# 4.3 [3, 4]- AND [4, 3]-CONFIGURATIONS

In the present section we shall survey the known facts concerning combinatorial and geometric [3, 4]- and [4, 3]-configurations.

The parameters of any combinatorial $(p_3, n_4)$ or $(n_4, p_3)$ configuration must satisfy the conditions $3p = 4n$, $p \geq 1 + 3 \cdot 3 = 10$ and $n \geq 1 + 4 \cdot 2 = 9$. Thus p must be divisible by 4 and n must be divisible by 3, so that the only possible configurations are those of the form $((4r)_3, (3r)_4)$ or $((3r)_4, (4r)_3)$, respectively, for $r = 3, 4, 5, \ldots$ . For combinatorial as well as geometric configurations, the existence of $((4r)_3, (3r)_4)$ implies by duality resp. polarity the existence of $((3r)_4, (4r)_3)$, and conversely. Hence it is sufficient in the following result to limit attention to one of the two cases.

**Theorem 4.3.1.** For each integer $r \geq 3$ there exists a combinatorial configuration $((4r)_3, (3r)_4$; topological and geometric $((4r)_3, (3r)_4)$ configurations exists for each $r \geq 4$.

**Proof.** We start with a combinatorial $(12_3, 9_4)$ configuration, given by the following configuration table.

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|
| A | A | A | L | L | L | M | M | M |
| B | G | K | B | G | K | B | G | K |
| C | F | J | J | C | F | F | J | C |
| D | E | H | E | H | D | H | D | E |

Table 4.3.1. A configuration table for a $(12_3, 9_4)$ configuration.

In order to complete the proof in case $r = 3$, we have to prove that no combinatorial configuration $(12_3, 9_4)$ can be realized by points and lines. For that we recall the result known as "Sylvester's problem", which we mentioned in Section 2.1 as Lemma 2.1.1.

To apply the Sylvester result to the question at hand, we note that in any combinatorial configuration $(12_3, 9_4)$ the 36 pairwise intersections of the 9 lines have to occur in

12 triplets — three intersections at each of the 12 points of the configuration. However, since (by Sylvester) in every topological or geometric configuration at least one such intersection is an "ordinary" one (which is therefore not a point of the configuration), there are not enough pairwise intersections to form 12 triplets.

On the other hand, it is possible to give a geometric realization of the dual configuration, but with two of the "lines" neither straight lines nor pseudolines. An example is shown in Figure 4.3.1.

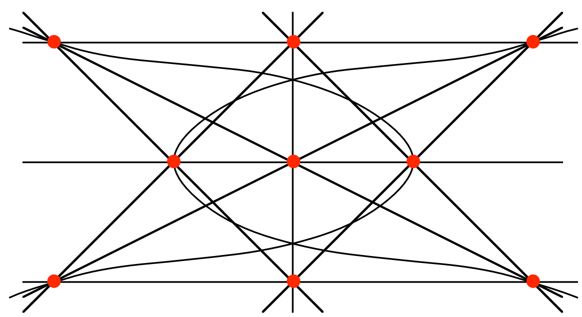

Figure 4.3.1. A realization of a $(9_4,12_3)$ configuration, dual to the one in Table 4.3.1; two of the "lines" are not straight. With a slight modification these two "lines" could have been chosen as circles.

For the remaining part of the proof of Theorem 4.3.1 we only have to exhibit appropriate geometric configurations of points and lines. The literature contains a number of papers devoted to the $(16_3, 12_4)$ configurations, or to the $(12_4, 16_3)$ configurations dual to them; several examples of the former kind are shown in Figure 4.3.2.

There appears to be no published mention of geometric $((4r)_3, (3r)_4)$ configurations with $r \geq 5$. However, examples of such configurations are very easy to produce. One method (see Figure 4.3.3) starts by placing $2r$ points equidistributed on a circle. Each of these points is connected to the one diametrally opposite to it, as well as to the two points separated from it by two other points. Adjoining the $2r$ triple intersections (whose existence is clear by the symmetry of the diagram) yields a $((4r)_3, (3r)_4)$ configuration, as required.

Other $((4r)_3, (3r)_4)$ configurations may be constructed by slight variations of this method; several are shown in Figure 4.3.4. In all these cases, the geometric existence of the configurations is an obvious consequence of the high degree of symmetry involved.

Although the configurations $(16_3, 12_4)$ and/or $(12_4, 16_3)$ have been studied for at least 150 years (starting not later than Hesse [H2] in 1848, in the "prehistoric" era of configurations), there still are many unresolved questions. It has been shown (or claimed – there seems to have been no independent verification) that there are precisely 574 combinatorial configurations $(12_4, 16_3)$, see Gropp [G14], [G16]. The large number of such configurations helps explain why there is no clarity on the question which (or, whether all) configurations $(124_4, 16_3)$ have geometric realizations in the Euclidean plane. Two additional aspects probably contribute to the lack of clarity: On the one hand, most of the relevant papers have been published in journals that are not well known nor widely available, many in Czech which is not too widely spoken; a large number of references is listed below. On the other hand, from the very beginning, these configurations have been

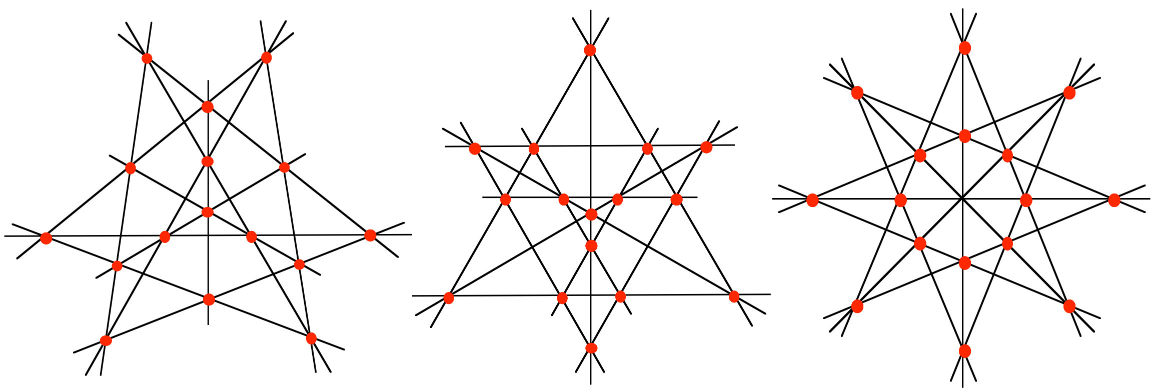

Figure 4.3.2. Three examples of configurations $(16_3, 12_4)$.

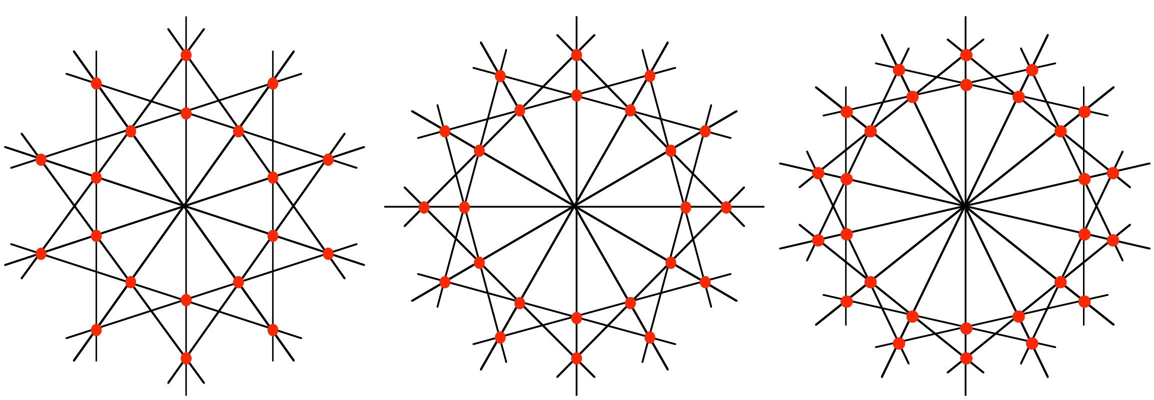

Figure 4.3.3. Examples of configurations $(20_3, 15_4)$, $(243_3, 18_4)$ and $(28_3, 21_4)$.

studied in close connection with the theory of cubic curves. This connection, in turn, is not too well known these days, and also makes it hard to know which parts of the claims of possibility of realization rely on configurations in the complex plane, and which claims of impossibility are due to the restriction of attention to configurations with vertices on cubic curves. See below for some relevant ideas.

From the duality in the projective plane it follows that geometric configurations $((3r)_4, (4r)_3)$ exist if and only if $r \geq 4$. One example of a $(12_4, 16_3)$ configuration is shown in Figure 4.3.5. In contrast to the very symmetric diagrams representing the $(16_3, 12_4)$ configurations, the diagrams of the $(12_4, 16_3)$ configurations shown in most publications are far from symmetric.

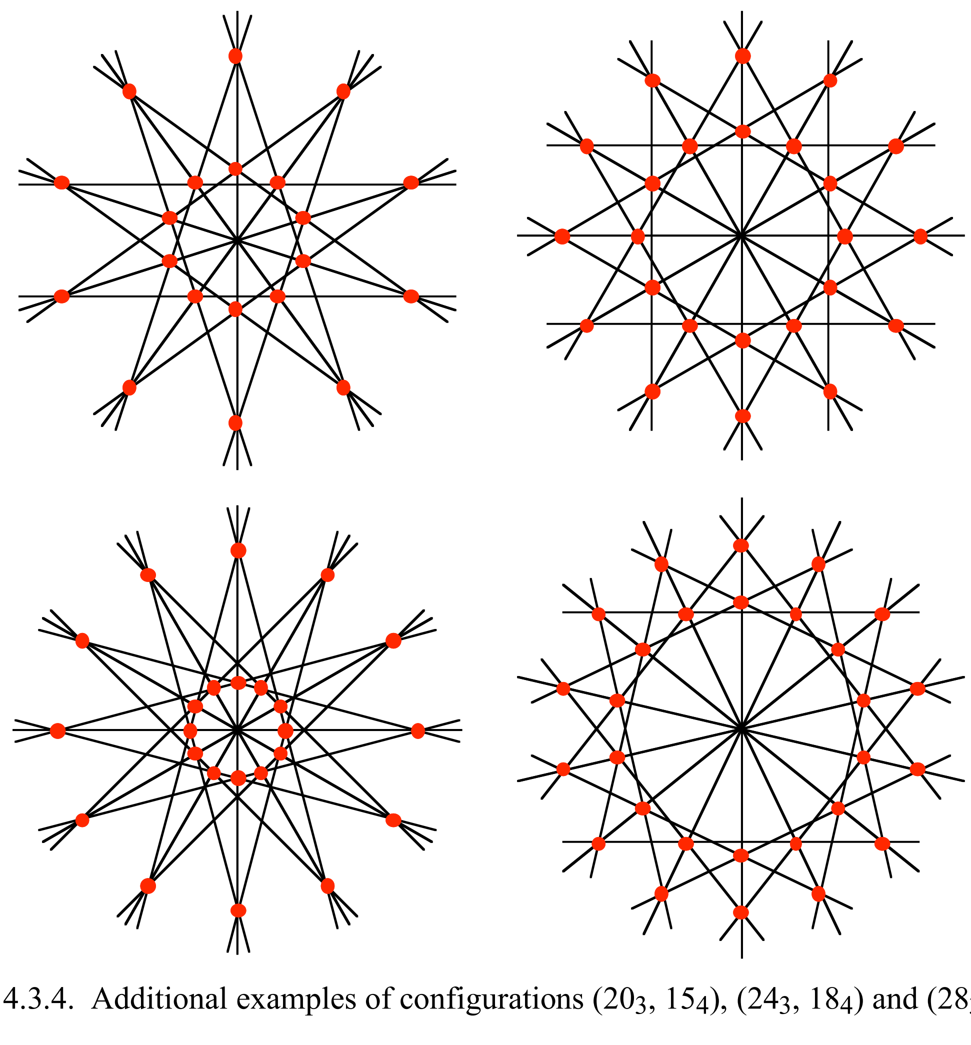

Figure 4.3.4. Additional examples of configurations $(20_3, 15_4)$, $(24_3, 18_4)$ and $(28_3, 21_4)$.

The reason for the difference is that projective duality does not in general preserve Euclidean symmetries — unless one considers the configurations in the extended Euclidean plane. In particular, all examples in Figures 4.3.3 and 4.3.4 have lines passing through the center of symmetry (taken at the origin) which have to be mapped to "ideal points" in order to preserve symmetry. If this is accepted, then it is easy to produce very symmetric $(16_3, 12_4)$ configurations, such as the one in Figure 4.3.6.

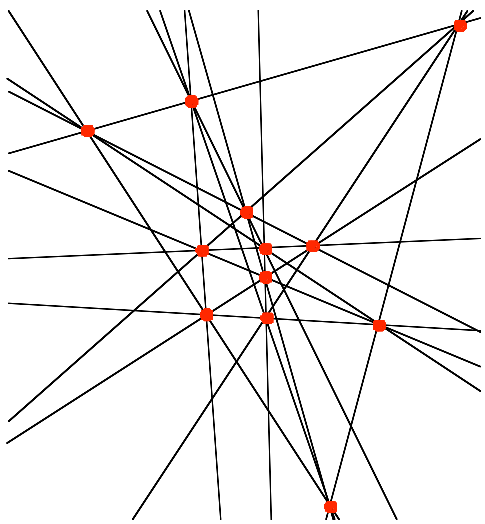

Figure 4.3.5. An example of a geometric configuration $(12_4,16_3)$.

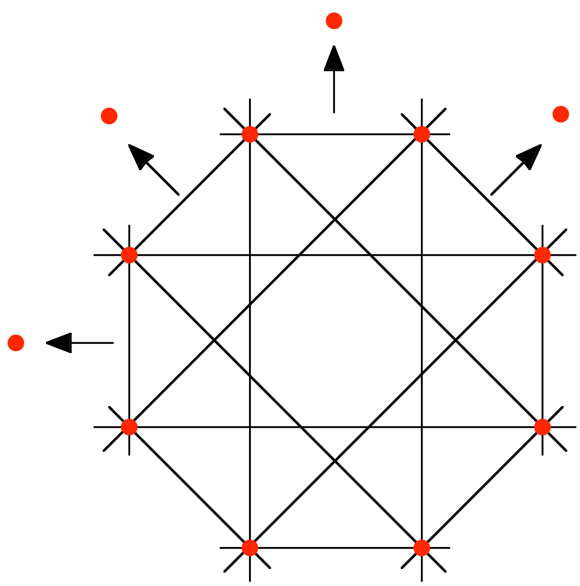

Figure 4.3.6. An example of a geometric configuration $(12_4,16_3)$ that is astral in the extended Euclidean plane.

Additional examples of quite symmetric ($12_4$, $16_3$) configurations are shown in Figure 4.3.7. These have vertices on cubic curves.

In order to give a feeling for the relation of cubic curves to configurations, we show another example in Figure 4.3.8. This is a geometric configuration ($12_4$,$16_3$) on a cubic curve, from the paper by V. Metelka [M17]. The equation of this cubic curve in homogeneous coordinates (x,y,z) is

$$z(x^2 + y^2) + x(x^2 - 3y^2) = 0$$

and the points are:

| | | | |
|---|---|---|---|
| M = (1,1,1) | N = (0,1,0) | O = (1,-1,1) | P = (1,-t,2) |
| Q = (1,t,2) | R = (t,1,0) | S = (-t,1 0) | T = (1,t-2,1-t) |
| U = (1,2-t,1-t) | V = (1,t+2,t+1) | W = (1,-t-2,t+1) | X = (1,0,-1) |

where $t = \sqrt{3}$.

As is well known, an easy way to see whether three points given in homogeneous coordinates are collinear is by checking whether the determinant formed by their coordinates is 0. Thus the assertions about which triplets are collinear (as indicated by Figure 4.3.8) can be algebraically verified.

As Metelka observed (this is the reason he considered the configuration "special") there are three additional lines that pass through three of the points. These three lines are indicated by the dashed lines in Figure 4.3.9. It is worth noting that the maximal number of collinear triplets determined by 12 point is 19 – this is one of frequently raised "orchard problems"; see more details at [B33].

As a clarification of what was briefly mentioned above regarding the use of cubic curves in looking for construction of configurations and related objects, in Figure 4.3.10 we show a diagram of a cubic curve on which are marked several values of the "degree" parameter. The following explanations are taken from the old paper [B33], from which the curve in Figure 4.3.10 was copied as well. References to texts that establish the properties in question are given in [B33]; the notation is the one that seems traditional in the literature.

A suitable projective image of each real non-singular cubic curve has an equation of the form

(1) $$y^2 = 4x^3 - g_2x - g_3$$

where $g_2$ and $g_3$ are real constants. The curve $C$ given by equation (1) may be parametrized by

(2) $$x = \wp(u), \quad y = d\wp(u)/du,$$

where $\wp(u)$ is the Weierstrass elliptic function defined by

$$u = \int_{\wp(u)}^{\infty} (4x^3 - g_2x - g_3)^{-1/2}\, dx.$$

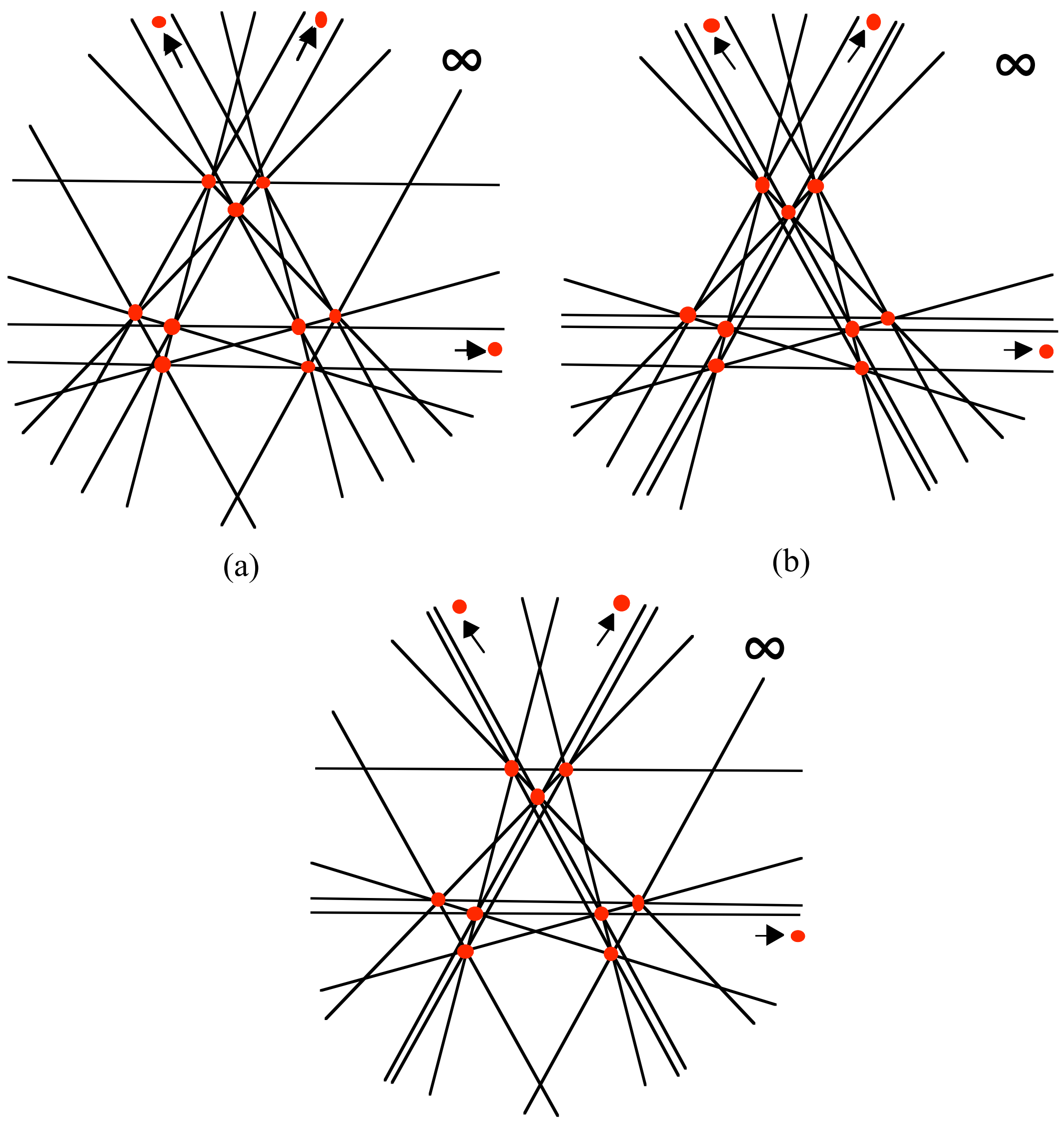


Figure 4.3.7. Three examples of quite symmetric configurations $(12_4, 16_3)$. The $\infty$ symbol is meant to indicate that the line at infinity is a line of the configuration.

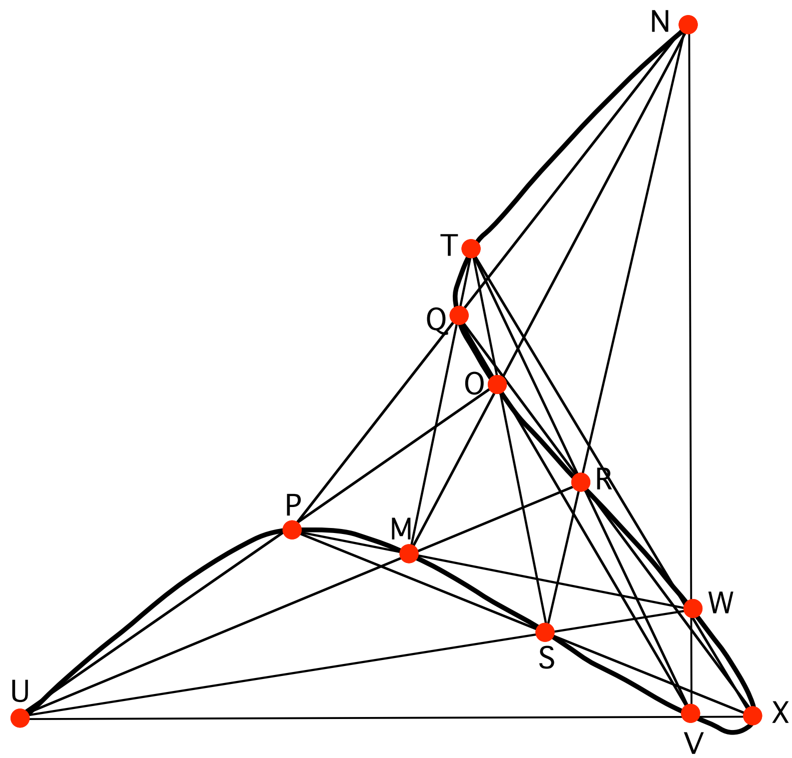


Figure 4.3.8. A configuration $(12_4, 16_3)$ with points on a cubic curve.

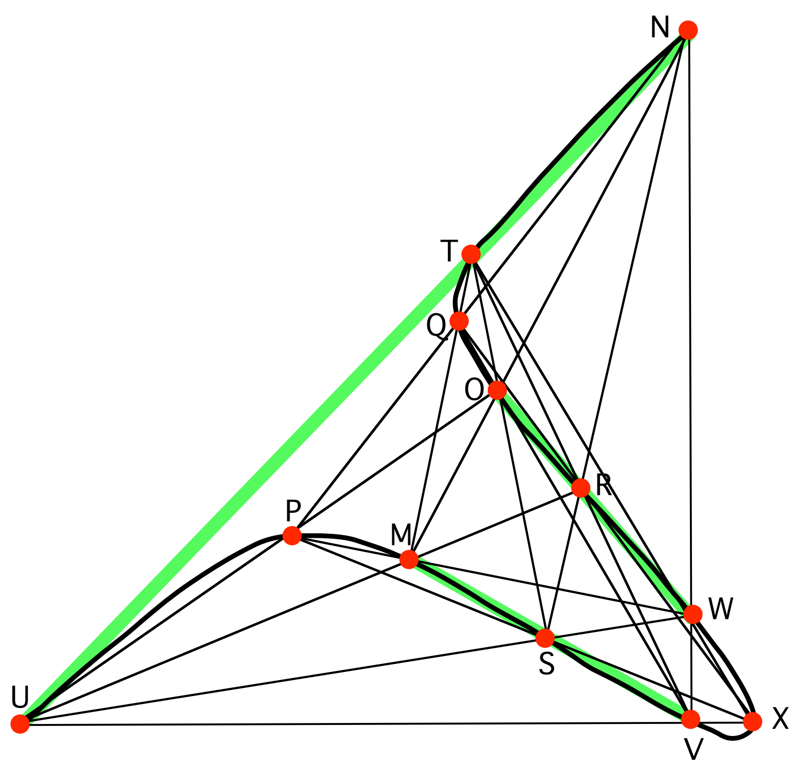


Figure 4.3.9. The points of the $(12_4, 16_3)$ configuration in Figure 4.3.8 determine three additional lines each incident with three of the points.

The Weierstrass elliptic function ℘(u) is a doubly-periodic meromorphic function of the **complex** variable u, and for real $g_2$, $g_3$ it has a real period that we shall denote $2\omega$ (as well as a purely imaginary period $2\omega'$). The parametrization (2) yields for real u the "odd circuit" (branch) of the cubic C. In case $D = g_2^3 - 27g_3^2 < 0$ this is the only real part of the curve C ("unipartite cubic"), while in case $D > 0$ the curve C has also an "even circuit" corresponding to the values $u = v + \omega'$, where v is real. (We shall be interested only in the "odd circuit".)

The importance of cubic curves for the present concerns is based on the following result of N. H. Abel:

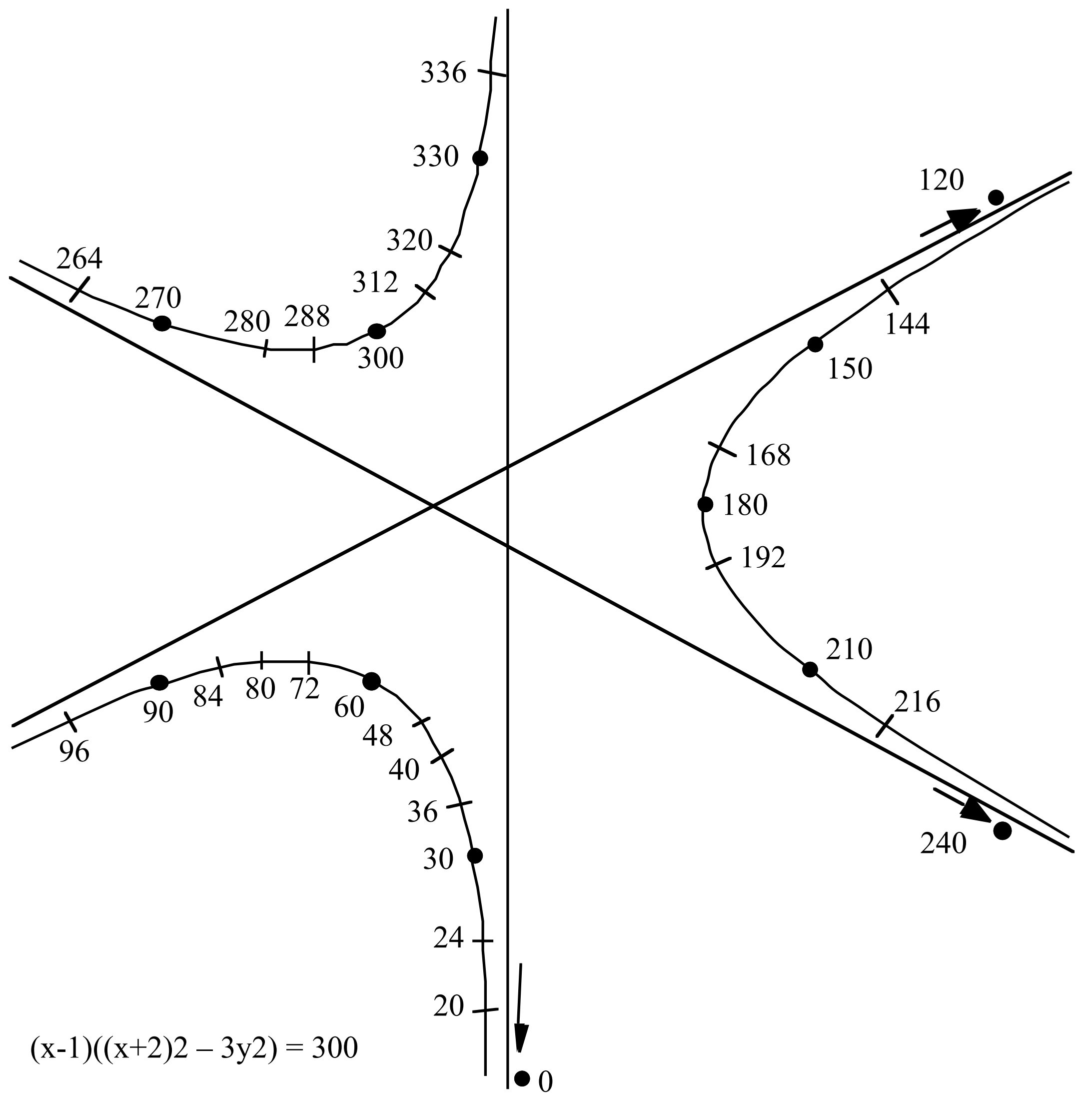


Figure 4.3.10. A cubic curve, with a parametrization derived from the Weierstrass ℘(u) function, as explained in the text.

Denoting by $P(u) = (\wp(u), d\wp(u)/du)$ the point on the cubic C given by (1), (2) and corresponding to the real parameter u, a necessary and sufficient condition for the collinearity of the points $P(u)$, $P(u')$, $P(u'')$ on the odd circuit of C is

$$u + u' + u'' \equiv 0 \pmod{2\omega}.$$

The curve we use is given by the equation $y^2 = 4x^3 - 1$, and by consulting appropriate tables or software we find that $\omega = 1.529954037...$ . As in much of the numerical work on the elliptic functions, we replace $2\omega$ by 360°; in Figure 4.3.10 we denote the points simple by their parameter-value in "degrees".

A practical weakness of the method is an inconvenient bunching of the points of interest. The situation can be improved by using a suitable projective transformation of the curve C; this goes back to W. K. Clifford in 1865. The "odd circuit" of C contains three collinear points of inflection $P(0)$, $P(2\omega/3)$, $P(4\omega/3)$. If we choose the line determined by these points as the "ideal line", and the points themselves to be in equiinclined directions, there results a very convenient and symmetric representation of C.

We are using the curve C with equation $y^2 = 4x^3 - 1$, for which the Clifford transformation may be achieved by

$$x = (2x^* + 1)/(2x^* - 2), \qquad y = 3y^*/(x^* - 1).$$

This results (on omitting the asterisks) in the equation

$$(x - 1)(3y^2 - (x + 2)^2) = \text{const.}$$

For better visibility we choose the constant as –300, yielding the curve in Figure 4.3.10. This curve is used in some of the exercises below.

The following is an extensive list of papers I am aware of that deal with $(16_3, 12_4)$ or $(12_4, 16_3)$ configurations. Some of them contain additional references to earlier papers. [B34], [D5], [G14], [H2], [M9], [M10], [M11], [M12], [M13], [M14], [M15], [M16], [M17], [M5], [M6], [M7], [R6], [Z1], [Z2], [Z3], [Z6].

A configuration $(15_4, 20_3)$ was described by Cayley [C2*] in 1846. There seems to be no other discussion in the literature of $((4r)_3, (3r)_4)$ configurations with $r \geq 5$ (or their duals).

*        *        *        *        *        *

The introduction of k-astral configurations helped develop the study of 3- and 4-configurations. It seems reasonable that investigations of [3,4]-configurations and similar objects would be advanced by moving from the concentration on the smallest cases to more general situations. As examples capable of various generalizations we show in Figure 4.3.11 and 4.3.12 configurations $(20_3,15_4)$ and $(15_4,20_3)$ with cyclic symmetry group $c_5$, and in Figure 4.3.12 a configuration $(18_4,24_3)$ with symmetry group $c_6$. It is clear that such configurations fit into infinite families for which the systematic investigation and notation still need to be developed.

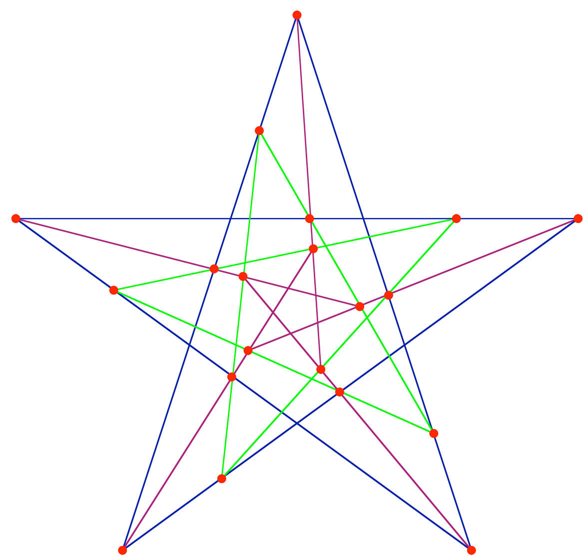

Figure 4.3.11. A [4,3]-astral $(20_3,15_4)$ configuration with symmetry group $c_5$.

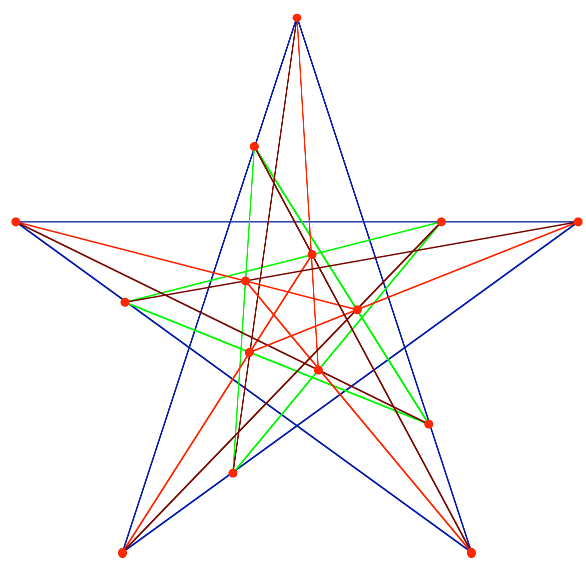

Figure 4.3.12. A [3,4]-astral $(15_4,20_3)$ configuration with symmetry group $c_5$.

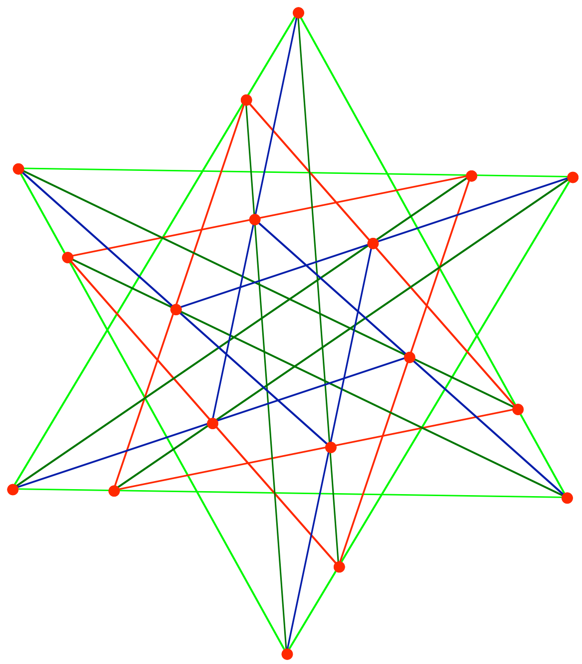

Figure 4.3.13. A [3,4]-astral $(18_4,24_3)$ configuration with symmetry group $c_6$.

**Exercises and problems 4.3.**

1. Show that each of the permutations (described by their cycle decompositions) (A)(L)(M)(BGK)(CFJ)(DEH) and (ABCD)(LGJHMKFE) maps the combinatorial configuration $(12_3, 9_4)$ of Table 4.3.1 onto itself. Deduce that the automorphisms of the configuration act transitively on its points as well as on its lines. Decide whether the configuration is flag-transitive? (**Flag** = pair consisting of a "point" and a "line" incident with it.)

2. Decide whether all combinatorial $(12_3, 9_4)$ configurations are isomorphic, that is, whether the configuration $(12_3, 9_4)$ is unique. (Hint: Delete a line and all its points.)

3. Prove that any geometric realization of the $(12_3, 9_4)$ configuration must contain at least two "lines" that are not straight.

4. Set up the configuration table of the configuration $(9_4, 12_3)$ dual to the configuration in Table 4.3.1. Decide whether this configuration can be geometrically realized with straight lines or with pseudolines.

5. Describe the configuration table of the $((4r)_3, (3r)_4)$ configuration constructed in the proof of Theorem 4.3.1.

6. Show that the two $(20_3, 15_4)$ configurations shown in Figures 4.3.3 and 4.3.4 are not isomorphic.

7. Decide whether any among the three configurations $(16_3, 12_4)$ in Figure 4.3.2 are isomorphic, and whether the two configurations $(24_3, 18_4)$ in Figure 4.3.4 are isomorphic.

8. Starting with 12 points equidistributed on a circle, how many $(24_3, 18_4)$ configurations can you construct that have different appearance? Are any two among them isomorphic?

9. For general r, starting with 2r points equidistributed on a circle, how many $((4r)_3, (3r)_4)$ configurations can you construct that have different appearance? Are any among them isomorphic?

10. Draw symmetric realizations in the extended Euclidean plane of the polars of the configurations in Figure 4.3.2.

11. Decide whether any among the three configurations $(12_4, 16_3)$ in Figure 4.3.7 are isomorphic.

12. Draw the polar configurations of the configurations in Figure 4.3.7.

13. Verify that those triplets shown as collinear in Figures 4.3.8 and 4.3.9 that contain the point U are, in fact, collinear.

14. Find in Figure 4.3.9 a configuration $(12_4,16_3)$ that contains the dashed lines, and decide whether it is isomorphic with the configuration in Figure 4.3.8.

15. Determine the group of automorphisms of the configuration in Figure 4.3.8.

16. On the cubic curve in Figure 4.3.10, find a configuration $(9_2,6_3)$, and a configuration $(12_3)$. Can you find any other configurations?

17. Decide whether the configurations in Figures 4.3.11 and 4.3.12 are duals of each other? If so, find a duality map. If not, find their duals.

18. Find the dual of the configuration in Figure 4.3.13.

19. Develop a theory — similar to the ones in Chapters 2 and 3 — of the [4,3]-configurations.