are not configuration points. (b) Adding these 18 points (green) yields a $(54_4, 36_6)$ configuration. Adding nine points at infinity (in the direction of quadruplets of parallel lines) yields a $(63_4,36_7)$. Adding instead the nine mirrors results in a $(54_5,45_6)$ configuration.

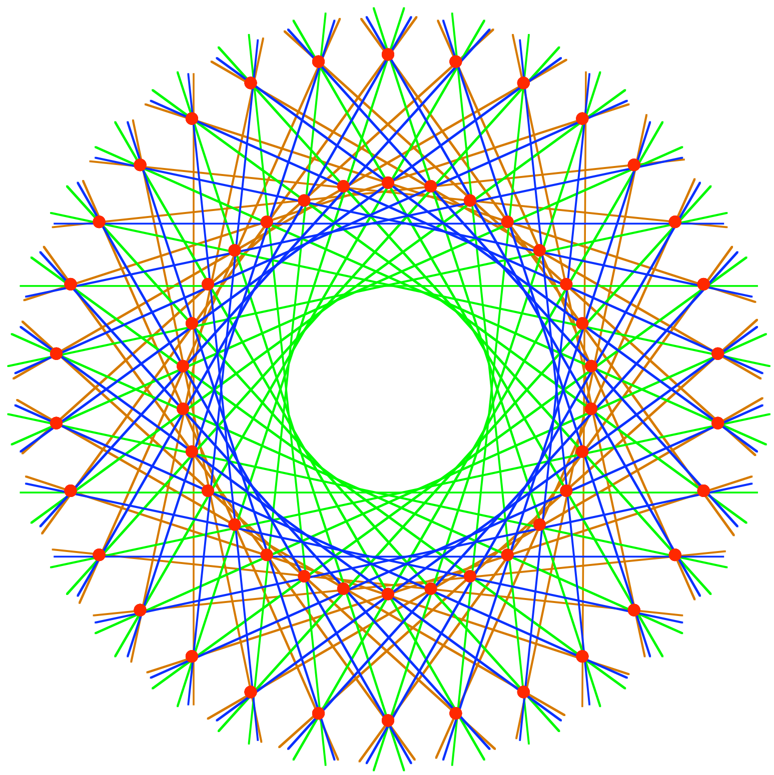


Figure 4.4.11. An astral $(60_6, 90_4)$ configuration, taken from [B5].

Exercises and problems 4.4

1. Decide whether any combinatorial configuration $(26_3, 13_6)$ can be realized geometrically or topologically.
2. Do there exist any geometric configurations $(2n_3, n_6)$ with n not a multiple of 3?
3. Draw a configuration $(45_4, 36_5)$.
4. Draw a configuration $(54_4, 36_6)$.
5. Draw a configuration $(49_3, 21_7)$.
6. Draw as small a configuration of type (p_8, n_3) as you can find.
7. By consulting the lists in Section 3.6, describe the other four astral $(60_6, 90_4)$ configurations.
8. Find configuration $(60_6, 90_4)$ that are not astral.