4.2 k-CONFIGURATIONS FOR $k \geq 6$

As justification for general existence statements for k-configurations with $k \geq 6$ we recall the configurations LC(k) introduced in Section 1.1. They illustrate the possibility of geometric configurations $((k^k)_k)$ for all k. Naturally, one may be interested in smaller examples, and there are systematic ways to find them, even though they yield configurations that are neither stimulating to look at, nor very small.

The first such construction, during the "prehistory period" of configurations, is due to Cayley [C2*] in 1846. Reflecting the spirit of the times, Cayley writes (in French, in a paper published in a German journal!):

> " *sans recourrir à aucune notion métaphysique à l'égard de la possibilité de l'éspace à quatre dimensions*, ..."

and proceeds to define configurations of flats of various dimensions spanned by families of points in general position; intersecting these with suitable planes he devises (for $k \geq 2$) configurations (n_{k+1}) where $n = (2k+1)!/k!(k+1)!$ Thus, what Cayley describes are geometric configurations (35_4), (126_5), (462_6), (1716_7), and so on. He also describes various unbalanced configurations, about which we shall report in Section 4.3.

Although Cayley's constructions yield smaller configurations than the LC(k), there are better construction methods that are easy generalizations of the ones we detailed in Section 3.3, considered there for the 4-configurations.

The (5m) construction which in Section 3.3 led from a configuration (m_3) to $((5m)_4)$ generalizes immediately: From any (m_k) configuration, taking k+1 copies that all intersect at the same points of a suitable line, and then adding m appropriate lines connecting corresponding points in these copies, we obtain a $(((k+2)m)_{k+1})$ configuration. Using the smallest configurations available, this construction leads from (9_3) to (45_4), from (18_4) to (108_5), from (48_5) to (336_6), from (110_6) to (880_7), and so on. Except for the last one, these are not the known minimal configurations — but in the last case this is the best available. Carrying out only the first step, with only k copies of the starting configuration, leads to a (k,k+1)-configuration. Taking a stack of k configurations and adding the lines connecting the corresponding points leads to a (k+1,k)-configuration. Some of the other methods in Section 3.3 generalize as well.

For 6-configurations we can do better than for general k. Figure 4.2.1 shows a 10-astral (110_6) configuration, and Figure 4.2.2 a 4-astral configuration (120_6); both were discovered by L. Berman (private communication).

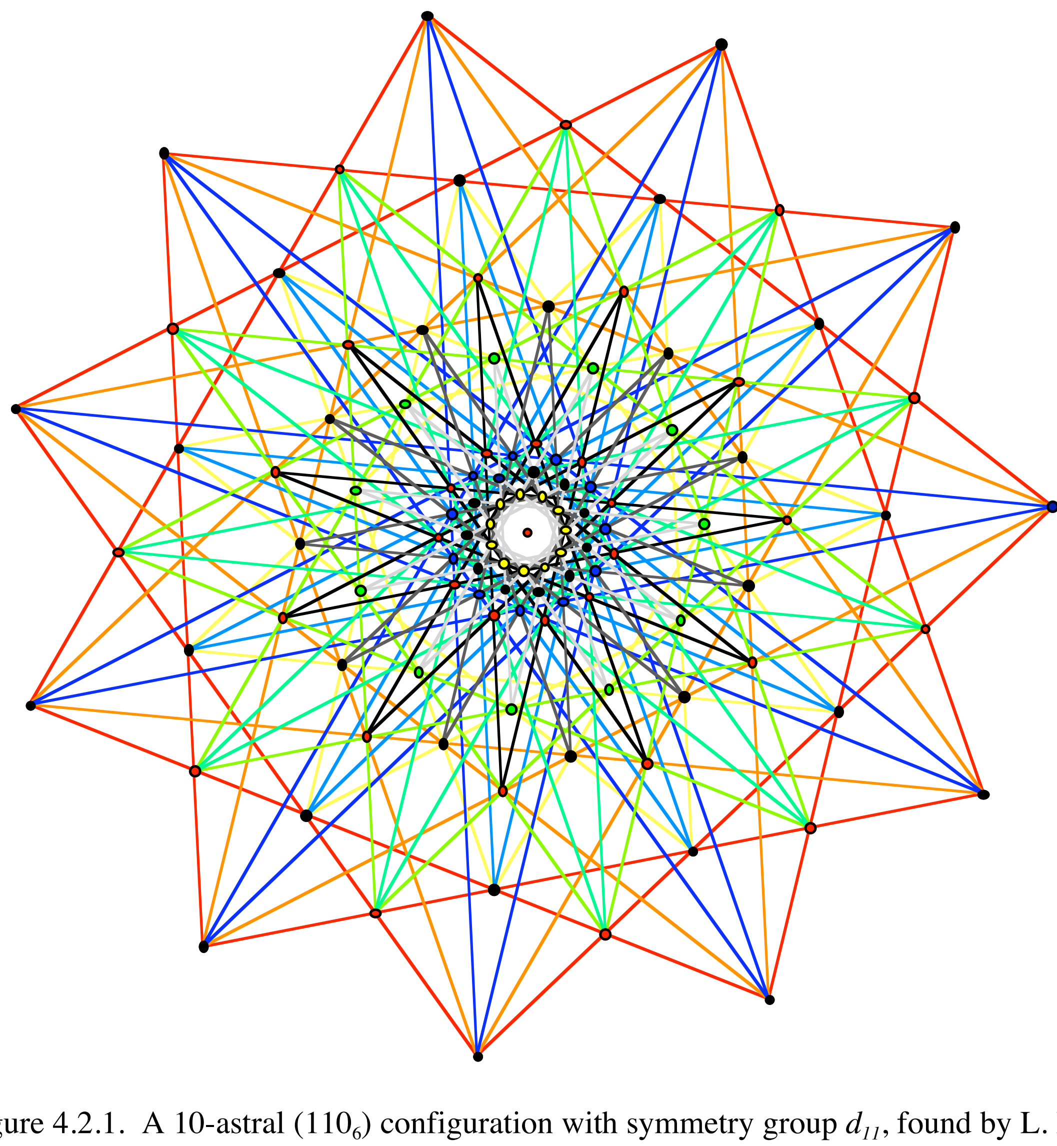

Figure 4.2.1. A 10-astral (110_6) configuration with symmetry group d_{11}, found by L. Berman.

On the other hand, there is negative information available concerning astral (that is, 3-astral) 6-configurations. As proved by Berman [B5], no such configurations exist, nor do any astral [2k,2h]-configurations for $k \geq 3, h \geq 3$.

The paucity of information on the topic of this section is clearly evidenced by its brevity, and the absence of references beyond [C2*] and [B5]. Notice that these are separated by more than a century and a half!

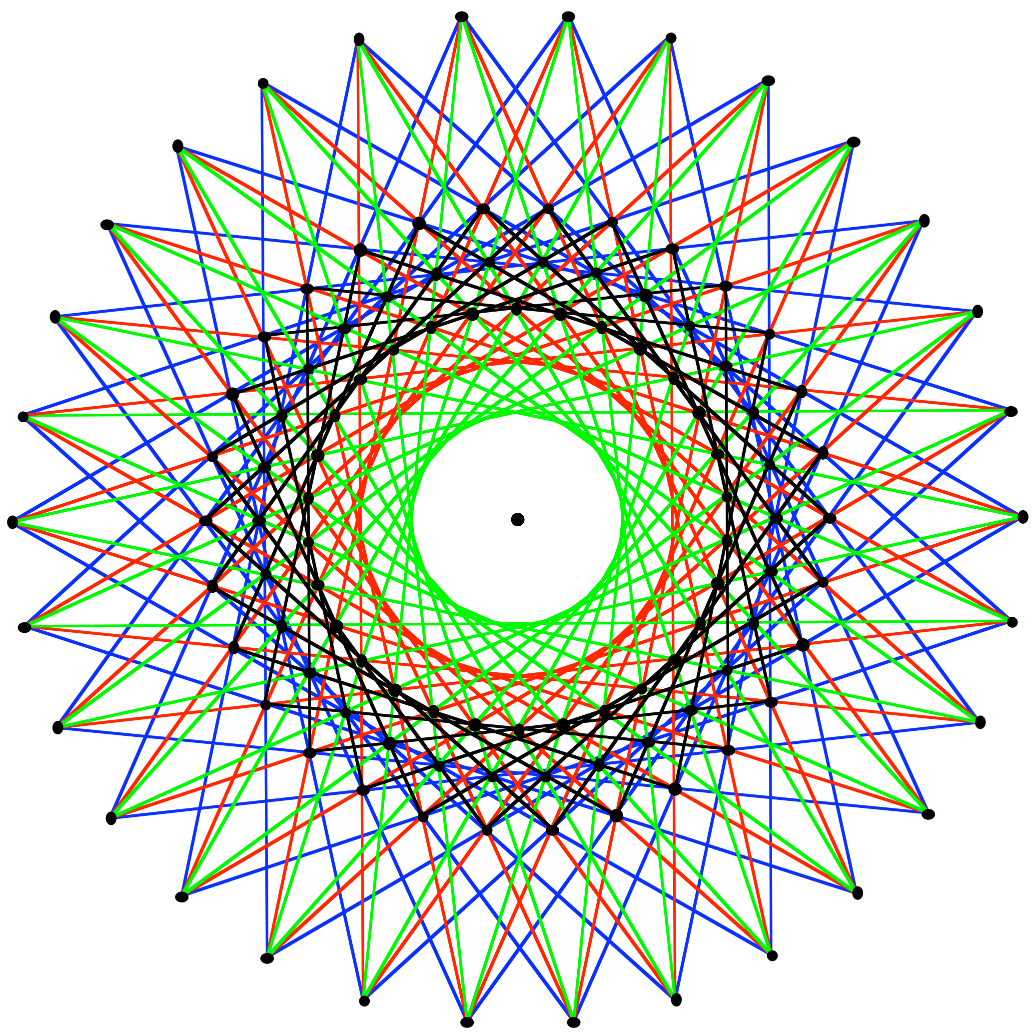

Figure 4.2.2. A 4-astral (120_6) configuration with symmetry group d_{12}, found by L. Berman.

Exercises and problems 4.2.

1. Decide whether there exist any other 4-astral 6-configurations.

2. Find some "small" topological 6-configurations.

3. Is there some systematic construction for 6-configurations that is analogous to the passage from 4-configurations to 5-configurations mentioned in Exercise 9 of section 4.1.

4. Find a visually intelligible 7-configuration.